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Performance of Euler-Maruyama, 2-Stage SRK and 4-Stage SRK in Approximating the Strong Solution of Stochastic Model

(Keberkesanan Kaedah Euler-Maruyama, Stokastik Runge-Kutta Peringkat 2, Stokastik Runge-Kutta Peringkat 4 dalam Mencari Penyelesaian Penghampiran Model Stokastik)

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ABSTRACT

Stochastic differential equations play a prominent role in many application areas including finance, biology and epidemiology. By incorporating random elements to ordinary differential equation system, a system of stochastic differential equations (SDEs) arises. This leads to a more complex insight of the physical phenomena than their deterministic counterpart. However, most of the SDEs do not have an analytical solution where numerical method is the best way to resolve this problem. Recently, much work had been done in applying numerical methods for solving SDEs. A very general class of Stochastic Runge-Kutta, (SRK) had been studied and 2-stage SRK with order convergence of 1.0 and 4-stage SRK with order convergence of 1.5 were discussed. In this study, we compared the performance of Euler-Maruyama, 2-stage SRK and 4-stage SRK in approximating the strong solutions of stochastic logistic model which describe the cell growth of C. acetobutylicum P262. The MS-stability functions of these schemes were calculated and regions of MS-stability are given. We also perform the comparison for the performance of these methods based on their global errors.

Keywords: 2-stage stochastic Runge-Kutta; 4-stage stochastic Runge-Kutta; Euler-Maruyama; stochastic differential equations

ABSTRAK

Persamaan pembezaan stokastik memainkan peranan penting dalam kebanyakan bidang seperti kewangan, biologi dan epidemiologi. Dengan menggabungkan elemen rawak ke atas sistem persamaan pembezaan biasa, pesamaan pembezaan stokastik muncul. Ini membawa kepada fenomena fizikal yang lebih kompleks berbanding dengan persamaan deteministik yang setara dengannya. Walau bagaimanapun, persamaan pembezaan stokastik tidak mempunyai penyelesaian analitik dan kaedah penyelesaian berangka merupakan cara terbaik untuk mengatasi masalah ini. Pada abad ini, banyak usaha telah dilakukan untuk mencari penyelesaian hampiran persamaan pembezaan stokastik. Bentuk am kelas Stokastik Runge-Kutta, SRK telah dikaji dan secara khusunya SRK peringkat 2 dengan pangkat penumpuan 1.5 dan SRK peringkat 4 dengan pangkat penumpuan 2.0 telah dibincangkan. Dalam kajian ini, kami melakukan perbandingan bagi melihat keberkesanan kaedah Euler-Maruyama, SRK peringkat 2 dan SRK peringkat 4 bagi mencari penyelesaian hampiran ke atas model logistik stokastik yang menerangkan kadar pertumbuhan sel C. acetobutylicum P262. Keberkesanan kaedah tersebut telah dibandingkan berdasarkan analisis stabiliti min kuasa dua dan ralat sejagat.

Kata kunci: Euler-Maruyama; persamaan pembezaan stokastik; stokastik Runge-Kutta peringkat 2; stokastik Runge-Kutta peringkat 4

INTRODUCTION

Modelling of physical phenomena and biological system by using stochastic differential equations (SDEs) has become an intensive research area over last few decades. By incorporating random elements into the deterministic differential equation system, the system of stochastic differential equations arises. These models may offer a far more realistic representation of the physical system, instead of a deterministic model. However, most of the SDEs do not have an explicit solution. Hence, there is a need for the development of reliable and efficient numerical integrators for such problems. The first attempt in this direction had been discovered by Maruyama in the 1950s. This scheme is known as Euler-Maruyamawhich has a strong and weak order convergence of 0.5 and 1.0, respectively. The order convergence of Euler method is quite low, thus there is a need for higher order numerical schemes.

One possible approach is to use the truncated Taylor series expansions. However, this approach needs more partial derivatives when more stochastic integral terms are added. The method of derivative-free scheme was introduced to replace the derivatives in Taylor approximations by finite difference. Milstein scheme, a derivative-free method of order 1.0 had been proposed by Milstein (1974). For multi-dimensional driving Wiener processes and non-commutative case, the double stochastic integral in the Milstein scheme needs to be computed. This leads to the method with derivatives and it is difficult to be implemented.

