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An Operator Defined by Convolution Involving the Generalised Hurwitz-Lerch Zeta Function

(Pengoperasi yang Ditakrif oleh Konvolusi Melibatkan Pengitlakan Fungsi Hurwitz-Lerch Zeta)

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ABSTRACT

In this article, we studied the generalised Hurwitz-Lerch zeta function. We defined a new operator and introduced a new class of function. Here, some interesting properties and sufficient conditions for subordination were also studied.

Keywords: Hadamard product; Hurwitz-Lerch zeta function; integral operator

ABSTRAK

Dalam kertas kerja ini, fungsi teritlak Hurwitz–Lerch zeta dikaji. Pengoperasi baharu dan kelas fungsi baharu diperkenalkan. Di sini beberapa sifat dan syarat cukup untuk subordinasi juga dikaji.

Kata kunci: Fungsi Hurwitz-LCerch zeta; hasil darab Hadamard; pengoperasi kamiran

INTRODUCTION

Let $U = \{z: z\in C \,|z|<1\}$ be the open unit disc and let A denotes the class of functions f normalised by:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disc U and satisfy the condition $f(0) = f'(0) - 1 = 0$.

Furthermore, we denote by T the subclass of A consisting of functions whose nonzero coefficients, from the second one, are negative and normalised by:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ a_k > 0. \tag{2}$$

Let Ω denote the class of functions $w(z)$ which are analytic in U with $w(0) = 0$ and $|w(z)|<1$. Further, let P denote the class of functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + \ldots$ which are analytic in U and satisfy the conditions $\Re(p(z)) > 0$ and

$$p(z) = \frac{1+w(z)}{1-w(z)}$$

for some $w(z) \in \Omega$. For $f_j \in A$ given by:

$$f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k,\ (j = 1,2)$$

the Hadamard product (or convolution) $f_1 * f_2$ and f_2 is defined by:

$$(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k.$$

Let f_1 and f_2 be analytic in U. We say that f_1 is subordinate to f_2, written $f_1 \prec f_2$ or $f_1(z) \prec f_2(z)$, if there exists a function $w(z) \in \Omega$ in U such that $f_1(z) = f_2(w(z))$.

Lemma 1: (Padmanabhan & Parvatham 1985). Let β, γ be complex numbers. Let $\phi(z)$ be convex univalent in U with $\phi(0) = 1$ and $\Re[\beta\phi(z) + \gamma] > 0, z \in U$ and $q \in A$ with $q(z) \prec \phi(z), z \in U$. If $p \in P$ is analytic in U, then:

$$p(z) + \frac{zp'(z)}{\beta q(z) + \gamma} \prec \phi(z) \Rightarrow p(z) \prec \phi(z).$$

Lemma 2: (Eenigenburg et al. 1983). Let ν, ζ be complex numbers. Let $\phi(z)$ be convex univalent in U with $\phi(0) = 1$ and $R[\nu\phi(z) + \zeta] > 0, z \in U$. If $p \in P$ is analytic in U, then:

$$p(z) + \frac{zp'(z)}{\nu p(z) + \zeta} \prec \phi(z) \Rightarrow p(z) \prec \phi(z).$$

Denote by $D^\lambda: A \to A$ the operator defined by:

$$D^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z) \qquad \lambda > -1.$$

It is obvious that, $D^0 f(z) = f(z)$ and $D^1 f(z) = zf'(z)$ and,

$$D^m f(z) = \frac{z\left(z^{m-1} f(z)\right)^{(m)}}{m!},\ m \in N_0 = N \cup \{0\}.$$

Observe that $D^m f(z) = z + \sum_{k=2}^{\infty} C(m,k) a_k z^k$, where $C(m,k) = \binom{k+m-1}{m}$.

The operator $D^n f$ is called the n th-order Ruscheweyh derivative of f introduced by Ruscheweyh (1975).

