ABSTRACT

Two explicit hybrid methods with algebraic order seven for the numerical integration of second-order ordinary differential equations of the form \( y'' = f(x, y) \) are developed. The algebraic order of these methods is the highest in comparison with other explicit hybrid methods of the same class. Numerical comparisons carried out show the advantage of the new methods.

Keywords: Algebraic order; explicit hybrid method; second-order ordinary differential equations

INTRODUCTION

There has been a great interest in the research of new methods for numerically solving the second-order ordinary differential equations of the form

\[
y'' = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0
\]

Such problems often arise in science and engineering field such as celestial mechanics, molecular dynamics, semi-discretizations of wave equations and so on. The second-order equation can be directly solved by using Runge-Kutta-Nystrom (RKN) methods or multistep methods. Several authors such as Hairer (1979), Cash (1981), Fatunla et. al (1999) and Chawla (1984) have proposed hybrid methods which are obtained from the idea underlying both the Runge Kutta and linear multistep methods.

In the development of numerical methods for solving (1), it is important to pay attention at the algebraic order of the method because this is the main factor to achieve high accuracy. If the theoretical solution of the problem has a periodic nature, then it is also essential to consider the phase-lag and dissipation. These are actually two types of truncation errors. The first is the angle between the analytical solution and the numerical solution while the second is the distance from a standard cyclic solution. A pioneering work on phase-lag property has been done by Brusa and Nigro (1980). Some of the current developments of hybrid methods which are implemented in constant step-size are contributed for example by Tsitouras (2003, 2006) and Franco (2006). Tsitouras has developed an explicit hybrid method of algebraic order seven with four stages per step intended for solving linear second-order problems of the form (1). He was also proposed two explicit hybrid methods of algebraic order six, one with six stages while the other with five stages per step, and has derived an implicit hybrid method of algebraic order eight with six stages per step. Meanwhile, Franco has derived explicit hybrid methods which reach up to order five and six with three and four stages per step, respectively. Other current research on hybrid methods include Coleman (2003) and Chan (2004) who have studied the hybrid methods theoretically through B-series and P-series, respectively.

In this paper, based on explicit hybrid methods in Franco (2006), our attempt is to derive explicit hybrid methods of algebraic order seven with five stages per step. These methods will be compared with existing methods.

EXPLICIT HYBRID METHODS

We consider the class of explicit hybrid methods established by Franco (2006):

\[
\begin{align*}
y &= y_{n-1}, \quad Y_2 = y_n, \\
Y_i &= (1+c) y_i - c y_{n-i} + h^i \sum_{j=1}^{i} a_{ij} f (x_j + c_i h, Y_j), i = 3, \ldots, s, \\
y_{n+1} &= 2 y_n - y_{n-1} + h \left[ b_1 f_{n+1} + b_2 f_n + \sum_{j=1}^{s} b_j f (x_j + c_i h, Y_j) \right]
\end{align*}
\]

(2)
where \( f_{n-1} \) and \( f_n \) represent \( f(x_{n-1}, y_{n-1}) \) and \( f(x_n, y_n) \), respectively. These methods require \( s - 1 \) function evaluations or stages per step and represented by the following table:

<table>
<thead>
<tr>
<th>-1</th>
<th>0 0 0 0 ... 0 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_s )</td>
<td>( a_{s1} ) ( a_{s2} ) 0 ... 0</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots ) ( \vdots ) ( \vdots ) ( \vdots )</td>
</tr>
<tr>
<td>( c_r )</td>
<td>( a_{r1} ) ( a_{r2} ) ... ( a_{rs} ) 0</td>
</tr>
<tr>
<td>( b_1 )</td>
<td>( b_2 ) ... ( b_{s-1} ) ( b_s )</td>
</tr>
</tbody>
</table>

