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A Simplified Conservative Runge–Kutta Method for Numerical Prediction of Fatigue Crack Growth

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ABSTRACT

Accurate numerical prediction of fatigue crack growth under cyclic loading is crucial for ensuring the safety and structural integrity of engineering structures. However, classical numerical integration schemes, such as the fourth-order Runge–Kutta (RK4) method, offer high accuracy but can fail to produce conservative crack size predictions when larger integration step sizes are used; as a result, they may underestimate crack growth and compromise safety margins. To address this issue, modified RK4 schemes have been proposed to enforce conservatism, but these approaches often introduce additional computational overhead or involve complex correction terms. In this study, a simplified conservative RK4 algorithm is proposed to enhance computational efficiency without sacrificing the conservative bias (overestimation) in crack growth prediction. The method integrates Paris law for fatigue crack growth with a reduced-order correction mechanism applied at each stage of the standard RK4 process. The performance of the proposed scheme is evaluated by analysing convergence behaviour, relative error, and conservative prediction regions across a range of Paris law exponents (m = 2 to 5). Simulation results demonstrate that the proposed scheme consistently yields upper-bound crack growth predictions across various step sizes, thereby providing an effective balance between structural safety and computational efficiency. This approach is particularly well-suited for high-cycle fatigue analysis, where the large number of load cycles demands efficient computation and a high level of prediction reliability.

Keywords: Fatigue crack growth; Paris law; Runge–Kutta method; conservative integration; numerical fatigue analysis

INTRODUCTION

Fatigue crack growth is a primary concern in the life-cycle assessment of metallic components subjected to cyclic loading (Suresh 1998; Murakami 2002). Cracks typically initiate at microstructural flaws or stress concentrators and propagate incrementally, often leading to structural failure if not detected and mitigated in time. To manage fatigue risk effectively, accurate numerical tools are required to simulate crack propagation throughout a component's service life.

TABLE 1. Nomenclature

Symbol	Description	Unit
a	Crack length	mm
a_0	Initial crack length	mm
a_{mod}	Crack length using proposed method	mm
a_{RK4}	Crack length using classical RK4 method	mm
C	Paris law material constant	mm/cycle
m	Paris law exponent	–
Y	Geometry correction factor	–
Σ	Applied stress range	MPa
ΔK	Stress intensity factor range	MPa·√mm
N	Number of fatigue loading cycles	cycle
h	Integration step size	cycle
$f(a)$	Crack growth rate function	mm/cycle
Δ	Conservative correction term	–
k_1, k_2, k_3, k_4	Runge–Kutta intermediate slopes	mm/cycle
rel_error	Relative error between methods	–

Among the most widely used models for predicting fatigue crack growth is Paris' Law (Paris and Erdogan 1963), which relates the crack growth rate per cycle to the range of the stress intensity factor. This model has played a pivotal role in simplifying fatigue crack analysis under constant amplitude loading conditions. However, crack growth is governed by more complex mechanisms, including microstructural barriers, residual stresses, load ratio effects, and crack closure phenomena (Elber 1970; Suresh 1998). While the stress intensity factor range used in Paris' Law captures the main driving force for crack extension, it does not fully account for transient, threshold, or nonlinear behavior observed in practice. In such cases, more advanced formulations like those of Forman, Walker, or NASGRO may be necessary.

In critical engineering applications such as aircraft fuselage panels, offshore oil structures, and railway axles fatigue failures often initiate from undetected microcracks and propagate silently over thousands of cycles before reaching a critical state (Ritchie 1999; Schijve 2009). Accurate numerical simulations are therefore essential not only for predicting the remaining life but also for planning inspections and meeting safety regulations. Importantly, many design codes require conservative predictions that systematically overpredict damage. Numerical methods capable of guaranteeing such conservatism, especially under coarse integration or uncertain loading, are vital to enhancing structural safety margins and enabling damage-tolerant design.

One of the most widely used models for predicting fatigue crack growth is Paris' Law (Paris and Erdogan 1963), which relates the crack growth rate per cycle to the range of the stress intensity factor

$$\frac{da}{dN} = C\Delta K^m \tag{1}$$

where:
- a is the crack length,
- N is the number of load cycles,
- C and m are material constants determined experimentally,
- ΔK is the stress intensity factor range.