An improvement to that, 2-stage and 4-stage stochastic Runge-Kutta, SRK was introduced by Burrage & Burrage (1996). They presented a very general class of explicit SRK with strong order convergence of 1.0 and 1.5.

In this paper, we focus on the comparison towards the relative performance of Euler-Maruyama (EM), 2-stage SRK (SRK2) and 4-stage SRK (SRK4) in approximating the strong solution of stochastic power law logistic model used to describe the cell growth of *C. acetobutylicum* P262. The outline of this paper is; in the next section, we briefly introduce stochastic differential equations used to model the cell growth of *C. acetobutylicum* P262. Descriptions of three numerical schemes are given in the following section in addition to mean-square stability presented in last section. Then, *MS* stability regions for EM, SRK2 and SRK4 were presented followed by plotting of global errors against step size for three different schemes. Numerical examples were carried out to simulate the solution of the resulting system of SDEs by means of strong order methods of 0.5, 1.0 and 1.5.

Then, the prediction quality of stochastic logistic model (SLM) and logistic model (LM) are presented.

STOCHASTIC LOGISTIC MODEL

The deterministic model used to explain cell growth of *C. acetobutylicum* P262 is given as:

$$dx(t) = \mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\,dt$$
$$x(t_0) = x_0,\ t \in [0,T] \tag{1}$$

where x is the cell concentration, μ_{max} represents the growth coefficient and x_{max} correspond to the maximum value of the cell growth. Arifah (2004) introduced a white noise perturbation on the coefficient parameter $\frac{\mu_{\max}}{x_{\max}}$ such that

$$b \to b + \sigma\frac{dW(t)}{dt} \tag{2}$$

where $b = \frac{-\mu_{\max}}{x_{\max}}$, σ is a diffusion coefficient and $W(t)$ is one dimensional stochastic process having scalar Wiener process components with increment $\Delta W(t) = W(t+\Delta t)$ which are independent Gaussian random variables $N(0, \Delta t)$. Model (1) in Ito form is SLM which is given by

$$dx(t) = \mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t)\,dt + \sigma x^2(t)\,dW(t) \tag{3}$$

or in the integral form it can be expressed by

$$x(t) = x_0 + \int_0^r \mu_{\max}\left(1-\frac{x(s)}{x_{\max}}\right)x(s)\,ds + \int_0^r \sigma x^2(s)\,dW(s). \tag{4}$$

The second integral in (4) represents stochastic integral with respect to a Wiener process and it cannot be interpreted as Riemann-Stieltjes integral. There are two ways to represent the stochastic integrals namely Itoand Stratonovich integral depending on the evaluation points of the integrand. Though our model is in to form, it can be converted to Stratonovich form and vice-versa by means of the following formula

$$\bar{f}(t,x_t) = f(t,x_t) - \frac{1}{2}g(t,x_t)\frac{\delta g}{\delta x}(t,x_t). \tag{5}$$

By employing (5) to drift coefficient in (3) we shall obtain stochastic logistic model (SLM) in Stratonovich form

$$dx(t) = \left[\mu_{\max}\left(1-\frac{x(t)}{x_{\max}}\right)x(t) - \sigma^2 x^3(t)\right]dt + \sigma x^2(t)\circ dW(t). \tag{6}$$

The equations (3) and (6) under different rules of calculus have the same solution. In the case of additive noise, the Ito and Stratonovich representations are equivalent. In order to avoid any confusion in notation, the symbol ∘ will be used to denote the Stratonovich form (i.e. $\circ\, dW(t)$).Throughout this paper only Stratonovich SDE will be considered. Please note that Runge-Kutta type schemes should not be used for Ito SDEs as they are generally not consistent with Ito calculus.

