NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED
BY A GENERALISED DIFFERENTIAL OPERATOR
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ABSTRACT

Let $S_H$ denote the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense preserving in the unit disk $U$. Earlier we have introduced a class of harmonic functions defined by a generalised differential operator. In this paper we introduce a new subclass of this class and obtain results on coefficient bounds, distortion and extreme points.

Keywords: Univalent functions; harmonic functions; generalised differential operator

1. Introduction

A continuous functions $f = u + iv$ is a complex harmonic function in a complex domain $\mathbb{C}$ if both $u$ and $v$ are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subseteq \mathbb{C}$ we can write $f(z) = h + \overline{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $D$ is that $|h'(z)| > |g'(z)|$ in $D$ (See Clunie & Shell-Small 1984). Denote by $S_H$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$. For $f = h + \overline{g} \in S_H$ we may express the analytic functions $h$ and $g$ as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_n| < 1.$$  \hspace{1cm} (1.1)

The class $T$ is defined as the subclass of $S_H$ consisting of all functions $f(z) = h + \overline{g}$, where $h$ and $g$ are given by

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n.$$  \hspace{1cm} (1.2)
Clunie and Shell-Small (1984) investigated the class \( S_{\gamma} \) as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on \( S_{\gamma} \) and its subclasses such that Silverman (1998), Silverman and Silvia (1999), and Jahangiri (1999) studied the harmonic univalent functions.

We denote by \( H^k(\alpha, \beta, \lambda, \delta, \gamma) \) the class of all function of the form (1.1) that satisfy the condition

\[
\Re \left( D_{\alpha, \beta, \lambda, \delta}^k f(z) \right) > 1 - |\gamma|, \quad z \in U
\]

(1.3)

where \( \gamma \in \mathbb{C}, k \in \mathbb{N}_0 \), \( D_{\alpha, \beta, \lambda, \delta}^k f(z) = D_{\alpha, \beta, \lambda, \delta}^k h(z) + D_{\alpha, \beta, \lambda, \delta}^k g(z) \) and \( D_{\alpha, \beta, \lambda, \delta}^k f(z) \) denote the operator introduced by Ramadan and Darus (2011) and given by

\[
D_{\alpha, \beta, \lambda, \delta}^0 f(z) = f(z)
\]

\[
D_{\alpha, \beta, \lambda, \delta}^1 f(z) = \left[ 1 - (\lambda - \delta)(\beta - \alpha) \right] f(z) + (\lambda - \delta)(\beta - \alpha) f'(z)
\]

\[
= z + \sum_{n=2}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n-1) + 1 \right] a_n z^n
\]

that is

\[
D_{\alpha, \beta, \lambda, \delta}^k f(z) = D_{\alpha, \beta, \lambda, \delta}^1 \left( D_{\alpha, \beta, \lambda, \delta}^{k-1} f(z) \right)
\]

\[
D_{\alpha, \beta, \lambda, \delta}^k f(z) = z + \sum_{n=2}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n-1) + 1 \right] a_n z^n,
\]

(1.4)

for \( \alpha \geq 0, \beta \geq 0, \lambda \geq 0, \delta \geq 0, \lambda > \delta, \beta > \alpha \) and \( k \in \{0, 1, 2, \ldots\} \).

**Remark 1.1.** (i) When \( \alpha = 0, \delta = 0, \lambda = 1, \beta = 1 \) we get Salagean differential operator (see Salagean 1983).

(ii) When \( \alpha = 0 \) we get Darus & Ibrahim differential operator (Darus & Ibrahim 2009).

(iii) And when \( \alpha = 0, \delta = 0, \beta = 1 \) we get Al-Oboudi differential operator (Al-Oboudi 2004).

Note that:

\[
H^0(0, 1, \lambda, 0, 1) \equiv H(\lambda) \text{ studied by Yalcin and Ozturk (2004).}
\]
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\[ H(0, 1, \lambda, 0, \gamma) \equiv H(\lambda, \gamma) \] studied by Janteng et al. (2007). Also we note that for the analytic part of the class \( H(0, 1, \lambda, 0, \gamma) \) was introduced and studied by Altintas and Ertekin (1992).

We further denote by \( TH^k(\alpha, \beta, \lambda, \delta, \gamma) \) the subclass of \( H^k(\alpha, \beta, \lambda, \delta, \gamma) \), where \( TH^k(\alpha, \beta, \lambda, \delta, \gamma) = T \cap H^k(\alpha, \beta, \lambda, \delta, \gamma) \).