Denote by $I_n : A \to A$ the operator defined by:

$$f_n(z)*f_n^{(-1)}(z) = \frac{z}{1-z}, \text{ where } f_n(z) = \frac{z}{(1-z)^{n+1}}, \; n \in N_0.$$

Then,

$$I_n f(z) = f_m^{(-1)}(z)*f_n(z) = \left[\frac{z}{(1-z)^{n+1}}\right]^{(-1)}$$

$$*f(z) = z + \sum_{k=2}^{\infty} \frac{n!k!}{(k+n-1)!} a_k z^k.$$

Note that $I_0 f(z) = zf'(z)$ and $I_1 f(z) = f(z)$. The operator I_n is called the Noor integral operator defined and studied by Noor (1999) and Noor and Noor (1999).

For $f \in A$, Salagean (1983) introduced the following operator called the Salagean operator:

$$D^n f(z) = f(z)*(z + \sum_{k=2}^{\infty} k^n z^k), \quad n \in N_0 = N \cup \{0\}.$$

Note that, $D_0 f(z) = f(z)$ and $D' f(z) = zf'(z)$.
Recently, Al-Shaqsi and Darus (2008) introduced the following linear operator:

$$D_\lambda^n f(z) = \left(G(n,z)\right)^{(-1)} * f(z),$$

where $G(n,z)$ is a polylogarithm function given by $G(n,z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ and $\sum_{k=1}^{\infty} \frac{z^k}{k^n} *(G(n,z))^{(-1)} = \frac{z}{(1-z)^{\lambda+1}} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{k!} z^{k+1}$, $\lambda > -1$. Then

$$(G(n,z))^{(-1)} = \sum_{k=1}^{\infty} k^n \frac{(k+\lambda-1)!}{\lambda!(k-1)!} a_k z^k.$$

Thus,

$$D_\lambda^n f(z) = z + \sum_{k=2}^{\infty} k^n \frac{(k+\lambda-1)!}{\lambda!(k-1)!} a_k z^k, \; n, \lambda \in N_0.$$

Now let us consider the generalised Hurwitz-Lerch zeta function:

$$\phi_\mu(z,s,\sigma) = \sum_{k=0}^{\infty} \frac{(\mu)_k}{k!} \frac{z^k}{(k+\sigma)^s}, z \in C, |z| < 1,$$

$$\sigma \in C/\{0,-1,-2,\ldots\}, \mu, s \in C, \tag{3}$$

introduced by Goyal and Laddha (1997). Here $(x)_k$ is Pochhammer symbol (or the shifted factorial, since $(1)_k = k!$) and $(\lambda)_k$ given in terms of the gamma functions can be written as:

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = x(x+1)\ldots(x+k-1) \quad \text{for } k = 1,2,3,\ldots$$

$$x \in R \;\; (x)_0 = 1.$$

Note that, the families and special cases of the Hurwitz-Lerch zeta function are studied by many authors (among them) Lin and Srivastava (2004) and Kanemitsu et al. (2000).

For $\sigma = 1$, the generalised Hurwitz-Lerch zeta function reduces to:

$$z\phi_\mu(z,s,1) = \sum_{k=1}^{\infty} \frac{(\mu)_{k-1} z^k}{(k-1)!k^s}.$$

We now introduce a function $[z\phi_\mu(z,s,1)]^{(-1)}$ given by:

$$[z\phi_\mu(z,s,1)]*[z\phi_\mu(z,s,1)]^{(-1)} = \frac{z}{(1-z)^{\lambda+1}} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k}{k!} z^{k+1}, \tag{4}$$

and obtain the following linear operator:

$$\theta_\mu^{\lambda,s} f(z) = \left[z\phi_\mu(z,s,1)\right]^{(-1)} * f(z). \tag{5}$$

From (4) we obtain $[z\phi_\mu(z,s,1)]^{(-1)} = \sum_{k=1}^{\infty} \frac{(\lambda+1)_{k-1}}{(\mu)_{k-1}} k^s z^k$.