The stress intensity factor for Mode I crack propagation under tensile loading is defined as:

$$\Delta K = Y.\Delta\sigma.\sqrt{\pi a} \tag{2}$$

where Y is a geometry-dependent factor and $\Delta\sigma$ is the applied stress range. The nonlinearity of Paris' Law, due to the dependence of ΔK on $\sqrt{a}$, necessitates the use of numerical methods for integration. (Hairer, Nørsett & Wanner 1993; Butcher 2003; LeVeque 2007)

Among available numerical solvers, the forward Euler and midpoint methods are commonly used due to their simplicity. However, their low-order accuracy and poor stability make them unsuitable for long-term crack propagation prediction. In contrast, the fourth-order Runge–Kutta (RK4) method offers a good balance between accuracy and computational cost (Butcher, 2003). Nevertheless, the standard RK4 formulation does not inherently ensure conservative crack length predictions, especially when larger integration step sizes are used—a critical limitation in safety-sensitive applications such as aircraft or pressure vessels.

This issue has been addressed in prior work by Amann and Lammering (2016), who proposed a modified RK4 scheme designed to enforce conservative crack length estimates. While effective, their method introduces additional correction computations at all RK stages, making it more complex for industrial or real-time implementation.

In this study, a simplified version of the conservative RK4 scheme is proposed to reduce computational overhead while maintaining overestimation behavior. Unlike Amann's approach, the correction is applied selectively

using an approximated term that avoids recomputing intermediate derivatives. Recent advancements, such as adaptive and stochastic Runge–Kutta solvers, further underscore the growing importance of numerical robustness and conservative prediction (Gomes, Beck & Garmbis 2023)—objectives central to the method presented here.

Existing conservative RK approaches often rely on multi-stage corrections or complex adaptive schemes (Amann & Lammering 2014; Zhou & Yi 2015). Although accurate, they may lack computational efficiency for large-scale simulations. By contrast, the proposed method applies a lightweight correction only at the final RK stage, preserving both accuracy and efficiency. This simplification is particularly valuable in applications like aerospace component design, where thousands of fatigue simulations are performed under uncertainty. Moreover, the method reinforces the importance of systematically overestimating crack growth to ensure safe lifetime predictions.

To address this trade-off, the present study proposes a simplified conservative RK4 algorithm that preserves the key strengths of the classical RK4 fourth-order convergence and stability while reducing computational overhead. The modification targets only the essential integration stages required to introduce conservativeness, thus offering a leaner numerical scheme suitable for large-scale or probabilistic fatigue simulations.

The proposed method is tested across various Paris law exponents $m \in [2,5]$ and its behavior is analyzed by examining the resulting crack growth trajectories, assessing the relative error compared to classical RK4 reference solutions, and mapping the conservative region boundaries as a function of the reduced integration step size. This comprehensive analysis provides insight into the method's accuracy, convergence characteristics, and ability to maintain conservative predictions under varying simulation conditions.

The results confirm that the proposed algorithm consistently overestimates crack growth across a broad range of parameters, making it an efficient and reliable tool for fatigue life prediction in safety-critical applications.

METHODOLOGY

FATIGUE CRACK GROWTH MODEL

The progression of fatigue cracks in metallic components under cyclic loading is governed by fracture mechanics models. One of the most widely adopted is Paris' law (Paris and Erdogan 1963), which relates the crack growth rate per cycle to the range of the stress intensity factor (SIF):

$$\frac{da}{dN} = C(\Delta K)^m \tag{3}$$

where:
- a is the crack length,
- N is the number of load cycles,
- C and m are material constants determined experimentally,
- ΔK is the SIF range across a fatigue cycle.

For a center-cracked plate in Mode I loading, the SIF range is commonly expressed as:

$$\Delta K = Y.\Delta\sigma.\sqrt{\pi a} \tag{4}$$

where:
- $\Delta\sigma$ is the cyclic stress range,
- Y is a dimensionless geometry correction factor dependent on crack and specimen shape.