2. Coefficients Bounds

**Theorem 2.1** Let \( f(z) = h + g \), with \( h \) and \( g \) be given by (1.1). Let

\[
\sum_{n=2}^{\infty} n \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) \right]^k \left( |a_n| + |b_n| \right) \leq |\gamma| - |b_1|,
\]

(2.1)

where \( a_1 = 1, \gamma \in \mathbb{C} \), \( \alpha \geq 0, \beta \geq 0, \lambda \geq 0, \delta \geq \lambda > \delta, \beta > \alpha \) and \( k \in \mathbb{N}_0 \). Then \( f \) is harmonic univalent sense preserving in \( U \) and \( f \in \mathbb{H}^k(\alpha, \beta, \lambda, \delta, \gamma) \).

**Proof:** For \( |z_1| \leq |z_2| \), we have by (2.1),

\[
|f(z_1) - f(z_2)| = |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)|
\]

\[
= \left| \left( z_1 - z_2 \right) + \sum_{n=2}^{\infty} a_n \left( z_1^n - z_2^n \right) - \sum_{n=1}^{\infty} b_n \left( z_1^n - z_2^n \right) \right|
\]

\[
\geq |z_1 - z_2| \left( 1 - |b_1| - \sum_{n=2}^{\infty} n \left( |a_n| + |b_n| \right) |z|^{n-1} \right)
\]

\[
\geq |z_1 - z_2| \left[ 1 - |b_1| - |z_2| \sum_{n=2}^{\infty} n \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) \right]^k \left( |a_n| + |b_n| \right) \right]
\]

\[
\geq |z_1 - z_2| \left[ 1 - |b_1| - |z_2| \left( |\gamma| - |b_1| \right) \right] > 0.
\]

Consequently, \( f \) is univalent in \( U \). We note that \( f \) is sense preserving in \( U \). This is because

\[
|h'(z)| \geq 1 - \sum_{n=2}^{\infty} n |a_n| |z|^{n-1} > 1 - \sum_{n=2}^{\infty} n |a_n|
\]

\[
\geq 1 - \sum_{n=2}^{\infty} n \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) + 1 \right]^k |a_n|.
\]
\[
\sum_{n=1}^{\infty} n \left[ \frac{(\lambda - \delta)(\beta - \alpha)(n-1)+1}{|\gamma|} \right] |b_n| \\
> \sum_{n=1}^{\infty} n |b_n| z^{-n} \approx |g'(z)|.
\]

Now we show that \( f \in H^k (\alpha, \beta, \lambda, \delta, \gamma) \). Using the fact that \( \text{Re} \{w\} > 1 - |\gamma| \) if and only if \( |\gamma| + w \geq 2 - |\gamma| - w \), it suffices to show that

\[
|\gamma + (D_{a,\beta,\lambda,\delta}^k (z)) - (D_{a,\beta,\lambda,\delta}^k g(z))| - 2 - |\gamma - (D_{a,\beta,\lambda,\delta}^k (z)) + (D_{a,\beta,\lambda,\delta}^k g(z))|
\]

\[
= |\gamma + \sum_{n=2}^{\infty} n \left[ \frac{(\lambda - \delta)(\beta - \alpha)(n-1)+1}{|\gamma|} \right] a_n z^{-n} - \sum_{n=1}^{\infty} n \left[ \frac{(\lambda - \delta)(\beta - \alpha)(n-1)+1}{|\gamma|} \right] b_n z^{-n} | \\
- 2 - |\gamma - \sum_{n=2}^{\infty} n \left[ \frac{(\lambda - \delta)(\beta - \alpha)(n-1)+1}{|\gamma|} \right] a_n z^{-n} + \sum_{n=1}^{\infty} n \left[ \frac{(\lambda - \delta)(\beta - \alpha)(n-1)+1}{|\gamma|} \right] b_n z^{-n} |
\]