The order conditions for this class of methods are given by Coleman (2003). The leading term associated with the local truncation error of a \( p \)-th order explicit hybrid method is:

\[
e_{p+1}(t) = \frac{\alpha(t)}{(p+2)!} [1 - (-1)^{p+2}] \left( b_1 b^p(\psi^*(t)) \right), t_j \in T_j, P(t_j) = p + 2.
\]

where \( T_j, \rho(t), \alpha(t) \) and \( \psi^*(t) \) are defined in Coleman (2003). The quantity:

\[
E = \sqrt{\sum_{j=1}^{n_{p+1}} e_{p+1}^2(t_j)},
\]

where \( n_{p+1} \) is the number of trees of order \( p + 2 \), is called the error norm for the \( p \)-th order method.

**BASIC THEORY**

Let \( H = \lambda h \) and \( e = (1 \ 1 \ 1)^T \). Applying the hybrid methods defined in (2) to equation, yields the recursion

\[
y_{n+1} - S(H)y_n + P(H)y_{n-1} = 0 \tag{3}
\]

where

\[
S(H) = 2 - H^2 b^2(1 + H^2 A)^{-1} (e + c), \quad P(H) = 1 - H^2 b^2(1 + H^2 A)^{-1} c
\]

The characteristic equation associated with (3) is:

\[
\xi^2 - S(H)\xi + P(H) = 0 \tag{4}
\]

According to Houwen and Sommeijer (1987), phase-lag is defined as the difference \( t = H - \theta(H) \) where \( H \) is the phase (or argument) of the exact solution of \( y'' = -\lambda^2 y \) and \( \theta(H) \) is the phase of the principal root of (4). For the hybrid methods corresponding the characteristic equation (4), the quantity:

\[
\phi(H) = H - \arccos \left( \frac{S(H)}{2 \sqrt{P(H)}} \right)
\]

is called phase-lag (or dispersion error) while the quantity

\[
d(H) = 1 - \frac{P(H)}{H^2}
\]

is called dissipation (or amplification) error. A hybrid method corresponding to (4) is said to have the phase-lag of order \( n \) if \( q(H) = O(H^{n+1}) \). If \( P(H) = 1 \) then \( d(H) = 0 \) and the method having this property is said to be zero dissipative or dissipative of order infinity. If \( P(H) \neq 1 \) then \( d(H) = O(H^{n+1}) \) and the method with this property is said to be dissipative of order \( m \). The interval \((0, H_s)\) is called the interval of periodicity of the method if:

\[
P(H) = 1 \quad \text{and} \quad |S(H)| < 2 \quad \text{for all} \quad H \in (0, H_s),
\]

where as the interval \((0, H_s)\) is called the interval of absolute stability if:

\[
|P(H)| < 1 \quad \text{and} \quad |S(H)| < 1 + P(H) \quad \text{for all} \quad H \in (0, H_s).
\]

**DERIVATION OF THE NEW METHODS**

We derive five-stage seventh order explicit hybrid methods algebraically. The new methods must satisfy the order conditions as given by Coleman (2003) with \( s = 6 \). There are 33 order conditions involved. By applying the simplifying condition:

\[
\sum_{j=1}^{6} a_{ij} c_j^2 + (-1)^{j+1} \frac{c_j}{(\alpha + 1)(\alpha + 2)} \quad \alpha = 0, 1, \tag{6}
\]

some order conditions which are combinations of the other order conditions are eliminated leaving 15 order conditions. Substituting in \( c_j = -1, c_1 = 0, a_{ij} = 0 \) and \( a_{ij} = 0 \) for \( j \neq i \) to the order conditions we get equations to be solved for the new methods. Other equations to be solved are obtained by substituting in \( c_j = -1, c_1 = 0, a_{ij} = 0 \) and \( a_{ij} = 0 \) for \( j \neq i \) to the simplifying conditions (6). The following are all equations to be solved for the new methods:

\[
b_1 + b_2 + b_3 + b_4 + b_5 + b_6 = 1, \tag{7}
\]