Due to the square root dependence on a, the equation becomes nonlinear and typically requires numerical integration to track crack growth across cycles. Bower, 2010 standardizes the experimental validation of fatigue crack growth models, reinforcing the importance of accurate numerical implementation in predictive simulations.

NUMERICAL INTEGRATION WITH CLASSICAL RK4 METHOD

The Runge–Kutta fourth-order method (RK4) is widely used to solve ordinary differential equations (ODEs) due to its balance between computational cost and accuracy (Butcher 2003; Hairer et al. 1993). Applied to Paris' law, the goal is to approximate the evolution of crack length $a(N)$ using:

$$\frac{da}{dN} = f(a) \tag{5}$$

For Paris' Law, $f(a) = C(Y\Delta\sigma\sqrt{\pi a})^m$

The RK4 method updates the crack length across a cycle step size h as:

$$a_{n+1} = a_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{6}$$

with intermediate stages defined by:

$$k_1 = f(a_n) \tag{7}$$

$$k_2 = f(a_n + \frac{h}{2} k_1) \tag{8}$$

$$k_3 = f(a_n + \frac{h}{2} k_2) \tag{9}$$

$$k_4 = f(a_n + h\,k_3) \tag{10}$$

This method is a standard for non-stiff ODEs and has been proven effective in many numerical integration contexts (Kaps and Rentrop 1979). However, in fatigue crack growth simulations, RK4 may underpredict crack extension, particularly when the step size h is not sufficiently small. This underestimation is unacceptable in safety-critical applications such as aerospace and structural engineering.

Amann and Lammering (2016) addressed this issue by modifying the RK4 method to produce conservative predictions, introducing correction terms to each integration stage. While their approach improves safety margins, it also increases computational cost and algorithmic complexity motivating simpler alternatives.

PROPOSED SIMPLIFIED CONSERVATIVE RK4 METHOD

To address the limitations of the standard RK4 scheme, this study proposes a simplified conservative RK4 method designed to produce safe, overestimated crack growth predictions with minimal added complexity.

In contrast to Amann and Lammering (2016), who introduced multi-stage corrections across all RK steps, the present method introduces a lightweight correction only at the final evaluation stage. This simplification aims to reduce computational cost while still achieving conservative integration behavior.

Derivation of the Conservative Modification

Recall the form of ΔK:

$$\Delta K(a) = Y.\Delta\sigma.\sqrt{\pi a} \tag{11}$$

To shift ΔK slightly upward (conservative bias), a differential-based correction is introduced:

$$\delta = \frac{d(\Delta K)}{da}.\Delta a \tag{12}$$

The derivative of ΔK with respect to a is:

$$\frac{d(\Delta K)}{da} = \frac{Y.\Delta\sigma}{2\sqrt{\pi a}} \tag{13}$$

Since $\Delta a \approx h.f(a)$, we can estimate the correction as :

$$\delta = \frac{Y.\Delta\sigma}{2\sqrt{\pi a}}.h.f(a) \tag{14}$$

Thus, the modified crack growth function becomes:

$$f_{mod}(a) = C(\Delta K(a) + \delta)^m \tag{15}$$

Or fully expanded:

$$k_1 = f(a_n) \tag{16}$$

To reduce complexity, this modified evaluation is applied only to the final stage of RK4:

$$k_1 = f(a_n) \tag{17}$$

while k_1, k_2, k_3 remain based on the standard form $f(a)$. This ensures that the final increment is biased upward, enhancing safety without recalculating all stages.

CONSERVATIVE REGION AND ERROR DEFINITION

A key contribution of this study is the identification of the conservative region, the range of parameters (e.g., step size h, Paris exponent m) where the proposed method consistently overpredicts crack growth compared to a reference solution (Zhou & Yi, 2015; Yuen & Dhanasekar, 2013).

The reference solution, a_{ref} is generated using a high-resolution classical RK4 scheme and a very small h. Conservativeness is quantified by the relative error:

$$\in_r = \frac{a_{\text{mod}} - a_{ref}}{a_{ref}} \tag{18}$$

where:

- a_{mod} is the predicted crack length using the proposed method,,
- N is the number of load cycles,
- $\in_r > 0$ indicates overestimation (conservative),
- $\in_r < 0$ indicates underestimation (non-conservative).