For $s, \lambda \in N_0$ and $\mu \in N$, we note that:

$$\theta_\mu^{\lambda,s} f(z) = z + \sum_{k=2}^{\infty} \frac{(\lambda+1)_{k-1}}{(\mu)_{k-1}} k^s a_k z^k, \tag{6}$$

or

$$\theta_\mu^{\lambda,s} f(z) = z + \sum_{k=2}^{\infty} \frac{(k+\lambda-1)!(\mu-1)!}{\lambda!(k+\mu-2)!} k^s a_k z^k, \tag{7}$$

which is equivalent to:

$$\theta_\mu^{\lambda,s} f(z) = z + \sum_{k=2}^{\infty} \frac{C(\lambda,k)}{\delta(\mu,k)} k^s a_k z^k, \tag{8}$$

where

$$C(\lambda,k) = \frac{(1+\lambda)_{k-1}}{(k-1)!} \quad \text{and} \quad \delta(\mu,k) = \frac{(\mu)_{k-1}}{(k-1)!}. \tag{9}$$

Note that $\theta_1^{\lambda,0} f(z)$ is the derivative operator introduced by Ruscheweyh (1975), $\theta_1^{0,s}$ is the derivative operator introduced by Salagean (1983), $\theta_{\mu+1}^{0,0} f(z)$ is the integral operator defined and studied by Noor (1999) and Noor and Noor (1999) and $\theta_1^{k,s}$ is introduced by Al-Shaqsi and Darus (2008). In particular, we note that $\theta_1^{0,0} f(z) = f(z)$ and $\theta_1^{0,1} f(z) = zf'(z)$.

In view of (1) and (6) we obtain:

$$z\left(\theta_\mu^{\lambda,s} f(z)\right)' = (\lambda+1)\theta_\mu^{\lambda+1,s} f(z) - \lambda\theta_\mu^{\lambda,s} f(z) \tag{10}$$

and

$$z\left(\theta_\mu^{\lambda,s} f(z)\right)' = \mu\theta_\mu^{\lambda,s} f(z) - (\mu-1)\theta_{\mu+1}^{\lambda,s} f(z). \tag{11}$$

The relations (10) and (11) play important and significant roles in obtaining our results.

Using the linear operator (6), we define the following class:

Definition. Let $f \in T$. A function $f \in M_{\mu}^{\lambda,s}(\alpha)$ if and only if:

$$\Re\left\{\frac{z\left(\theta_{\mu}^{\lambda,s} f(z)\right)'}{\theta_{\mu}^{\lambda,s} f(z)}\right\} > \alpha.$$

where $0 \le \alpha < 1$ $\lambda, s \in N_0$ and $\mu \in N$.

Note that $M_1^{0,0}(\alpha) = S^*(\alpha)$.

In this paper, basic properties of the class $M_{\mu}^{\lambda,s}(\alpha)$ will be given such as the coefficient estimates and growth and distortion properties. In addition, sufficient conditions for subordination are also obtained.

Main Results

Theorem 1: Let $f \in T$. Then $f(z) \in M_{\mu}^{\lambda,s}(\alpha)$ if and only if:

$$\sum_{k=2}^{\infty} k^s (k-\alpha) \frac{C(\lambda,k)}{\delta(\mu,k)} |a_k| \le (1-\alpha), \tag{12}$$

where $C(\lambda,k)$ and $\delta(\mu,k)$ as given in (9).

Proof. Let $f(z) \in M_{\mu}^{\lambda,s}(\alpha)$ and we will prove that (12) holds. Note that,

$$\Re\left(\frac{z\left(\theta_{\mu}^{\lambda,s} f(z)\right)'}{\theta_{\mu}^{\lambda,s} f(z)}\right) = \Re\left(\frac{z - \sum_{k=2}^{\infty} \frac{C(\lambda,k)}{\delta(\mu,k)} k^{s+1} a_k z^k}{z - \sum_{k=2}^{\infty} \frac{C(\lambda,k)}{\delta(\mu,k)} k^s a_k z^k}\right) > \alpha.$$

Let $z \to 1^-$ through real values, then we obtain:

$$1 - \sum_{k=2}^{\infty} k^{s+1} \frac{C(\lambda,k)}{\delta(\mu,k)} |a_k| \ge \alpha\left(1 - \sum_{k=2}^{\infty} \frac{C(\lambda,k)}{\delta(\mu,k)} k^s |a_k|\right).$$

This gives (12).

