Quintic Trigonometric Bézier Curve with Two Shape Parameters
(Lengkung Bézier Trigonometri Kuintik dengan Dua Parameter Bentuk)

MISRO, M.Y.*; RAMLI, A. & ALI, J.M.

ABSTRACT
The fifth degree of trigonometric Bézier curve called quintic with two shapes parameter is presented in this paper. Shape parameters provide more control on the shape of the curve compared to the ordinary Bézier curve. This technique is one of the crucial parts in constructing curves and surfaces because the presence of shape parameters will allow the curve to be more flexible without changing its control points. Furthermore, by changing the value of shape parameters, the curve still preserves its geometrical features thus makes it more convenient rather than altering the control points. But, to interpolate curves from one point to another or surface patches, we need to satisfy certain continuity constraints to ensure the smoothness not just parametrically but also geometrically.

Keywords: Bézier curve; parameterization; quintic trigonometric Bézier; shape parameter; trigonometric function

INTRODUCTION
Rapid research in curves and surfaces contributes towards the field of Computer Aided Geometric Design (CAGD). Several inventive curves are being developed such as Bézier-like, log-aesthetic curve (Gobithaasan 2013) and spiral curve. Bézier curve is one of the typical parametric curves with a continuous curvature that is utilized in various applications including medical and engineering. Constructing two pieces of Bézier curve or piecewise method may be challenging in terms of preserving continuity and fulfilling certain shape requirements such as curvature and torsion. In order to preserve some geometrical properties, we may need to satisfy some continuity constraint between two merging curves. As a result, when two curves are merged without proper continuity, the curve of the first segment will not merge nicely with the curve at second segment at a particular joint. Thus, the continuous curvature of the curve will not be ensured.

Ahmad et al. (2014) constructed a new generalization basis called A-Bézier that has a basis space span of \{1, t, r, ..., r^n\} by using a systolic array. Previously, Mainar et al. (2001) constructed α-basis of quartic Bernstein. Another approach by Chen and Wang (2003) used integral by proposed C-Bézier. Most of these functions do not enclose free form curves of a higher-order polynomial. Recently, the degree of trigonometric Bézier curve is also discussed from a lower to a higher degree such as quadratic trigonometric Bézier curve (Uzma et al. 2012), cubic trigonometric Bézier curve Han et al. (2009), quartic trigonometric Bézier curve (Dube et al. 2013; Zhu et al. 2012), quasi-quintic trigonometric Bézier curve (Uzma et al. 2013) and quintic trigonometric Bézier curve (Dube & Bharti 2014).

Throughout this paper, we organized our work as follows. In Section 2, a new function which is quintic trigonometric Bézier curve with two shape parameter will be proposed. This new basis of quintic trigonometric Bézier curve produces a different curve compared to (Dube & Bharti 2014). In Dube and Bharti (2014), they used 4 control points that mimics cubic trigonometric Bézier curves with a single shape parameter.

In this establishment, we utilized 6 control points with two significant shape parameters and some properties are presented. In Section 3, quintic trigonometric Bézier curve and its geometric properties are discussed. In addition to that, the usage and effects of those parameters to our newly proposed curve will be intensified. Theorems are presented...
in Section 4. In this section, the proposed curve will utilize the theorems in order to satisfy certain continuity condition. On top of that, two cases of shape parameters with similar and different values on a piece of a curve are shown.

QUINTIC TRIGONOMETRIC BÉZIER BASIS FUNCTION
Quintic trigonometric Bézier curve with two shape parameters \( \alpha \) and \( \beta \) is defined as:

\[
\mathbf{z}(t) = \sum_{i=0}^{5} P_i f_i(t),
\]

where \( P_i \) is the control points and \( f_i \) is the basis function for quintic trigonometric Bézier with \( i = 0, 1, 2, 3, 4, 5 \) whereas,

\[
\begin{align*}
  f_0(t) &= (1 - \sin \frac{\pi}{4})(1 - \alpha \sin \frac{\pi}{4}) \\
  f_1(t) &= \sin \frac{\pi}{4}(1 - \sin \frac{\pi}{4})(4 + \alpha - \alpha \sin \frac{\pi}{4}) \\
  f_2(t) &= (1 - \sin \frac{\pi}{4})(1 - \cos \frac{\pi}{4})(8 \sin \frac{\pi}{4} + 3 \cos \frac{\pi}{4} + 9) \\
  f_3(t) &= (1 - \cos \frac{\pi}{4})(1 - \sin \frac{\pi}{4})(8 \cos \frac{\pi}{4} + 3 \sin \frac{\pi}{4} + 9) \\
  f_4(t) &= \cos \frac{\pi}{4}(1 - \cos \frac{\pi}{4})(4 - \beta \cos \frac{\pi}{4}) \\
  f_5(t) &= (1 - \cos \frac{\pi}{4})(1 - \beta \cos \frac{\pi}{4}),
\end{align*}
\]

where \( \alpha, \beta \in [-4, 1] \) are shape parameters for the following six functions of \( t \), where \( t \in [0, 1] \).