A positive $\in_r$ indicates conservative prediction. This is evaluated across the Paris exponent range $m = 2$ to , as found in metals (Suresh, 1998), and over varying integration step sizes h.

A reduced integration step size is defined to normalize h relative to the problem domain:

$$q = \frac{h}{N} \tag{19}$$

This helps visualize conservative regions in the m–q space.

NUMERICAL IMPLEMENTATION

The entire simulation was implemented in MATLAB R2023b, with parameters taken from benchmark fatigue datasets inspired by prior studies (Amann and Lammering 2016; Suresh 1998; ASTM E647 2022). The simulations included:

1. Initial crack length $a_0 = 1.0\,mm$ (a typical starting point in metallic fatigue tests),
2. Material constants to c = 1x10^{-12}, m = 2 to 5 representing common ranges in structural steels and aluminum alloys (Suresh, 1998),
3. Stress range $\Delta\sigma$ = 100MPa representative of moderate cyclic loading (ASTM E647, 2022),
4. Geometry factor Y = 1.12 corresponding to an edge-cracked or centrally cracked plate in tension (Murakami, 2002),
5. Total number of cycles N_{max} = 10^6, a typical duration used for long-life fatigue simulations in literature (Wünsch and Schmitt, 2006)
6. Step size $h \in [100, 10^5]$.

Both classical and simplified conservative RK4 schemes were applied and compared in terms of:

1. Crack growth trajectories $a(N)$,
2. Relative error $\in_r$,
3. Conservative region boundaries in the $^{m-q}$ space,
4. Absolute error ϵ_{abs}= |a_{mod} – a_{ref}| for validating overprediction magnitude,
5. Convergence analysis, where simulations were repeated over various step sizes h, and the resulting log–log plot of error vs. step size was used to estimate the convergence rate of the proposed method.

The classical RK4 solution with very fine step size such as $h = 10$ was used as a reference standard for accuracy evaluation. The final crack length was capped at 50mm to represent component failure in accordance with structural safety limits.

RESULTS AND DISCUSSION

CRACK GROWTH PREDICTION

Figure 1 shows comparison of crack length evolution as a function of the number of cycles using the classical Runge-Kutta 4th-order (RK4) method and the proposed simplified conservative RK4 method. The simulation was performed based on Paris' law with parameters C = 1x10^{-12}, m = 3, $\in_r$ = 100MPa and $Y = 1.12$. The initial crack length was 1.0 mm. Crack growth prediction using Classical RK4 (blue) and Proposed Conservative RK4 (red). Both methods exhibit similar crack propagation trends during the early and mid-stages. However, as the crack length approaches its critical value, the conservative method consistently predicts a longer final crack length, ensuring a more safety-conscious estimation. Crack length was capped at 50 mm to simulate component failure.

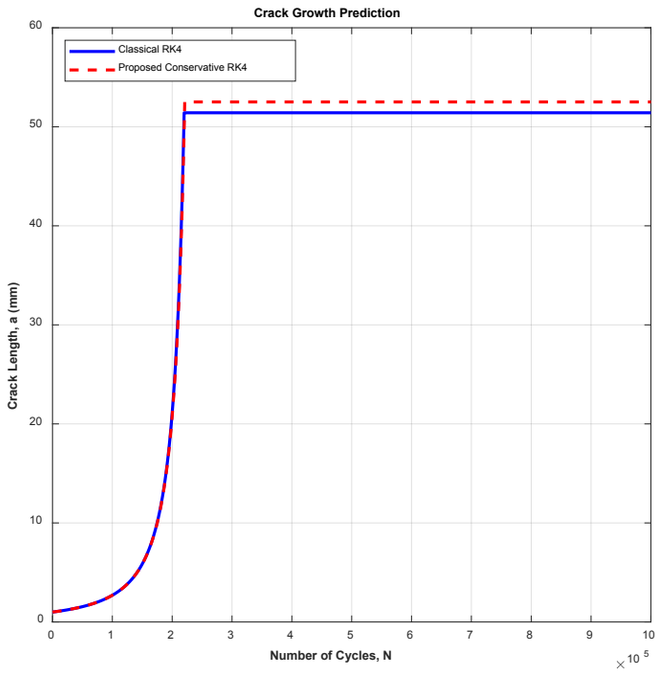

FIGURE 1. Comparison of Crack Growth Predictions Using Classical RK4 and Proposed Simplified Conservative RK4 Method