NUMERICAL METHODS FOR SDES

Three numerical schemes were adopted for solving SDE in (6) namely EM, SRK2 and SRK4. The simplest numerical method for solving SDE is EM which can be represented by the following formula

$$y_{n+1} = y_n + f(y_n)\,\Delta t + g(y_n)\,\Delta W_n \tag{7}$$

where f is a drift coefficient, g is a diffusion coefficient

$$\Delta t = t_{n+1} - t_n,\ \Delta W_n = W(t_{n+1}) - W(t_n). \tag{8}$$

ΔW_n be generated numerically by using pseudo-random number generator which involves sampling from independent, normal distributed random variables with mean zero and variance, Δt. The order convergence of EM method is quite low and so more efficient methods are needed. Rumelin (1982) presented a so-called *s*-stage explicit SRK for SDE which is based on the increment of Wiener process, ΔW_n. A simple generalisation of SRK methods to SDEs is

$$Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j) + J_1 \sum_{j=1}^{s} b_{ij} g(Y_j),\ i = 1, \ldots, s$$
$$y_{n+1} = y_n + h\sum_{j=1}^{s} \alpha_j f(Y_j) + J_1 \sum_{j=1}^{s} \gamma_j g(Y_j) \tag{9}$$

where $A = (a_{ij})_{s\times s}$ and $B = (b_{ij})_{s\times s}$ are matrices of real elements while $\alpha^T = (\alpha_1, \ldots, \alpha_s)$ and $\gamma^T = (\gamma_1, \ldots, \gamma_s)$ are row vectors $\in \Re^S$. The stochastic component comes from J_1 integral, where $J_1 = \int_{t_n}^{t_{n+1}} \circ dW$. Burrage and Burrage (1996) refined (9) by introducing other stochastic elements apart from J_1. Arbitrary matrix $\mathbf{Z}$ and vector $\mathbf{z}^T$ were introduced whose elements themselves are random variables. Hence, the general family of s-stage SRK is formulated as follows

$$Y_i = y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j) + \sum_{j=1}^{s} Z_{ij} b_{ij} g(Y_j), \quad 1,\ldots,s$$
$$y_{n+1} = y_n + h\sum_{j=1}^{s} \alpha_j f(Y_j) + \sum_{j=1}^{s} z_j g(Y_j). \tag{10}$$

Z_{ij} and z_j can be written as a linear combination of p different random variables $\theta_1,\ldots,\theta_p$ as follows

$$Z_{ij} = \sum_{l=1}^{p} b_{ij}^l \theta_l, \quad i,j = 1,\ldots,s$$
$$z_j = \sum_{l=1}^{p} r_j^l \theta_l, \quad j = 1,\ldots,s \tag{11}$$

An explicit SRK with strong order of 1.0 and 1.5was developed by letting $p = 2$. For $p = 2$ we have $\theta_1 = J_1$ and $\theta_2 = \frac{J_{10}}{h}$, where $J_1 = \int_{t_n}^{t_{n+1}} \circ dW(s)$ and $J_{10} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \circ dW(u) \circ ds$. The random variable J_{10} has the following representation

$$\frac{J_{10}}{h} = \frac{\sqrt{h}}{2}\left(G_1 + \frac{G_2}{\sqrt{3}}\right) \tag{12}$$

where G_1 and G_2 are standard normal distribution. Ans-stage SRK can be written as

$$Y_1 = y_n + h\sum_{j=1}^{s} a_{ij} f(Y_j) + \sum_{j=1}^{s}\left(b_{ij}^{(1)} J_1 + b_{ij}^{(2)} \frac{J_{10}}{h}\right) g(Y_j), \quad i = 1,\ldots,s$$
$$Y_{n+1} = y_n + h\sum_{j=1}^{s} \alpha_j f(Y_j) + \sum_{j=1}^{s}\left(\gamma_j^{(1)} J_1 + \gamma_j^{(2)} \frac{J_{10}}{h}\right) g(Y_j). \tag{13}$$