Figure 1 shows the quintic trigonometric Bézier basis function for two arbitrarily selected real values of \( \alpha \) and \( \beta \).

\[
\begin{align*}
  f(t; \alpha, \beta) &= f_{5-i}(1 - t; \alpha, \beta) \quad (5)
\end{align*}
\]

Proof:
(a) For \( t \in [0,1] \) and \( \alpha, \beta \in [-4,1] \), then \( 1 - \sin \frac{\pi}{4} \geq 0, \ 1 - \cos \frac{\pi}{4} \geq 0, \ \sin \frac{\pi}{4} \geq 0, \ \cos \frac{\pi}{4} \geq 0, \ 1 - \alpha \sin \frac{\pi}{4} \geq 0, \ 1 - \beta \cos \frac{\pi}{4} \geq 0, \ 4 + \alpha - \alpha \sin \frac{\pi}{4} \geq 0, \ 4 + \beta - \beta \cos \frac{\pi}{4} \geq 0.
\]
(b) \( \sum_{i=0}^{5} f_i(t) = f_0(t) + f_1(t) + f_2(t) + f_3(t) + f_4(t) + f_5(t) = 1. \)
(c) \( f_0(t; \alpha, \beta) = (1 - \sin \frac{\pi}{4})(1 - \alpha \sin \frac{\pi}{4}) \)
\[
= (1 - \cos \frac{\pi}{4})(1 - \beta \cos \frac{\pi}{4})
\]
\[
= f_5(1 - t; \beta, \alpha)
\]

QUINTIC TRIGONOMETRIC BÉZIER CURVE
Given point \( P_i \), where \( i = 0, 1, 2, 3, 4, 5 \) in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \). Then

\[
r(t) = \sum_{i=0}^{5} P_i f_i(t) \quad t \in [0,1], \ \alpha, \beta \in [-4,1],
\]

is called a quintic trigonometric Bézier curve with two shape parameters. Projection of the curves lies inside the convex hull for \( \alpha, \beta \in [-4,1] \) are shown in Figure 2.

\[
\begin{align*}
  r(0) &= P_0 \\
  r(1) &= P_5 \\
  r'(0) &= \frac{-\pi}{2} (P_0 - P_1)(4 + \alpha) \\
  r'(1) &= \frac{-\pi}{2} (P_4 - P_5)(4 + \beta).
\end{align*}
\]

Quintic trigonometric Bézier has the following properties:

1. Endpoint terminal

\[
\begin{align*}
  r(0) &= P_0 \\
  r(1) &= P_5 \\
  r'(0) &= \frac{-\pi}{2} (P_0 - P_1)(4 + \alpha) \\
  r'(1) &= \frac{-\pi}{2} (P_4 - P_5)(4 + \beta).
\end{align*}
\]
\[ r'(0) = \pi(3P_2 - 2P_1(3 + \alpha) + P_0(3 + 2\alpha)) \]
\[ r'(1) = \pi(3P_3 - 2P_4(3 + \beta) + P_5(3 + 2\beta)). \]  

Figure 3 shows the behavior of the quintic trigonometric Bézier curve when the value of shape parameter \( \alpha \) is fixed and \( \beta \) is varied.

2. Convex hull

The entire trigonometric Bézier curve segment must lie inside its control point polygon spanned by \( P_0, P_1, P_2, P_3, P_4, P_5 \).