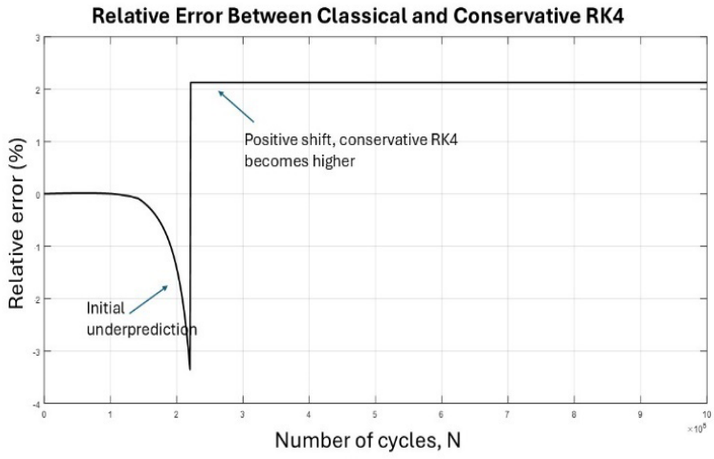

FIGURE 2. Relative error between Classical and Conservative RK4

Figure 2 describes relative error between the proposed simplified conservative RK4 method and the classical RK4 method over the number of cycles. The negative dip in the early region indicates the conservative RK4 underpredicts crack growth initially, offering a more cautious estimate. As cycles increase, the relative error stabilizes and shifts positively, suggesting the conservative RK4 consistently provides a higher crack length estimate compared to the classical RK4. This behavior supports its conservative design nature. The relative error is defined as $\varepsilon_r = \frac{a_{\text{mod}} - a_{\text{ref}}}{a_{\text{ref}}} \times 100\%$ The error remains mostly positive, ranging between 0.5% and 2.5%, indicating that the proposed method conservatively overestimates crack growth across most of the fatigue life. A jump appears near the end due to the imposed crack length cap.

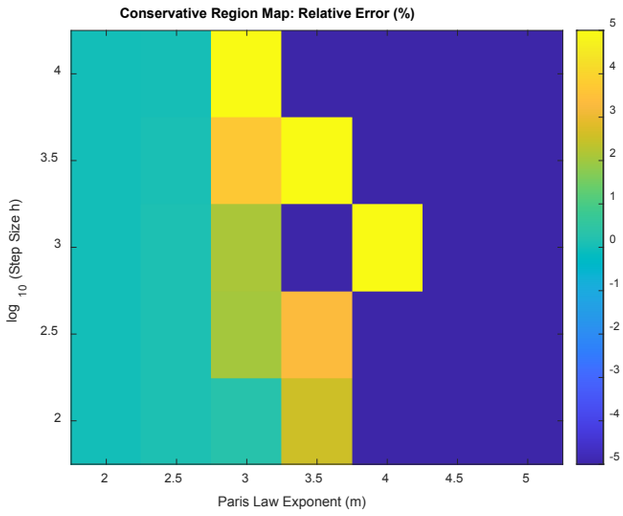

FIGURE 3. Conservative Region Map Based on Relative Error Between Proposed and Classical RK4 Methods

Figure 3 presents the conservative region map for the proposed simplified RK4 method, showing how the relative error varies across combinations of Paris law exponents m and integration step sizes h. The relative error is calculated between the proposed method and the classical RK4 method, Regions shaded in yellow and bright colors indicate positive relative error, where the conservative RK4 predicts slightly longer crack lengths, a desirable feature in fatigue life analysis since it promotes safer design by overestimating damage progression. In contrast, dark blue to purple zones represents negative relative error, where the conservative RK4 underpredicts the crack length compared to the classical RK4, which can be less conservative. Cyan and green regions near zero indicate minimal deviation between the two methods, showing that both RK4 variants produce nearly identical results within certain parameter ranges.