\[
\geq 2|\gamma| - \sum_{n=2}^{\infty} n \left[ \frac{(\lambda - \delta)(\beta - \alpha)(n-1)+1}{|\gamma|} \right] |a_n| z^{-n} - \sum_{n=1}^{\infty} n \left[ \frac{(\lambda - \delta)(\beta - \alpha)(n-1)+1}{|\gamma|} \right] |b_n| z^{-n} \\
- \sum_{n=2}^{\infty} n \left[ \frac{(\lambda - \delta)(\beta - \alpha)(n-1)+1}{|\gamma|} \right] |a_n| z^{-n} - \sum_{n=1}^{\infty} n \left[ \frac{(\lambda - \delta)(\beta - \alpha)(n-1)+1}{|\gamma|} \right] |b_n| z^{-n} \\
- \sum_{n=1}^{\infty} n \left[ \frac{(\lambda - \delta)(\beta - \alpha)(n-1)+1}{|\gamma|} \right] |a_n| z^{-n} - \sum_{n=1}^{\infty} n \left[ \frac{(\lambda - \delta)(\beta - \alpha)(n-1)+1}{|\gamma|} \right] |b_n| z^{-n}
\]
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\[
= 2|\gamma| - 2\sum_{n=2}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) + 1 \right] |a_n|^n - 1
\]

\[
-2\sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) + 1 \right] |b_n|^n - 1
\]

\[
\geq 2\left\{ |\gamma| - \left( \sum_{n=2}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) + 1 \right] |a_n| \right. \\
\left. + \sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) + 1 \right] |b_n| \right) \right\} \geq 0,
\]

by (2.1). The harmonic mappings

\[
f(z) = z + \sum_{n=2}^{\infty} \frac{|\gamma|x_n}{n \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) + 1 \right]^k} z^n
\]

\[
+ \sum_{n=1}^{\infty} \frac{|\gamma|y_n}{n \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) + 1 \right]^k} z^n.
\]

(2.2)

where \(\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1\), show that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.2) are in \(H^k(\alpha, \beta, \lambda, \delta, \gamma)\) because

\[
\sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) + 1 \right] \left( |a_n| + |b_n| \right)
\]

\[
= 1 + |\gamma| \left\{ \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| \right\} = 1 + |\gamma|.
\]

The restriction placed in Theorem 2.1 on the moduli of coefficients of \(f(z) = h + \overline{g}\), enables us to conclude for arbitrary rotation of the coefficients of \(f\) that the resulting functions would still be harmonic univalent and \(f \in H^k(\alpha, \beta, \lambda, \delta, \gamma)\). We next show that the condition (2.1) is also necessary for functions \(f\) in \(TH^k(\alpha, \beta, \lambda, \delta, \gamma)\).
Theorem 2.2 Let \( f(z) = h + \overline{g} \), with \( h \) and \( g \) be given by (1.1). Then \( f \in \mathcal{H}(\alpha, \beta, \lambda, \delta, \gamma) \) if and only if
\[
\sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n-1) \right]^{k} \left( |a_{n}| + |b_{n}| \right) \leq |\gamma| - |\beta|,
\] (2.3)
where \( a_{1} = 1, \gamma \in \mathbb{C}, \alpha \geq 0, \beta \geq 0, \lambda \geq 0, \delta \geq 0, \lambda > \delta, \beta > \alpha \) and \( k \in \mathbb{N}_{0} \).

Proof: We first suppose that \( f \in \mathcal{H}(\alpha, \beta, \lambda, \delta, \gamma) \), then by (1.3) we have
\[
\text{Re} \left\{ \left( D_{\alpha,\beta,\lambda,\delta}^{k} h(z) \right) - \left( D_{\alpha,\beta,\lambda,\delta}^{k} g(z) \right) \right\}
= \text{Re} \left\{ 1 - \sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n-1) + 1 \right]^{k} |a_{n}| z^{n-1} \right. \\
- \sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n-1) + 1 \right]^{k} |b_{n}| z^{n-1}
\]
\[
> 1 - |\gamma|.
\]
If we choose \( z \) to be real and let \( z \to 1^{-} \), we get
\[
1 - \sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n-1) + 1 \right]^{k} |a_{n}| - \\
\sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n-1) + 1 \right]^{k} |b_{n}| \geq 1 - |\gamma|,
\]
which is precisely the assertion (2.3).
Conversely, suppose that the inequality (2.3) holds true. Then we find from the equation (1.3) that
\[
\text{Re} \left\{ \left( D_{\alpha,\beta,\lambda,\delta}^{k} h(z) \right) - \left( D_{\alpha,\beta,\lambda,\delta}^{k} g(z) \right) \right\}
= \text{Re} \left\{ 1 - \sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n-1) + 1 \right]^{k} |a_{n}| z^{n-1} \right. \\
- \sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n-1) + 1 \right]^{k} |b_{n}| z^{n-1}
\]
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\[
\geq 2 - \sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) + 1 \right]^k \left( |a_n| + |b_n| \right) z^{n-1}
\]

\[
> 2 - \sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) + 1 \right]^k \left( |a_n| + |b_n| \right) \geq 1 - |\gamma|
\]

provided that the inequality (2.3) is satisfied.