Burrage and Burrage scheme for 2-stage SRK has the following form

$$Y_1 = y_n$$
$$Y_2 = y_n + \frac{2}{3} h f(Y_n) + \frac{2}{3}\Delta W_n g(Y_n)$$
$$y_{n+1} = y_n + \left(\frac{1}{4} f(Y_1) + \frac{3}{4} f(Y_2)\right) + \Delta W_n \left(\frac{1}{4} g(Y_1) + \frac{3}{4} g(Y_2)\right) \tag{14}$$

while 4-stage SRK with strong order of 1.5 can be written in tableaus form as follows

$$A = \begin{pmatrix} 0 & & & \\ \frac{1}{2} & 0 & & \\ 0 & \frac{1}{2} & 0 & \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \alpha^T = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}\right)$$

$$B^{(1)} = \begin{pmatrix} 0 & & & \\ -0.72429163 & & & \\ 0.42373534 & -0.19944370 & 0 & \\ -1.5784755 & 0.84010034 & 1.73837510 & \end{pmatrix}$$

$$B^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2.7002000410 & 0 & 0 & 0 \\ 1.757261649 & 0 & 0 & 0 \\ -2.918524118 & 0 & 0 & 0 \end{pmatrix} \tag{15}$$

$\gamma^{(1)T} = (-0.78007, 0.073637, 1.4865, 0.21992)$
$\gamma^{(2)T} = (1.69395, 1.63610, -3.02400, -0.306049)$.

Mean Square-Stability Analysis

Numerical stability analysis for SDEs is far more complex than ODEs. Consider a scalar test equation of Stratonovich form with complex number λ ($\Re\lambda < 0$) and μ,

$$dX(t) = \lambda X(t)dt + \mu X(t)dW(t), \quad t \in [0, T]$$
$$X(0) = 1 \tag{16}$$

The exact solution of (16) is

$$X(t) = \exp\left\{\left(\lambda - \frac{1}{2}\mu^2\right)t + \mu W(t)\right\} \tag{17}$$

Saito & Mitsui (1996) defined $\bar{Y}_n$ by

$$\bar{Y}_n = E|X_n|^2 \tag{18}$$

When we apply numerical scheme to (16) and take the mean square norm, we obtain one-step difference equation of

$$\bar{Y}_{n+1} = R(\bar{h}, k)\bar{Y}_n \tag{19}$$

$R(\bar{h}, k)$ is called stability function of the scheme and clearly $\bar{Y}_n \to 0$ as $n \to \infty$ if and only if $|R(\bar{h}, k)| < 1$. We will calculate the stability function of three numerical schemes considered in this study and their respective regions of *MS*-stability. Let $\bar{h} = h\lambda$ and $k = \frac{-\mu^2}{\lambda}$,it can be easily derived that Euler Maruyama which is given by equation (7) have the stability function

$$R(\bar{h}, k) = |1 + \bar{h}|^2 + |k\bar{h}| \tag{20}$$

The derivation of *MS*-stability function for SRK2 is given as below

$$X_1 = X_n$$
$$X_2 = X_n + \frac{2}{3} h f(X_n) + \frac{2}{3} J_1 g(X_n)$$
$$X_{n+1} = X_n + h\left(\frac{1}{4} f(X_1) + \frac{3}{4} f\left(X_1 + \frac{2}{3} h f(X_1) + \frac{2}{3} J_1 g(X_1)\right)\right)$$
$$+ J_1\left(\frac{1}{4} g(X_1) + \frac{3}{4} g\left(X_1 + \frac{2}{3} h f(X_1) + \frac{2}{3} J_1 g(X_1)\right)\right). \tag{21}$$

Substitute (16) into (21) we have

$$X_{n+1} = X_n\left(1+\lambda h+\frac{\lambda^2 h^2}{2}+J_1\mu h\lambda+J_1\mu+\frac{\mu^2 J_1^2}{2}\right). \qquad (22)$$

By squaring both sides of (22), we obtain

$$X_{n+1}^2 = X_n^2\left(\frac{1}{4}h^4\lambda^4+h^3\lambda^3+2h^2\lambda^2+2h\lambda+\frac{1}{4}\mu_1^4+J_1^4+\left(\mu^3+h\lambda\mu\right)J_1^3\right.$$
$$\left.+\left(2\mu^2+3h\lambda\mu^2+\frac{3}{2}h^2\lambda^2\mu^2\right)J_1^2+\left(2\mu+3h^2\lambda^2+4h\lambda\mu+h^3\lambda^3\mu\right)J_1\right). \qquad (23)$$