Conversely, suppose that (12) holds and we will prove that $f(z) \in M_{\mu}^{\lambda,s}(\alpha)$. For $|z| = 1$, we get:

$$\le \frac{\sum_{k=2}^{\infty} (k-1) k^s \frac{C(\lambda,k)}{\delta(\mu,k)} |a_k|}{1 - \sum_{k=2}^{\infty} k^s \frac{C(\lambda,k)}{\delta(\mu,k)} |a_k|} \le 1-\alpha.$$

This shows that the value of $\frac{z\left(\theta_{\mu}^{\lambda,s} f(z)\right)'}{\theta_{\mu}^{\lambda,s} f(z)}$ lies in a circle centered at $w = 1$ whose radius $1 - \alpha$. Hence f satisfies (12).

Remark. Theorem 1 is sharp for function of the form:

$$f(z) = z - \frac{\delta(\mu,k)(1-\alpha)}{(k-\alpha)k^s C(\lambda,k)}, \quad 0 \le \alpha < 1,\ \lambda, s \in N_0,$$

$\mu \in N$ and $k \ge 2$.

Corollary 1. Let $f(z)$ $M_{\mu}^{\lambda,s}(\alpha)$, then,

$$a_k \le \frac{\delta(\mu,k)(1-\alpha)}{(k-\alpha)k^s C(\lambda,k)}, \quad 0 \le \alpha < 1,\ \lambda,\ s \in N_0,$$

$\mu \in N$ and $k \ge 2$.

Theorem 2: Let $f(z) \in M_{\mu}^{\lambda,s}(\alpha)$, then for $|z| \le r < 1$, we have:

$$r - r^2 \frac{\mu(1-\alpha)}{(\lambda+1)(2-\alpha)2^s} \le |f(z)| \le r + r^2 \frac{\mu(1-\alpha)}{(\lambda+1)(2-\alpha)2^s}, \tag{13}$$

and,

$$1 - r \frac{\mu(1-\alpha)}{(\lambda+1)(2-\alpha)2^{s-1}} \le |f'(z)| \le 1 + r \frac{\mu(1-\alpha)}{(\lambda+1)(2-\alpha)2^{s-1}}. \tag{14}$$

Proof. By Theorem 1, we have:

$$\sum_{k=2}^{\infty} a_k \le \frac{\mu(1-\alpha)}{(2-\alpha)2^s(1+\lambda)}.$$

Hence,

$$|f(z)| \le r + \sum_{k=2}^{\infty} |a_k| r^k \le r + \frac{\mu(1-\alpha)}{(\lambda+1)(2-\alpha)2^s} r^2$$

and,

$$|f(z)| \ge r - \sum_{k=2}^{\infty} |a_k| r^k \ge r - \frac{\mu(1-\alpha)}{(\lambda+1)(2-\alpha)2^s} r^2.$$

Thus (13) is true. In addition,

$$|f'(z)| \le 1 + 2r \sum_{k=2}^{\infty} |a_k| \le 1 + \frac{\mu(1-\alpha)}{(\lambda+1)(2-\alpha)2^{s-1}} r.$$

Furthermore,

$$|f'(z)| \ge 1 - 2r \sum_{k=2}^{\infty} |a_k| \ge 1 - \frac{\mu(1-\alpha)}{(\lambda+1)(2-\alpha)2^{s-1}} r.$$

By making use of Lemmas 1 and 2, we prove the following subordination.

Theorem 3: Let be convex univalent in U with $\phi(0) = 1$ and $\Re\phi(z) \ge 0$. If $f(z) \in A$ satisfies the condition

$$\frac{1}{1-\gamma}\left(\frac{z\left(\theta_{\mu}^{\lambda,s} f(z)\right)'}{\theta_{\mu}^{\lambda,s} f(z)} - \gamma\right) \prec \phi(z).$$

Then,

$$\frac{1}{1-\gamma}\left(\frac{z\left(\theta_{\mu}^{\lambda,s} f(z)\right)'}{\theta_{\mu}^{\lambda,s} f(z)} - \gamma\right) \prec \phi(z)$$

for $\lambda > -1$, $0 \le \gamma < 1$.