**Corollary 2.3** If \( f \in T^k H^k (\alpha, \beta, \lambda, \delta, \gamma) \), then

\[
\sum_{n=2}^{\infty} (|a_n| + |b_n|) \leq \frac{\gamma - |\gamma|}{2[(\lambda - \delta)(\beta - \alpha) + 1]^k}.
\]

**Corollary 2.4** Suppose that \( \gamma, \gamma^* \in \mathbb{C} \) such that \( |\gamma| < |\gamma^*| \). Then for \( \alpha \geq 0, \beta \geq 0, \lambda \geq 0, \delta \geq 0, \lambda > \delta, \beta > \alpha \) and \( k \in \mathbb{N}_0 \) we have

\( T^k H^k (\alpha, \beta, \lambda, \delta, \gamma) \subset T^k H^k (\alpha, \beta, \lambda, \delta, \gamma^*) \).

### 3. Distortion Bounds and Extreme Points

In this section, we shall obtain distortion bounds for functions in \( T^k H^k (\alpha, \beta, \lambda, \delta, \gamma) \) and also provide extreme points for this class.

**Theorem 3.1** If \( f \in T^k H^k (\alpha, \beta, \lambda, \delta, \gamma) \), for \( \gamma \in \mathbb{C} \), \( \alpha \geq 0, \beta \geq 0, \lambda \geq 0, \delta \geq 0, \lambda > \delta, \beta > \alpha \) and \( k \in \mathbb{N}_0 \) and \( \|f\| = r > 1 \), then

\[
|f(z)| \leq (1 + b_1) r + \frac{|\gamma| - |\gamma|}{2[(\lambda - \delta)(\beta - \alpha) + 1]^k} r^2,
\]

and

\[
|f(z)| \geq (1 - b_1) r - \frac{|\gamma| - |\gamma|}{2[(\lambda - \delta)(\beta - \alpha) + 1]^k} r^2.
\]

**Proof:** We only prove the second inequality. The argument for the first inequality is similar and will be omitted. Let \( f \in T^k H^k (\alpha, \beta, \lambda, \delta, \gamma) \). Taking the absolute value of \( f \), we obtain

\[
|f(z)| \geq (1 - b_1) r - \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^n \geq (1 - b_1) r - \sum_{n=2}^{\infty} (|a_n| + |b_n|) r^2
\]
\[(1-b_1)r - \frac{1}{2[(\lambda-\delta)(\beta-\alpha)+1]} \sum_{k=2}^{\infty} \left( (\lambda-\delta)(\beta-\alpha)(n-1)+1 \right)^k \sum_{k=2}^{\infty} \left( (\lambda-\delta)(\beta-\alpha)(n-1)+1 \right)^k \times \]

\[\left( |a_n| + |b_n| \right)^2 \]

\[\sum_{k=2}^{\infty} \left( (\lambda-\delta)(\beta-\alpha)(n-1)+1 \right)^k \times \left( |a_n| + |b_n| \right)^2.\]

The bounds given in Theorem 3.1 for functions \( f(z) = h + \bar{g} \), of the form (1.2) also hold for functions of the form (1.1) if the coefficient condition (2.1) is satisfied. The functions

\[ f(z) = z + |\bar{h}_1|z - \frac{|\bar{g}|-|\bar{h}_1|}{2[(\lambda-\delta)(\beta-\alpha)+1]}z^2 \]

and

\[ f(z) = (1-|\bar{h}_1|)z - \frac{|\bar{g}|-|\bar{h}_1|}{2[(\lambda-\delta)(\beta-\alpha)+1]}z^2, \]

for \(|\bar{h}_1| < 1\) show that the bounds given in Theorem 3.1 are sharp.

The following covering result follows from the second inequality in Theorem 3.1.