Note that $E(J_1) = E(J_1{}^3) = 0$, $E(J_1{}^2) = h$, $E(J_1{}^4) = 3h^2$. By taking expectation of both sides of (24), we obtain

$$E\left(X_{n+1}\right)^2 = E\left(X_n\right)^2\left(1+\frac{1}{4}h^4+\lambda^4+h^3\lambda^3+2h^2\lambda^2+\frac{3}{4}\mu^4h^2\right.$$
$$\left.+2h\mu^2+3h^2\lambda\mu^2+\frac{3}{2}h^3\lambda^2\mu^2\right)$$

Thus, the stability function for SRK2 is

$$R\left(\bar{h},k\right)=1+\frac{\bar{h}^4}{4}+\left(1+\frac{3k}{2}\right)\bar{h}^3+\left(2+3k+\frac{3}{4}k^2\right)\bar{h}^2+\left(2k+2\right)\bar{h}. \qquad (24)$$

With the same approach, *MS*-stability function for SRK4 can be calculated as

$$X_1 = X_n$$
$$f(X_1) = \lambda X_n,\ g(X_1) = \mu X_n$$
$$X_2 = X_n + a_{21}hf(X_1) + \left(J_1 b_{21}^{(1)}+\frac{J_{10}}{h}b_{21}^{(2)}\right)g\left(X_1\right)$$
$$f(X_2) = \lambda X_2,\ g(X_2) = \mu X_2$$
$$X_3 = X_n + ha_{32}f(X_2)$$
$$+ J_1\left(b_{31}^{(1)}g\left(X_1\right)+b_{32}^{(1)}g\left(X_2\right)\right)+\frac{J_{10}}{h}b_{31}^{(2)}g\left(X_1\right)$$
$$f(X_3) = \lambda X_3,\ g(X_3) = \mu X_3 \qquad (25)$$

$$X_4 = X_n + a_{43}hf\left(X_3\right)+J_1\left(b_{41}^{(1)}g\left(X_1\right)+b_{42}^{(1)}g\left(X_2\right)\right.$$
$$\left.+b_{43}^{(1)}g\left(X_3\right)\right)+\frac{J_{10}}{h}b_{41}^{(2)}g\left(X_1\right)$$
$$f(X_4) = \lambda X_4,\ g(X_4) = \mu X_4$$
$$X_{n+1} = X_n + (\alpha_1 f(X_1) + \alpha_2 f(X_2) + \alpha_3 f(X_3) + \alpha_4 (X_4))h$$
$$+ (\gamma_1{}^{(1)}g(X_1)+\gamma_2{}^{(1)}g(X_2)+\gamma_3{}^{(1)}g(X_3)+\gamma_4{}^{(1)}g(X_4))J_1$$
$$+(\gamma_1{}^{(2)}g(X_1)+\gamma_2{}^{(2)}g(X_2)+\gamma_3{}^{(2)}g(X_3)+\gamma_4{}^{(2)}g(X_4))\frac{J_{10}}{h}. \qquad (26)$$

The stability function of R4 is given by the following expression

$$R\left(\bar{h},k\right)=1+2(1+k)\bar{h}+\left(2+3.269893207k+1.71057106k^2\right)\bar{h}^2$$
$$+\left(\frac{4}{3}+2.863798646k+1.925252077k^2+0.270309581k^3\right)\bar{h}^3$$
$$+\left(\frac{2}{3}+1.645042404k+1.3704558k^2-0.319843114k^3+0.101922k^4\right)\bar{h}^4$$
$$+\left(\frac{1}{4}+0.6354256573k+0.23985221k^2-0.00553023635k^3\right)\bar{h}^5$$
$$+\left(\frac{5}{72}+0.1478030732k-0.008416501k^2\right)\bar{h}^6$$
$$+\left(\frac{1}{72}+0.02157131671k\right)\bar{h}^7+\frac{\bar{h}^8}{576}. \qquad (27)$$

Numerical Experiment& Discussion

The above analysis can be justified by a numerical example

$$dX(t) = (-100+100i)Xdt + 10XdW(t),\ t \in [0, T]$$
$$X(0) = 1. \qquad (28)$$

then, $k=\frac{1}{(1-i)}$ and $|k|=\frac{1}{(2)}$. In order to visualise the domain of *MS*-stability, we plot the stability region for $\left|R\left(\bar{h},k\right)\right|<1$.