\[
-b_1 + b_1 c_1 + b_2 c_1 + b_2 c_1 + b_4 c_1 = 0, \tag{8}
\]

\[
-b_1 + b_1 c_1 + b_2 c_1 + b_2 c_1 + b_4 c_1 = \frac{1}{4}, \tag{9}
\]

\[
-b_1 + b_2 c_1 + b_3 c_1 + b_3 c_1 + b_4 c_1 = 0, \tag{10}
\]

\[
b_1 + b_2 c_1 + b_4 c_1 + b_4 c_1 + b_6 c_1 = \frac{1}{4}. \tag{11}
\]

\[
b_1 a_{21} + b_2 a_{21} + b_3 a_{21} + b_4 a_{21} + b_4 c_1 + b_2 c_1 = \frac{1}{16}. \tag{12}
\]

\[
-b_1 + b_2 c_1 + b_4 c_1 + b_4 c_1 + b_6 c_1 = 0. \tag{13}
\]
From equation (26), we already have \( a_{31} \) in term of \( c_3 \).

Solving equations (7) — (11) and (13) for \( b_1 \), we get

\[
b_1 = \frac{5c_5 c_3 c_6 + 5c_5 c_3 c_7 + 2c_5 + 5c_5 c_3 c_8 + 2c_5 + 5c_5 c_3 c_9 + 2c_5}{30 (1 + c_5) (1 + c_5) (1 + c_5)}.
\]

Substituting (30) — (35) into equation (16) and solving the resulted equation for \( c_6 \) we get

\[
c_6 = \frac{-28 c_6 c_5 - 28 c_5 c_6 + 15 + 70 c_5 c_6 + 28 c_5 - 28 c_5 c_6 + 28 c_6}{14 (5c_5 c_6 - 5c_6 c_5 - 5c_5 c_6 + 2c_5 - 5c_6 c_5 + 2c_6 + 2c_5 - 2)}.
\]

From equations (27) — (29), we have

\[
a_{41} = a_{43} c_3 - \frac{c_3^2 + c_3}{6},
\]

\[
a_{51} = a_{53} c_3 + a_{54} a_{54} - \frac{c_3^2 + c_3}{6},
\]

\[
a_{61} = a_{63} c_3 + a_{64} c_3 + a_{65} a_{65} - \frac{c_3^2 + c_3}{6}.
\]
Substituting (37) — (39) into equations (12), (14) and (17), then solving the resulted equations for \(a_{43}\), we get
\[
a_{43} = \frac{28c_1c_4 + 14c_5 + 14c_6 + 15c_7(c_1 + 1)(-c_4 + c_5)}{168(c_1 + 1)c_2 (-5c_1 + 5c_2 - 5c_3 + 2c_4 + 2c_5 - 2c_6 + 5c_7 + 2c_8).}
\]

Substituting (37) — (39) into equations (12), (14), (15) and (18), then solving the resulted equations for \(a_{54}\) we get
\[
a_{54} = \frac{14c_1 + 28c_5c_2 - 28c_4 + 37}{5040 \left(1 + c_4\right) \left(-c_4 + c_5\right) c_2 \left(-c_4 + c_5\right) b_4}.
\]

Substituting (37) — (39) into equation (12), then the resulted equation is multiplied by \(c_6\) and denoted equation A. Equation (14) is subtracted from equation A. Solving the resulted equation for \(a_{53}\) we get:
\[
a_{53} = \frac{-360b_6a_5c_3 - 26b_6a_5c_2 + 26b_6a_5c_1 + 26b_6a_5c_0 + 26b_6a_5c_0 + 26b_6a_5c_0}{360b_6c_2 \left(-c_4 + c_5\right) \left(1 + c_4\right)}.
\]