3. Symmetry

\{P_0, P_1, P_2, P_3, P_4, P_5\} and \{P_5, P_4, P_3, P_2, P_1, P_0\} define the same trigonometric Bézier curve in different parameterizations, i.e.,

\[ r(t; \alpha, \beta; P_0, P_1, P_2, P_3, P_4, P_5) = r(1 - t; \alpha, \beta; P_5, P_4, P_3, P_2, P_1, P_0) \]  

where \( 0 \leq t \leq 1, -4 \leq \alpha, \beta \leq 1 \)

4. Geometric invariance

\[ r(t; \alpha, \beta; P_0 + y, P_1 + y, P_2 + y, P_3 + y, P_4 + y, P_5 + y) = r(t; \alpha, \beta; P_0, P_1, P_2, P_3, P_4, P_5) + y \]

\[ r(t; \alpha, \beta; P_0 \ast m, P_1 \ast m, P_2 \ast m, P_3 \ast m, P_4 \ast m, P_5 \ast m) = r(t; \alpha, \beta; P_0, P_1, P_2, P_3, P_4, P_5) \ast m \]

where \( 0 \leq t \leq 1, -4 \leq \alpha, \beta \leq 1 \) where is an arbitrary vector in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \), \( m \) is an arbitrary \( d \times d \) matrix where \( d = 5 \) or 6.

We can see the effect of shape parameter if we let \( \beta \) to be fixed like in Figure 4.

2. Convex hull

The entire trigonometric Bézier curve segment must lie inside its control point polygon spanned by \( P_0, P_1, P_2, P_3, P_4, P_5 \).

3. Symmetry

\{P_0, P_1, P_2, P_3, P_4, P_5\} and \{P_5, P_4, P_3, P_2, P_1, P_0\} define the same trigonometric Bézier curve in different parameterizations, i.e.,

\[ r(t; \alpha, \beta; P_0, P_1, P_2, P_3, P_4, P_5) = r(1 - t; \alpha, \beta; P_5, P_4, P_3, P_2, P_1, P_0) \]  

where \( 0 \leq t \leq 1, -4 \leq \alpha, \beta \leq 1 \)

4. Geometric invariance

\[ r(t; \alpha, \beta; P_0 + y, P_1 + y, P_2 + y, P_3 + y, P_4 + y, P_5 + y) = r(t; \alpha, \beta; P_0, P_1, P_2, P_3, P_4, P_5) + y \]

\[ r(t; \alpha, \beta; P_0 \ast m, P_1 \ast m, P_2 \ast m, P_3 \ast m, P_4 \ast m, P_5 \ast m) = r(t; \alpha, \beta; P_0, P_1, P_2, P_3, P_4, P_5) \ast m \]

where \( 0 \leq t \leq 1, -4 \leq \alpha, \beta \leq 1 \) where is an arbitrary vector in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \), \( m \) is an arbitrary \( d \times d \) matrix where \( d = 5 \) or 6.

We can see the effect of shape parameter if we let \( \beta \) to be fixed like in Figure 4.

Figure 3 shows the behavior of the quintic trigonometric Bézier curve when the value of shape parameter \( \alpha \) is fixed and \( \beta \) is varied.

Figure 3. Quintic trigonometric curve with \( \alpha = -3 \) and \( \beta = -3 \) (dotted blue), 0 (black), 1 (dashed red)

Figure 4. Quintic trigonometric Bézier curve with \( \alpha = -3 \) (dotted red), 0 (black), 1 (blue) and \( \beta = 1 \)

CURVATURE DISTRIBUTION OF QUINTIC TRIGONOMETRIC BÉZIER CURVE

Curvature distribution for Figure 5 shows that the curvature profile inherits symmetrical form when we used the same value of shape parameter. When one of the variables of shape parameter either \( \alpha \) or \( \beta \) are fixed, it will still retain the pattern of the curvature, but the value of the amplitude of the curvature may increase or decrease as in Figures 6 and 7.

Figure 5. A curve with \( \alpha = \beta \) (left) and its curvature distribution (right)
Shape control of quintic trigonometric Bézier curve makes the construction of the curve much easier and flexible. Discussion and advantages will be presented in the next section.

**SHAPE CONTROL OF THE QUINTIC TRIGONOMETRIC BÉZIER CURVE**

For $t \in [0, 1]$ where we can write (2) as follows:

$$r(t) = \sum_{i=0}^{5} P_i f_i(t) = P_0 f_0(t) + P_1 f_1(t) + P_2 f_2(t) + P_3 f_3(t) + P_4 f_4(t) + P_5 f_5(t).$$

Shape parameter $\alpha$ will only affect curve on the control point of $P_2 - P_1$ and $P_1 - P_0$, where $\beta$ affect $P_5 - P_4$ and $P_4 - P_3$, respectively. In fact, from Figure 1, we can also predict the following behavior of the curves.

As $\alpha$ increases, the curve moves in the direction of the edge $P_2 - P_1$ and $P_1 - P_0$, and as $\beta$ increases, the curve moves in the direction of the edge $P_5 - P_4$ and $P_4 - P_3$. While the shape parameter $\alpha = \beta$, the curve moves in the direction of; or, the opposite direction to the edge $P_2 - P_1$ and $P_1 - P_0$ depending on the values of the parameter itself. This nature is called as geometric effect. For close curve the changing effect of shape parameter we can see in Figure 8.