Figure 1 represents the regions of absolute stability at $|k|=\frac{1}{\sqrt{2}}$ in versus plane for EM, SRK2 and SRK4 schemes. We can observe that SRK4 is superior in stability compared to other schemes. Then, we compute the global errors at the right-end point T with $M = 500$ different repeated simulations of sample paths.

The global errors against step size are plotted on log-log scale. The results were illustrated in Figure 2. It shows that the absolute error decreases as the step size decreases. It can be seen that SRK4 has lower global error, thus shows

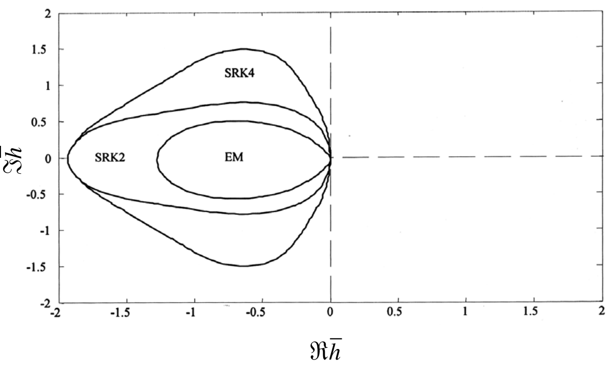

FIGURE 1. The region at $|k|=\frac{1}{\sqrt{2}}$ of absolute stability of EM,SRK2 and SRK4 schemes

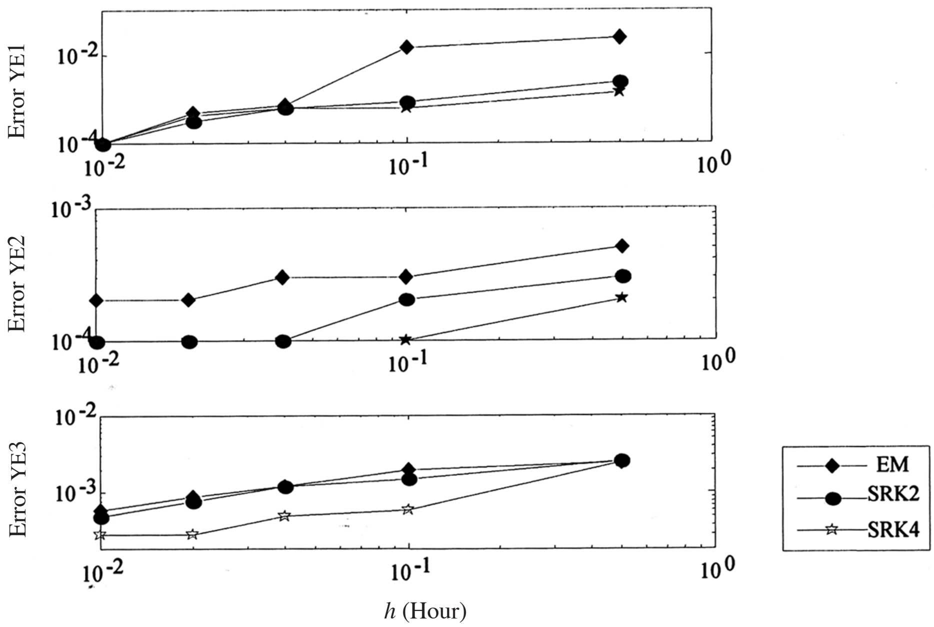

FIGURE 2. Strong convergence of EM, SRK2 and SRK4 methods to stochastic model (6)

the better performance compared to SRK2 and EM. Then, we add for illustration a brief numerical example that will indicate the performance SRK4 in comparison to SRK2 and EM. Let us consider the population growth model taken from Oksendal (2003). Stratonovich interpretation of this model is given by:

$$dX(t) = rX(t)dt + \alpha X(t)\circ dW(t), \text{ for } t\in[0,T]. \quad (29)$$

The analytical solution of (29) is

$$X(t) = X(t_0)\exp(rt + \alpha W(t)). \quad (30)$$

To construct a numerical example, we choose $r = 0.1$, $\alpha = 0.005$ and $T = 1000$.