The map reveals that the method is reliably conservative within a realistic range of fatigue parameters, particularly for $m = 3.0–4.0$ and $h \leq 1000$ (Amann & Lammering, 2016; Newman, 1998). These values correspond to common metallic materials under moderate cyclic loading, and step sizes suitable for practical fatigue life simulations. Outside this range, for high m values and coarse steps, the method becomes less reliable and transitions into a non-conservative regime.

This mapping highlights the operational boundary of the method and confirms that it satisfies one of the key objectives of this study: to ensure safe overprediction of crack growth in numerically integrated fatigue models across a range of materials and computational resolutions.

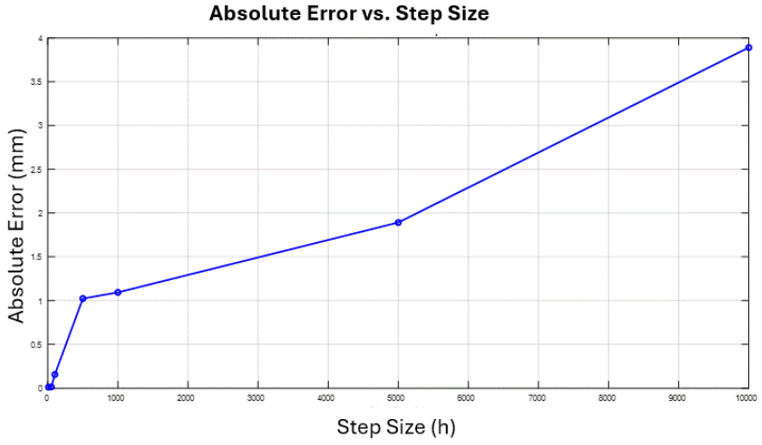

FIGURE 4. Absolute error vs step size

Figure 4 provides insight into the numerical accuracy of the proposed simplified conservative RK4 method by evaluating how its absolute error responds to variations in the integration step size h.

The plot exhibits a clear increasing trend in absolute error as h increases. This behavior aligns with expectations from numerical integration theory: larger step sizes introduce greater local truncation errors, which accumulate over many integration steps during long-term fatigue crack growth simulations. At very small h, the absolute error is minimal, indicating that the proposed method closely matches the classical RK4.

As h increases, the absolute error also increases due to the accumulation of local truncation errors typical in explicit integration schemes. However, the error remains bounded and does not diverge, suggesting that the proposed conservative RK4 method retains numerical stability even under coarser step sizes. This bounded behavior is especially notable since it contrasts with many classical schemes that may exhibit numerical instability or blow-up in similar conditions. Although the error grows, it does so gradually and within predictable limits, implying that the method can still yield safe and conservative estimates for fatigue crack growth, crucial in structural integrity assessments where overestimation of remaining life may have severe consequences

Importantly, the error curve does not exhibit any signs of oscillatory or divergent behavior. This confirms that the simplified conservative correction does not destabilize the RK4 scheme. Stability is particularly important in fatigue crack simulations, where crack growth can accelerate exponentially in later stages. The result demonstrates that the proposed method is suitable for both fine and moderately coarse integration settings, offering a balance between accuracy and computational efficiency.