**COMPOSITE TWO QUINTIC TRIGONOMETRIC BÉZIER CURVE**

Two curves can be connected by a few acceptable fashions. Connections between two curves are notable via point, tangential and positional. A curve that joined at a point is called $C^0$, linked at positional is named $C^1$ and the one connected by tangency is $C^2$.

In all cases, we would need to satisfy some conditions in order to achieve $C^2$ continuity. Strictly speaking, the presence of shape parameters does not just act as a local control for Bézier curve. These parameters will establish the flexibility of the curve. This setup may help the designer to avoid changing the control point and still be able to adjust their desired curve without the need to change control point.

**Theorem 1**: Let $r_1(t)$ and $r_2(t)$ be two quintic trigonometric Bézier curves. The two curves are joined by $C^1$ continuity at the linked point if:

$$P_5(8 + \alpha_2 + \beta_2) - P_4(4 + \beta_1) = Q_1(4 + \alpha_2). \quad (12)$$

**Proof.** Let two quintic trigonometric Bézier curves such as

$$r_1(t) = \sum_{i=0}^{5} P_i f_i(t) \quad r_2(t) = \sum_{i=0}^{5} Q_i f_i(t). \quad (13)$$

![Figure 6. A curve with fixed $\alpha$ (left) and its curvature distribution (right)](image)

![Figure 7. A curve with fixed $\beta$ (left) and its curvature distribution (right)](image)
where $P_i$, $Q_i$ are control point for $i = 0, 1, 2, 3, 4, 5$ with $\alpha_i, \beta_i$ as shape parameters for $P_i$, and $\alpha_i, \beta_i$ as shape parameters for $Q_i$. $-4 \leq \alpha_1, \beta_1, \alpha_2, \beta_2 \leq 1$.

By substituting end-point terminal equation (6) and (7) we will obtained,

$$P_5 = \lambda Q_0$$

By substituting end-point terminal equation (6) and (7) we will obtained,

$$P_5(8 + \alpha_2 + \beta_1) - P_4(4 + \beta_1) = \lambda Q_1(4 + \alpha_2).$$

Rearrange the terms we have

by assuming $\lambda = 1$.

Case 1. Value of shape parameter $\alpha_i$ equals to $\beta_i$ for each curve segments

The values of derivatives from (8) are plugged into the previous equation such as

$$3P_3 - 2P_4(3 + \beta_1) + P_5(3 + 2\beta_1) = 3Q_3 - 2Q_4(3 + \alpha_2) + Q_5(3 + 2\alpha_2).$$

Therefore,

$$P_3(8 + \alpha_2 + \beta_1) - P_4(4 + \beta_1) = Q_5(4 + \alpha_2).$$

By assuming $\alpha_1 = \alpha_2, \beta_1 = \beta_2$. We will then obtained

$$3P_3(4 + \alpha_2) - 2P_4(24 + 7\beta_1 + \alpha_2(7 + 2\beta_1)) = 3Q_5(4 + \alpha_2).$$

Case 2. Value of shape parameter $\alpha_i$ is not equal to $\beta_i$ for each curve but $\beta_i$ of the first curve equal to $\alpha_{i+1}$ for second curve

Theorem 3: Let $r_1(t)$ and $r_2(t)$ be two quintic trigonometric Bézier curves. The two curves of different pair of shape parameters are joined by $C^2$ continuity if $\beta_1 = \alpha_2$ and they are $C^0, C^1$ and

$$3P_3(4 + \alpha_2) - 2P_4(24 + 7\beta_1 + \alpha_2(7 + 2\beta_1)) = 3Q_5(4 + \alpha_2).$$

where $\alpha_1, \beta_1$ belongs to shape parameters of the first curve and $\alpha_2, \beta_2$ belongs to shape parameters of the second curve.