Substituting (37) — (42) into equations (15) and (20), then solving the resulted equations for \(a_{64}\) and \(a_{65}\) we get:
\[
a_{64} = \frac{14c_1 - 14c_5 + 3 - 14c_4}{2520b_3c_3 \left(c_1c_3 + c_2c_4 - c_3c_4 - c_1c_3 - c_2c_4 + c_3c_4 + c_3^2 + c_4^3\right)}.
\]

Substituting (37) — (44) into equation (19) and solving the resulted equation for \(a_{65}\) we get:
\[
-37c_1 - 6c_2 + 43c_3 - 28c_4c_5 - 5040b_6a_5c_1c_1' + 5040b_6a_5c_1c_2' + 28c_4c_5 - 28c_4c_5 - 5040b_6a_5c_1c_2' + 28c_4c_5 - 28c_4c_5 - 5040b_6a_5c_1c_2' + 28c_4c_5 - 28c_4c_5 - 5040b_6a_5c_1c_2' + 28c_4c_5 - 28c_4c_5 - 5040b_6a_5c_1c_2'
\]

Now the only free parameter left is \(c_5\). We obtain two methods depending on the value of \(c_5\). As for the error norm, \(E\) of our methods, the computation of \(E\) has been done using trees related with the sum in the order conditions (46) to (53) listed in Table 1 (Coleman 2003).

<table>
<thead>
<tr>
<th>Order conditions</th>
<th></th>
</tr>
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<tbody>
<tr>
<td>(\sum_{i,j} b_i c_j a_{ij} c_i^j = 0)</td>
<td>(47)</td>
</tr>
<tr>
<td>(\sum_{i,j} b_i c_j a_{ij} c_i^j = 0)</td>
<td>(48)</td>
</tr>
<tr>
<td>(\sum_{i,j} b_i c_j a_{ij} c_i^j = 1/180)</td>
<td>(49)</td>
</tr>
<tr>
<td>(\sum_{i,j} b_i c_j a_{ij} c_i^j = 1/1080)</td>
<td>(50)</td>
</tr>
<tr>
<td>(\sum_{i,j} b_i a_j c_i a_{ij} c_i^j = 0)</td>
<td>(51)</td>
</tr>
<tr>
<td>(\sum_{i,j} b_i a_j c_i a_{ij} c_i^j = 1/2160)</td>
<td>(52)</td>
</tr>
<tr>
<td>(\sum_{i,j} b_i a_j c_i a_{ij} c_i^j = 0)</td>
<td>(53)</td>
</tr>
</tbody>
</table>

Strategies employed in choosing the free parameter for our methods are:

1. Minimize the function \(P\) which is given by \(P = \sqrt{\sum s_i^2}\).

Here, \(s_i\)'s represent expressions obtained by subtracting the right sides of order conditions (46) to (53) from the left sides.

2. Increase the dissipation order.

For our first method, we select the parameter \(c_5\) so that \(P\) is as small as possible obtaining the values \(c_5 = \frac{28521}{50000} E = 6.755534178017401 \times 10^{-4}\). This method has an interval of absolute stability \((0, 3.341)\) while the phase-lag and dissipation error are respectively given by:

\(q(H) = \left\lfloor -2.3585671 \times 3670272000000 + 555412997 \times 12845952000000 \right\rfloor \sqrt{5}\)

\(H^4 = O(H^4)\) and

\(d(H) = 1.921345174 \times 10^{-5} H^4 + O(H^6)\)
Thus, this method which will be denoted as EHM7(8, 7) has the phase-lag of order 8 and dissipative of order 7.