**Corollary 3.2** If \( f \in T H^k (\alpha, \beta, \lambda, \delta, \gamma) \), then

\[ \left\{ w : |w| < 1 - \frac{|\gamma|-1-2[(\lambda-\delta)(\beta-\alpha)+1]}{2[(\lambda-\delta)(\beta-\alpha)+1]}|\bar{h}_1| \right\} \subset f(U). \]

**Theorem 3.3** \( f \in T H^k (\alpha, \beta, \lambda, \delta, \gamma) \) if and only if \( f \) can be expressed as

\[ f(z) = \sum_{n=1}^{\infty} (Y_n h_n + Y_n g_n), \]

where \( z \in U \),

\[ h_1(z) = z, h_n(z) = z - \frac{|\gamma|}{n[(\lambda-\delta)(\beta-\alpha)(n-1)+1]}z^n, (n = 2, 3, ...), \]
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\[ g_n = z - \frac{|\gamma|}{n \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) + 1 \right]} \bar{z}^n, (n = 1, 2, \ldots), \]

\[ \sum_{n=1}^{\infty} (Y_n + Y'_n) = 1, Y_n \geq 0 \text{ and } Y'_n \geq 0. \]

In particular, the extreme points of \( T^k H^k (\alpha, \beta, \lambda, \delta, \gamma) \) are \( \{h_n\} \) and \( \{g_n\} \).

Proof: Note that for \( f \) we may write

\[ f(z) = \sum_{n=1}^{\infty} (Y_n h_n + Y_n g_n) \]

\[ = \sum_{n=1}^{\infty} (Y_n + Y'_n) z - \sum_{n=1}^{\infty} n \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) \right]^k Y_n \bar{z}^n \]

\[ - \sum_{n=1}^{\infty} \frac{|\gamma|}{n \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) \right]^k} Y_n \bar{z}^n. \]

Now the first part of the proof is complete, since by Theorem 2.2

\[ \sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) \right]^k \frac{|\gamma|Y_n}{n \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) \right]^k} \]

\[ + \sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) \right]^k \frac{|\gamma|Y'_n}{n \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) \right]^k} \]

\[ = |\gamma| \sum_{n=1}^{\infty} (Y_n + Y'_n) - Y_1 = |\gamma| - Y_1 \leq |\gamma|. \]

Conversely, suppose that \( f \in T^k H^k (\alpha, \beta, \lambda, \delta, \gamma) \). Then

\[ \sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) + 1 \right]^k \left( |a_n| + |b_n| \right) \leq 1 + |\gamma|. \]

Setting

\[ Y_n = \frac{n \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) + 1 \right]^k}{|\gamma|} |a_n| |\gamma| \neq 0, 0 \leq Y_n \leq 1, (n = 2, 3, \ldots) \]
\[ Y_n = \frac{n \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) + 1 \right]^k}{\gamma} \]

and \( Y_1 = 1 - \sum_{n=2}^{\infty} Y_n + \sum_{n=2}^{\infty} Y_n \), we obtain

\[ f(z) = \sum_{n=1}^{\infty} \left( Y_n h_n + Y_n g_n \right) \]
as required.

4. Closure Theorem

Let the functions \( f_j(z) \) be defined, for \( j = 1, 2, \ldots, m \) by

\[ f_j(z) = z - \sum_{n=2}^{\infty} \left| a_{n,j} \right| z^n + \sum_{n=2}^{\infty} \left| b_{n,j} \right| z^n, z \in U. \quad (4.1) \]

**Theorem 4.1** Let the functions \( f_j(z) \) defined by (4.1) be in the class \( TH^k(\alpha, \beta, \lambda, \delta, \gamma) \) for every \( j = 1, 2, \ldots, m \). Then the functions \( \Psi(z) \) defined by

\[ \Psi(z) = \sum_{j=1}^{m} t_j f_j(z) \left( t_j \neq 0 \right), \quad (4.2) \]
is also in the class \( TH^k(\alpha, \beta, \lambda, \delta, \gamma) \), where \( \sum_{j=1}^{m} t_j = 1 \).