Numerical approximation for step size $h = 0.1$ and their respective analytical solution are given in Figure 3. This simple example visually demonstrates that higher order method can considerably improve the accuracy of the simulation.

MATHEMATICAL MODEL *C. ACETOBUTYLICUM* P262

We estimated the value of μ_{max} and σ by using Levenberg Marquardt algorithm (Haliza et al. 2009). Thus, stochastic logistic model (SLM) for respective YE1, YE2 and YE3 are as below

$$\text{YE1: } dx(t) = \left(0.4848\left(1-\frac{x(t)}{3.5250}\right)x(t) - 7.84\times10^{-6}x^3(t)\right) dt + 0.0028x^2(t)\circ dW(t) \quad (31)$$

$$\text{YE2: } dx(t) = \left(0.5056\left(1-\frac{x(t)}{0.9490}\right)x(t) - 1.6129\times10^{-4}x^3(t)\right) dt + 0.0127x^2(t)\circ dW(t) \quad (32)$$

$$\text{YE3: } dx(t) = \left(0.6354\left(1-\frac{x(t)}{4.2950}\right)x(t) - 2.704\times10^{-5}x^3(t)\right) dt + 0.0052x^2(t)\circ dW(t) \quad (33)$$

SRK4 is used to simulate the strong solution of SLM since it has higher order of convergence and superior in stability comparing with other schemes. The results of (31), (32) and (33) are presented in Figure 4. The prediction quality of the models can be assessed by using root mean square error (RMSE)

$$\text{RMSE} = \sqrt{\frac{\sum_{i=1}^{n}(y_1 - x_i)^2}{n}} \quad (34)$$

where y_i is the experimental data and x_i is the predicted solution. The obtained RMSE for YE1, YE2 and YE3 are shown in Table 1.

It can be seen that numerical solution of SLM described more precise the experimental data, presenting low values of RMSE for YE1, YE2 and YE3. The dynamics of *C. acetobutylicum* P262 differ when SLM is used. As time evolves, the system is subjected to an intrinsic variability of the competing within species and deviations from exponential growth arise. It happens as a result of the nutrient level and toxin concentration achieves a value