For our second method, $c_4$ is selected so that the dissipative order is increased. To do this, we first establish conditions related with dissipation error that have to be satisfied by the five-stage explicit hybrid methods. The expressions $S(H)$ and $P(H)$ for the five-stage explicit hybrid methods are polynomials in $H$ given by:

$$S(H) = 2 - u_1 H^2 + u_2 H^4 - u_3 H^6 + u_4 H^8 - u_5 H^{10},$$

$$P(H) = 1 - t_1 H^2 + t_2 H^4 - t_3 H^6 + t_4 H^8 - t_5 H^{10},$$

where $u_i$ and $t_j$ are expressions in terms of coefficients of the method. Expanding equation (5) in Taylor series, we obtain:

$$\frac{d}{dt}\left(\frac{1}{2} t_1 H^2 + \left(-\frac{1}{2} t_2 + \frac{1}{8} t_4\right) H^4 + \left(\frac{1}{2} t_3 - \frac{1}{4} t_5 + \frac{1}{16} t_7\right) H^6 + O(H^8)\right) = 0.$$

By setting the coefficients of $H^2$, $i \geq 1$ to zero, we obtain conditions for the explicit hybrid methods to have the dissipation of order 9. We note that this is the highest attainable dissipation order for our method.

**Dissipation of order 9:** Conditions to be satisfied are

$$\frac{1}{2} t_1 = 0, \frac{1}{2} t_3 - \frac{1}{4} t_5 + \frac{1}{16} t_7 = 0,$$

and

$$\frac{1}{2} t_4 + \frac{1}{4} t_6 + \frac{3}{16} t_8 + \frac{5}{128} t_10 = 0.$$

Substituting (30) — (45), and $c_4 = \frac{1}{2} + \frac{1}{2} \sqrt{5}$ into the equations corresponding to the above conditions and solving the resulted equations for $c_4$ yields $c_4 = \frac{406}{134647} \approx 0.00593$, and the error norm is $E = 6.1564599162350 \times 10^{-7}$. This method has an interval of absolute stability $(0, 2.843]$. The phase-lag and dissipation error for this method are given by:

$$\phi(H) = -\frac{13}{7257600} H^3 + O(H^4)$$

and $d(H) = -2631856949 \times 10^{-7} H^0 + O(H^0)$. Thus, this method which will be denoted as EHM7(8, 9) has the phase-lag of order 8 and dissipative of order 9. Most of the values of coefficients of EHM7(8, 7) and EHM7(8, 9) methods are in a form of surds and are too long to be reported in this paper. A complete list of these values will be given to the reader upon request.

**PROBLEMS TESTED**

The following are some second-order problems with exact solutions taken from the literature that have been used to evaluate the performance of our methods.

**Problem 1** In-homogenous problem

$$y'' = \sin(x), y(0) = 1, y'(0) = 1,$$

Solution: $y(x) = \cos(10\pi) + \sin(10\pi) + \sin(x)$.

**Problem 2** Homogeneous problem

$$y'' = \sin(x), y(0) = 1, y'(0) = 1,$$

Solution: $y(x) = \sin(x)$.

**Problem 3** Nonlinear oscillatory problem

$$y'' = -4x^3 y - \frac{2y}{\sqrt{y^2 + x^2}}, y(0) = 0, y'(0) = 0,$$

Solution: $y(x) = 0$.

**Problem 4** Orbit problem

$$y'' = -\frac{x^2}{\sqrt{y^2 + x^2}^3}, y(0) = 0, y'(0) = 0,$$

Solution: $y(x) = \sin(x)$.

**Problem 5** Almost orbit problem

$$z''(x) + z(x) = e^x, z(0) = 1, z'(0) = \sin(x), z(1) = 0.9995, z \in C, x \in [0, 0.20],$$

The theoretical solution is $z(x) = (1 - 0.0005ix)e^x$. In this paper, we solve the equivalent system

$$y'' = y + \frac{1}{1000} \cos(x), y(0) = 1, y'(0) = 0,$$

with the theoretical solution: $y(x) = \cos(x) + 0.005x \sin(x), y(1) = \sin(x) - 0.0005 \cos(x)$. We choose $\varepsilon = 10^{-3}$. Solution: $y(x) = \cos(x) + \varepsilon \sin(x)$.