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z\left(\theta_{\mu}^{\lambda,s} f(z)\right)'}{\theta_{\mu}^{\lambda,s} f(z)} - \gamma\right), \tag{15}$$

where $p \in P$. By using (10) in (15) and differentiating logarithmically, we get:

$$\frac{1}{1-\gamma}\left(\frac{z\left(\theta_{\mu}^{\lambda+1,s} f(z)\right)'}{\theta_{\mu}^{\lambda+1,s} f(z)} - \gamma\right) \prec p(z) + \frac{zp'(z)}{(\lambda+1)q'(z)}.$$

where and $q(z) \prec \phi(z)$. Hence by applying Lemma 1, we obtain the required result.

Theorem 4: Let $\phi(z)$ be convex univalent in U with $\phi(0) = 1$ and $\Re\phi(z) \geq 0$. If $f(z) \in A$ satisfies the condition:

$$\frac{1}{1-\gamma}\left(\frac{z\left(\theta_{\mu}^{\lambda,s} f(z)\right)'}{\theta_{\mu}^{\lambda,s} f(z)} - \gamma\right) \prec \phi(z),$$

then,

$$\frac{1}{1-\gamma}\left(\frac{z\left(\theta_{\mu+1}^{\lambda,s} f(z)\right)'}{\theta_{\mu+1}^{\lambda,s} f(z)} - \gamma\right) \prec \phi(z),$$

for $\lambda > -1$, $\mu \geq 0$, $0 \leq \gamma < 1$.

The proof of Theorem 4 is similar to Theorem 3 by making use of (11) and Lemma 2.

Theorem 5: Let $\phi(z)$ be convex univalent in U with $\phi(0) = 1$ and $\Re\phi(z) \geq 0$. If $f(z) \in A$ satisfies the condition:

$$\frac{1}{1-\gamma}\left(\frac{z\left(\theta_{\mu}^{\lambda,s} f(z)\right)'}{\theta_{\mu}^{\lambda,s} f(z)} - \gamma\right) \prec \phi(z) \quad (0 \leq \gamma < 1; z \in U),$$

then,

$$\frac{1}{1-\gamma}\left(\frac{z\left(\theta_{\mu}^{\lambda,s} \Psi(z)\right)'}{\theta_{\mu}^{\lambda,s} \Psi(z)} - \gamma\right) \prec \phi(z) \quad (0 \leq \gamma < 1; z \in U),$$

where Ψ be the integral operator introduced by Bernardi (1969) and given by:

$$\Psi(z) = \frac{c+1}{z^{c}} \int_{0}^{z} t^{c-1} f(t)\,dt \quad (c > -1). \tag{16}$$

Proof. Let

$$p(z) = \frac{1}{1-\gamma}\left(\frac{z\left(\theta_{\mu}^{\lambda,s}(z)\right)'}{\theta_{\mu}^{\lambda,s} \Psi(z)} - \gamma\right),$$

where $p(z) \in P$. From (16), we have,

$$z\left(\theta_{\mu}^{\lambda,s} \Psi(z)\right)' = (c+1)\theta_{\mu}^{\lambda,s} f(z) - c\theta_{\mu}^{\lambda,s} \Psi(z). \tag{17}$$

Then by using (17), we get,

$$(1-\gamma)\,p(z) + c + \gamma = \frac{(c+1)\theta_{\mu}^{\lambda,s} f(z)}{\theta_{\mu}^{\lambda,s} \Psi(z)}. \tag{18}$$

Taking logarithmic derivatives in both sides of (18), we obtain,

$$p(z) + \frac{zp'(z)}{c+\gamma+(1-\gamma)\,p(z)} = \frac{1}{1-\gamma}\left(\frac{z\left(\theta_{\mu}^{\lambda,s} f(z)\right)'}{\theta_{\mu}^{\lambda,s} f(z)} - \gamma\right).$$

Finally, by using Lemma 2, we obtain that,

$$\frac{1}{1-\gamma}\left(\frac{z\left(\theta_{\mu}^{\lambda,s} \Psi(z)\right)'}{\theta_{\mu}^{\lambda,s} \Psi(z)} - \gamma\right) \prec \phi(z) \quad (0 \leq \gamma < 1;\ z \in U).$$

The generalised operator introduced can be applied in solving other problems such as the Fekete-Szego problems (Ajwely & Darus 2011; Al-Abbadi & Darus 2011) and the Hankel determinant problems (Darus & Faisal 2010).

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