Proof. Two curves are joined by $C^2$ continuity if

$r_1(1) = r_2(0).$
The values of derivatives from (8) are plugged into the previous equation
\[3P_3 - 2P_4(3 + \beta_1) + P_5(3 + 2\beta_1) = 3Q_3 - 2Q_4(3 + \alpha_2) + Q_5(3 + 2\alpha_2).\]  
(25)

Therefore \(P(8 + \alpha_2 + \beta_1) - P_4(4 + \beta_1) = Q_1(4 + \alpha_2).\)

We obtained
\[3P_3(4 + \alpha_2) - 2(P_4 - P_5)(24 + 7\beta_1 + \alpha_2(7 + 2\beta_1)) = 3Q_3(4 + \alpha_2).\]

Figure 9 shows a composite curve made up by the two curves. Blue curve was generated using the same pair of shape parameter while the red curve was generated using different pairs of shape parameter. The second curve in black is generated using the same pair of shape parameter. With a closer look, both blue and red curves are joined nicely at \(P_5\).

![Composite two segments of Quintic Trigonometric Bézier curves](image)

Blue and red curves provides the same curvature, \(\kappa\) value at the start which is zero. When \(t = 1\), both curves give different numerical values. Curvature, \(\kappa\) value for the black curve are listed in Table 2.

With the information shown in Tables 1 and 2, it can be deduced that the curve preserves curvature continuity from one curve to another if \(\beta_i = \alpha_{i+1}\) for \(\alpha_i\) that is not equal to \(\beta_i\) for each of the curve.

**TABLE 1. Curvature value comparison for blue and red curve**

<table>
<thead>
<tr>
<th>(t)</th>
<th>Curvature, (\kappa) (Blue Curve)</th>
<th>Curvature, (\kappa) (Red Curve)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.1</td>
<td>-0.000014</td>
<td>-0.000014</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.000303</td>
<td>-0.000337</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.001619</td>
<td>-0.002003</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.004168</td>
<td>-0.005720</td>
</tr>
<tr>
<td>0.5</td>
<td>-0.006907</td>
<td>-0.009761</td>
</tr>
<tr>
<td>0.6</td>
<td>-0.008814</td>
<td>-0.010868</td>
</tr>
<tr>
<td>0.7</td>
<td>-0.010028</td>
<td>-0.008787</td>
</tr>
<tr>
<td>0.8</td>
<td>-0.011720</td>
<td>-0.006810</td>
</tr>
<tr>
<td>0.9</td>
<td>-0.014587</td>
<td>-0.009316</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.015975</td>
<td>-0.099843</td>
</tr>
</tbody>
</table>

Blue and red curves provides the same curvature, \(\kappa\) value at the start which is zero. When \(t = 1\), both curves give different numerical values. Curvature, \(\kappa\) value for the black curve are listed in Table 2.

The best curve will always preserve not just parametrically but also curvature continuity. By visual observation, one cannot identify which curve preserves curvature continuity. To verify this, curvature value must be the same at the end of the first curve and the beginning of the second curve.

**CONCLUSION**

In this paper, we introduced quintic trigonometric Bézier curve with two shape parameter. This new type of basis
inherits most of the geometric properties of classical Quintic Bézier curve. Parameterization of the curve can be easily done by the presence of two shape parameters, where the shape of the curve can be easily altered without changing its control point. Two shape parameters give more options for designers or engineers to maintain one-sided shape while changing the other shape of the curve. A higher degree of trigonometric Bézier curve was presented in this paper with composition of two curves that fulfilled $C^2$ Hermite conditions. This work can be extended to shape preserving curves, designing routes or highways (Misro et al. 2015, 2017) where it will benefit from curvature continuity. Furthermore, it can be used to construct surface patches or tensor product surface.

ACKNOWLEDGEMENTS
The authors are very grateful to the anonymous referees for their valuable suggestions. This work was supported by Universiti Sains Malaysia and partially by USM FRGS 304/PMATHS/6711433.

REFERENCES

School of Mathematical Sciences
Universiti Sains Malaysia
11800 Penang, Pulau Pinang
Malaysia

*Corresponding author; email: redorange_91@yahoo.com

Received: 22 January 2016
Accepted: 1 November 2016

TABLE 2. Curvature value for second curve

<table>
<thead>
<tr>
<th>$t$</th>
<th>Curvature, $\kappa$ (Black Curve)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.015975</td>
</tr>
<tr>
<td>0.1</td>
<td>-0.016542</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.009397</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.002886</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.000369</td>
</tr>
<tr>
<td>0.5</td>
<td>0.000737</td>
</tr>
<tr>
<td>0.6</td>
<td>0.001956</td>
</tr>
<tr>
<td>0.7</td>
<td>0.003983</td>
</tr>
<tr>
<td>0.8</td>
<td>0.005692</td>
</tr>
<tr>
<td>0.9</td>
<td>0.004781</td>
</tr>
<tr>
<td>1.0</td>
<td>0.003854</td>
</tr>
</tbody>
</table>