**RESULTS**

The methods that have been used in the comparisons are denoted by:

1. TSI6: The sixth-order explicit hybrid method with five stages found in Tsitouras (2003)
2. TSI7: The seventh-order explicit hybrid method with four stages derived in Tsitouras (2002)
3. RKNH2: The second-order explicit RKN method with five stages derived by van Der Houwen and Sommeijer (1987)
4. EHM7(8, 7): The first seventh-order explicit hybrid method with five stages derived in this paper.
5. EHM7(8, 9): The second seventh-order explicit hybrid method with five stages derived in this paper.

The numerical results shown in Figures 1, 3 and 4 have been computed with step-size \( h = 0.1/2^i \), \( i = 0, 1, \ldots, 4 \) for EHM7(8, 7), EHM7(8, 9), TSI7 and RKNH2 methods, and with step-size \( h = 0.01/2^i \), \( i = 2, 3, \ldots, 6 \) for TSI6. In Figure 2, the numerical results for EHM7(8, 7), EHM7(8, 9), TSI7 and RKNH2 methods have been computed with \( h = 1/2^i \), \( i = 0, 1, \ldots, 4 \) while for TSI6, \( h = 0.1/2^i \), \( i = 0, 1, \ldots, 4 \). Figure 5 presents the numerical results for EHM7(8, 7), EHM7(8, 9), TSI7 and RKNH2 using \( h = 1/2^i \), \( i = 0, 1, \ldots, 4 \) while for TSI6 method, \( h = 0.1/2^i \), \( i = 4, 5, \ldots, 8 \). In Figure 6, the step-sizes used in the computation of the numerical results for EHM7(8, 7), TSI7, EHM7(8, 9) and RKNH2 methods are \( h = 0.1/2^i \), \( i = 0, 1, \ldots, 4 \) while for TSI6 method, \( h = 0.1/2^i \), \( i = 1, 2, \ldots, 5 \). In Figure 1 and Figure 2, curves of \( \log_{10} \text{end-point error} \) versus step-size are depicted. The formula of the maximum global error (MAXGE) is \( \text{MAXGE} = \| y(x) - y^n \| \) where \( y(x) \) is the exact solution and \( y^n \) is the numerical solution. Figure 3 – 6 display the curves of \( \log_{10} \text{MAXGE} \) versus step-size whereas Figure 7 shows the total time (in seconds) required by each method to solve each problem over various step-sizes. The horizontal grid tick-marks in Figure 7 represent the problems solved where:

1. 0 means “Problem 1”.
2. 1 means “Problem 2”.
3. 2 means “Problem 3”.
4. 3 means “Problem 4”.
5. 4 means “Problem 5”, and
6. 5 means “Problem 6”.

Discussion and Conclusion

From the numerical results in Figures 1 to 6, we observe that EHM7(8, 9) method is the most efficient for solving Problems 1, 2, 4, and 6 of all methods being compared. On the other hand, TSI6 is the least efficient method. For Problem 6, EHM7(8, 7) and EHM7(8, 9) both perform well whereas for Problem 3, EHM7(8, 7) method gives the best performance. From Figure 7, it is obvious that the total time required by TSI6 to solve each problem over various step-sizes is longer than that required by other methods.
considered. It is because the computation of TSI6 code needs smaller step-sizes which results in more number of function evaluations in order to gain accuracy. It is also clear that for Problems 3, 4 and 5, the computations of RKHN2 and TSI7 codes are faster than that of EHM7(8, 7) and EHM7(8, 9) codes whereas for Problem 6, TSI7 is the fastest of all codes considered. In conclusion, the new methods perform more efficiently than TSI6, TSI7 and RKHN2 methods. Furthermore, the new method with reduced dissipation error is preferable for solving most of the problems used in this paper. All codes are designed using Microsoft Visual C++ version 6.0 in HP computer with Intel(R)Core(TM)2Duo CPU P8600@2.40GHz.

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