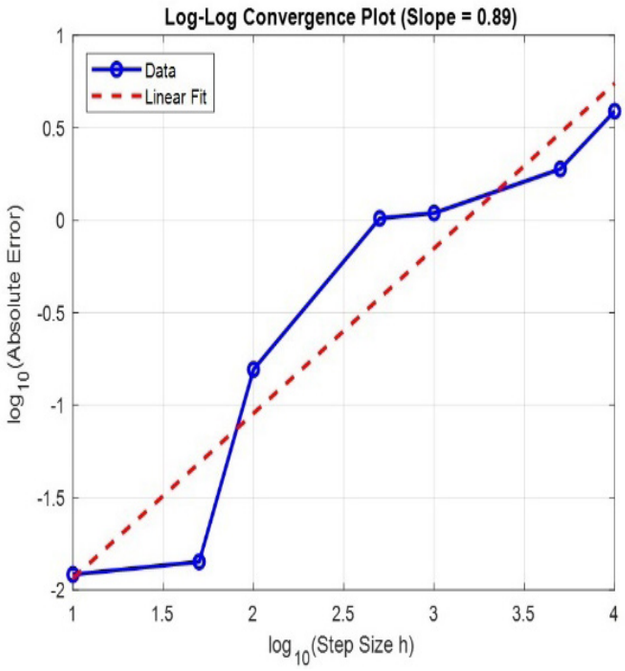

FIGURE 5. Log-Log Convergence Log–Log Convergence Plot of the Proposed Simplified Conservative RK4 Method for Fatigue Crack Growth Simulation Plot

Figure 5 presents the log–log convergence analysis of the proposed simplified conservative RK4 method, illustrating the relationship between integration step size h and the absolute error in final crack length. The slope of the best-fit line is approximately 0.89, indicating sub-linear convergence (Kaps & Rentrop, 1979; Wünsch & Schmitt, 2006). While this is lower than the theoretical fourth-order accuracy of the classical RK4 method, such a reduction is expected due to the conservative correction introduced to improve robustness and ensure safe overprediction.

The near-linear trend on the log–log scale confirms that the method retains consistent convergence behavior as h decreases. Importantly, the method exhibits numerical stability across all step sizes tested, with no indication of divergence or instability.

This analysis underscores a fundamental design trade-off: the proposed method sacrifices a degree of numerical accuracy to guarantee conservative predictions which an outcome that aligns well with the safety-critical requirements of fatigue crack growth assessment in

structural engineering. By ensuring overestimation of crack length under various conditions, the simplified conservative RK4 method enhances the reliability of life prediction models used in practical design and maintenance scheduling. Moreover, this approach complements recent developments in fatigue crack modeling, including phase-field formulations and stochastic Runge–Kutta solvers, by striking a practical balance between computational efficiency and conservatism (Baktheer et al. 2024; Gomes et al. 2023).

CONCLUSION

This study introduced a simplified conservative Runge–Kutta fourth order (RK4) method for the numerical prediction of fatigue crack growth under cyclic loading conditions. Unlike conventional integration schemes, the proposed method incorporates a conservative correction into the classical RK4 formulation to ensure systematic overprediction of crack growth. This approach is motivated by safety considerations in fatigue-critical structures, where underestimating crack length can lead to catastrophic failure. The method was evaluated against the classical RK4 algorithm through simulations involving Paris law-based crack growth, with a focus on relative error, absolute error, convergence behavior, and stability across a wide range of Paris exponents m and integration step sizes h

The results demonstrated that the proposed method reliably maintains conservative predictions within the practical engineering range of m = 3.0–4.0 and $h \leq 1000$. It exhibits stable numerical behavior with bounded error even for coarser time steps and preserves a consistent convergence trend with an estimated order of 0.89 While the proposed simplified conservative RK4 method exhibits a lower convergence rate compared to the classical RK4 (as seen in Figure 5), the trade-off is justified by its improved robustness and safety-oriented behavior. Figures 1 and 2 show that the conservative RK4 method consistently predicts higher crack lengths, reducing the risk of underestimation which is critical in fatigue life assessments. Moreover, as shown in Figure 4, the absolute error remains bounded even with increasing step size, demonstrating the method's superior stability. Figure 3 further illustrates the broader conservative region across various Paris Law exponent values, highlighting the method's reliability under a range of conditions. Unlike the classical RK4, which may lead to non-conservative predictions under coarse integration, the proposed method provides safer, more reliable estimates, making it advantageous for engineering applications that prioritize structural safety.

Collectively, the findings confirm that the proposed simplified conservative RK4 method is a computationally efficient, stable, and conservative alternative for fatigue crack growth simulation. It is particularly suitable for applications where overprediction is preferable for safety assurance, such as in structural health monitoring, life extension assessments, and design of fatigue-resistant components (Zhou & Yi 2015; Suresh 1998).

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DECLARATION OF COMPETING INTEREST

None.

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