Proof: According to the definition of \( \Psi \), we can write

\[ \Psi(z) = z - \sum_{n=2}^{\infty} \left( \sum_{j=1}^{m} t_j \left| a_{n,j} \right| \right) z^n - \sum_{n=2}^{\infty} \left( \sum_{j=1}^{m} t_j \left| b_{n,j} \right| \right) z^n. \quad (4.3) \]

Further, since functions \( f_j(z) \) are in \( TH^k(\alpha, \beta, \lambda, \delta, \gamma) \), for every \( j = 1, 2, \ldots, m \) we get

\[ \sum_{n=2}^{\infty} n \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) \right]^k \left( \left| a_{n,j} \right| + \left| b_{n,j} \right| \right) \leq 1 + \gamma, \]

for every \( j = 1, 2, \ldots, m \). We can see that

\[ \sum_{n=2}^{\infty} n \left[ (\lambda - \delta)(\beta - \alpha)(n - 1) + 1 \right]^k \left( \sum_{j=1}^{m} t_j \left( \left| a_{n,j} \right| + \left| b_{n,j} \right| \right) \right) \]
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\[
= \sum_{j=1}^{m} \left( \sum_{n=1}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha)(n-1) + 1 \right] \left( |a_{n,j}| + |b_{n,j}| \right) \right)
\]

\[
\leq (1 + |\gamma|) \sum_{j=1}^{m} = 1 + |\gamma|,
\]

by Theorem 2.2, we have \( \Psi(z) \in T_{\mu} H^{k}(\alpha, \beta, \lambda, \delta, \gamma) \).

5. An Applications of Neighbourhood

Following Yalcin and Ozturk (2004), we defined the \( n \)-neighbourhood of a function \( f \in S_{H}^{*} (S_{H}^{*} \text{ class of starlike harmonic functions in } U) \) by

\[
N_{\mu}(f) = \left\{ F \in H : F(z) = z + \sum_{n=2}^{\infty} A_{n} z^{n} + \sum_{n=1}^{\infty} B_{n} z^{n} \text{ and } \sum_{n=2}^{\infty} k \left( |a_{n} - A_{n}| + |b_{n} - B_{n}| \right) + |b_{1}| \leq \mu \right\}.
\]

In particular, for the identity function \( I(z) = z \), we immediately have

\[
N_{\mu}(I) = \left\{ f : F(z) = z - \sum_{n=2}^{\infty} |a_{n}| z^{n} + \sum_{n=1}^{\infty} |b_{n}| z^{n} \text{ and } \sum_{n=2}^{\infty} n \left( |a_{n}| + |b_{n}| \right) + |b_{1}| \leq \mu \right\}.
\]

Theorem 5.1 Let

\[
\mu = \frac{|\gamma|}{2 \left[ (\lambda - \delta)(\beta - \alpha) + 1 \right]} + \frac{2 \left[ (\lambda - \delta)(\beta - \alpha) + 1 \right]^{k} - 1}{2 \left[ (\lambda - \delta)(\beta - \alpha) + 1 \right]^{k}} |b_{1}|.
\]

Then, \( T_{\mu} H^{k}(\alpha, \beta, \lambda, \delta, \gamma) \subset N_{\mu}(I) \).

Proof: Let \( f \in T_{\mu} H^{k}(\alpha, \beta, \lambda, \delta, \gamma) \). Then the proof follows since, by (2.1), we have

\[
\sum_{n=2}^{\infty} n \left( |a_{n}| + |b_{n}| \right) + |b_{1}| \]
\[
\leq |b_1| + \frac{1}{2((\lambda - \delta)(\beta - \alpha) + 1)^{k}} \sum_{n=2}^{\infty} \left[ (\lambda - \delta)(\beta - \alpha) + 1 \right]^{k} \left( |\gamma_n| + |\delta_n| \right)
\]

\[
\leq |b_1| + \frac{1}{2((\lambda - \delta)(\beta - \alpha) + 1)^{k}} |\gamma| |b_1|
\]

\[
\leq \frac{|\gamma|}{2((\lambda - \delta)(\beta - \alpha) + 1)^{k}} + \frac{2((\lambda - \delta)(\beta - \alpha) + 1)^{k} - 1}{2((\lambda - \delta)(\beta - \alpha) + 1)^{k}} |b_1| = \mu.
\]

Hence \( f \in \mathcal{N}_\mu(I) \).

Some other work related to harmonic functions and differential operators can be found in Al-Shaqsi and Darus (2008; 2007; 2006), Al-Shaqsi et al. (2010), Darus and Al-Shaqsi (2006), and many elsewhere.

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**References**


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