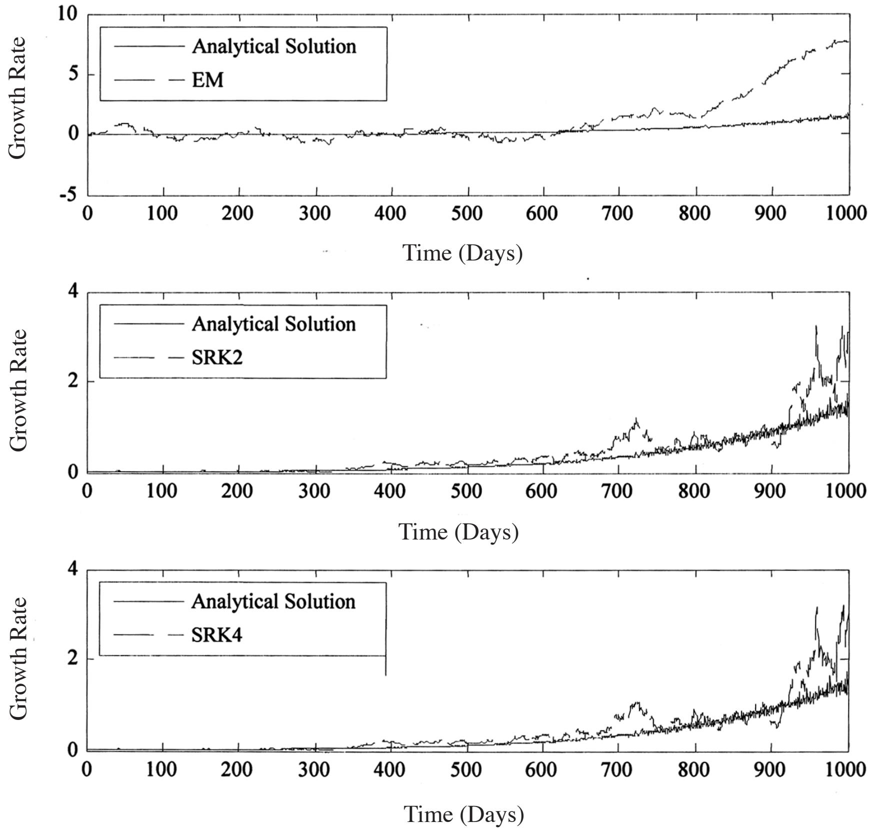

FIGURE 3. Analytical solution and numerical solution of (29) using three different methods

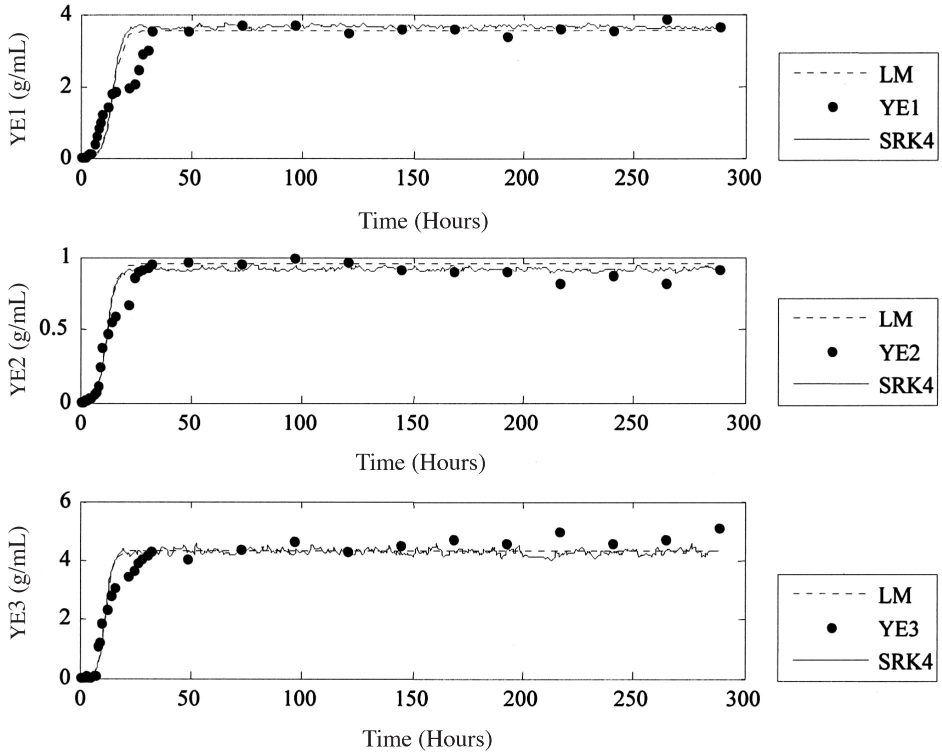

FIGURE 4. Results for LM, Experimental Data and SLM of (31), (32) and (33) for YE1, YE2, and YE3

TABLE 1. RMSE for YE1, YE2 and YE3

Model	RMSE		
	YE1 $n = 500$	YE2 $n = 500$	YE3 $n = 500$
Stochastic logistic model (SLM)	0.4820	0.0771	0.4421
Logistic model (LM)	0.5483	0.1058	0.5420

which can no longer support the maximum growth rate. So, the stochastic fluctuations mainly affect the logistic growth of *C. acetobutylicum* P262 and we can conclude that cell growth in fermentation process can be modelled by using SLM.

Conclusion

Stochastic logistic model may represent idealistic situations to describe the behaviour of cell growth proliferations in *C. acetobutylicum* P262. The dynamics of *C. acetobutylicum* P262 differ when the stochastic logistic model is used. Furthermore, we can conclude that 4-stage SRK shows better performance compared to 2-stage SRK and Euler-Maruyama methods.

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