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Approximate-Analytic Solution of Hyperchaotic Finance System by Multistage Approach

(Penyelesaian Analitik Anggaran Sistem Kewangan Hiperkalut Menggunakan Pendekatan Multitahap)

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ABSTRACT

This paper devotes to constructing an approximate analytic solution for the hyperchaotic finance model. The model describes the time variation of the interest rate, the investment demand, the price exponent, and the average profit margin. The multistage homotopy analysis method (MHAM) and multistage variational iteration method (MVIM) are utilized to generate the analytical solutions. The solutions are presented in terms of continuous piecewise functions without interpolation. These procedures prove their applicability for this kind of model due to rapidly convergent series solutions with easily computable terms, iterates, and efficiently obtained by applying it over multiple time intervals. We also provide the convergences theorem of the MHAM. Numerical comparisons are displayed with the results obtained by MHAM, MVIM, and the fourth-order Runge-Kutta method to demonstrate the validity and effectivity of this procedure.

Keywords: Finance system; hyperchaotic system; multistage homotopy analysis method; multistage variational iteration method

ABSTRAK

Kertas ini membincangkan pembinaan penyelesaian analitik anggaran bagi model kewangan hiperkalut. Model ini melibatkan variasi masa kadar faedah, permintaan pelaburan, eksponen harga dan margin untung purata. Kaedah analisis homotopi multitahap (KAHM) dan kaedah lelaran variasi multitahap (KLVM) dibina dan digunakan untuk mendapatkan penyelesaian yang berbentuk fungsi cebisan selanjar tanpa interpolasi. Kemampuan kedua-dua kaedah ini terbukti berhasil untuk model seperti ini kerana penyelesaian sirinya didapati cepat menumpu dan sebutan dalam penyelesaiannya mudah dihitung di dalam lelaran. Ketepatan penyelesaian dapat diperoleh melalui penerapannya dalam beberapa selang masa berganda. Kertas ini turut memperuntukkan teorem penumpuan KAHM. Perbandingan berangka bagi keputusan yang diperoleh daripada KAHM, KLVM dan kaedah Runge-Kutta peringkat keempat dipaparkan untuk menunjukkan kesahihan dan keberkesanan kedua-dua kaedah baharu ini.

Kata kunci: Kaedah analisis homotopi multitahap (KAHM); kaedah lelaran variasi multitahap (KLVM); sistem berkekalutan hiper; sistem kewangan

Introduction

There was economic chaos in 1985, which had a huge permanent impact on the current Western economy leading to disorder in the current economic system (Grandmont 1985). It should be remarked that the chaotic economic behavior entails that the financial system in question is inherently indefinite. Another paradigm chaotic finance system (Ma & Chen 2001) was made public in 2001. The system depicts the differences in time of three state variables: Interest rate, x, investment demand γ, and price index z. The variables that influence changes in x is contradiction of the investment market, the difference between savings and investment, and the price reorganization. The rate of change y is relative to the level of investment, and it is proportional to the inversion with investment costs and rates of interest. Changes in z, on the one hand, are controlled by a contradiction between supply and demand on the commercial market and influenced by inflation rates. As mentioned in Yu et al. (2012), in 2007, the US subprime mortgage crisis triggered a global economic downturn, and the financial system once again showed a butterfly effect and a chaotic effect. The system is investigated in this backdrop. Because this global economic crisis has the potential to trigger severe depression, this chaotic financial system reflects the economic phenomenon. Hence, the factual background and source of the chaotic financial system become hyperchaotic finance system. The hyperchaotic finance model adopted from a model stated in Rossler (1979) followed the four-variable oscillator. It contains but one nonlinear term of quadratic type (xz) on its right-hand side. Wu and Chen (2010) built a hyperchaotic financial system that described the time variations of four state variables: the interest rate x, the investment demand y, the price exponent z, and the average profit margin w. It is well known that these factors affecting the interest rates are related to investment demand and price index and the average profit margin and the average profit margin as well interest rate. In this work, the four-dimensional hyperchaotic finance system investigating is expressed as follows (Cao 2018)

$$\dot{x} = z + (y - \alpha)x + w, \tag{1}$$

$$\dot{y} = 1 - \beta y - x^2, \tag{2}$$

$$\dot{z} = -x - \gamma w, \tag{3}$$

$$\dot{w} = -\mu xy + \varphi w, \tag{4}$$

where $\dot{v} = \text{dv / dt}$; $\alpha \geq 0$ represents a saving amount; $\beta \geq 0$ denotes investment cost per year, and $\gamma \geq 0$ represents the elasticity of commercial market demand, and the end μ, φ are new positive constant parameters (Kocamaz et al. 2018). We fix $\alpha = 0.9$, $\beta = 0.2$, $\gamma = 1.5$, $\mu = 0.2$ and select φ as the governing parameter according to the two criteria in Cao (2018). In the case of $\alpha = 0.9$, $\beta = 0.2$, $\gamma = 1.5$ and $\varphi = 0.17$, the Lyapunov exponents that determined by Wolf procedure (Ma & Chen 2001), are 0.034432, 0.018041,0 and -1.1499.

Research on the hyperchaotic finance system is in the attention of researchers. Jahanshahi et al. (2019) published a dynamic and entropy analysis of a hyperchaotic financial system with coexisting attractors. Xu et al. (2020) analyzed a controlled hyperchaotic finance system with energy-bounded disturbance under the delayed feedback controller. Chen et al. (2021) designed a series of controls to realize global asymptotic synchronization and controllers to realize global exponential synchronization to synchronize the hyperchaotic finance system fully.

Finding an efficient method for hyperchaotic systems has been undertaking active researchers as Alcin (2020), Emiroglu et al. (2021), He et al. (2019), and Rangkuti and Alomari (2021). Since exact solutions to the Hyperchaotic system are challenging, analytical and numerical methods must be applied. For example, RK4 is one of the most used methods for those kinds the systems (Alcin 2020; Emiroglu et al. 2021; Rangkuti & Alomari 2021). Numerical-analytical methods with a multistage approach become an effective way to solve nonlinear problems such as the multistage optimal homotopy asymptotic method (MOHAM) (Shah et al. 2020), the multistage differential transformation method (MDTM) (Aljahdaly et al. 2021), and the multistage successive approximation method (MSAM) (Prabowo & Mungkasi 2021).

Now, it is essential to find the powerful and precise method for solving hyperchaotic systems. Alomari et al. (2009) suggested homotopy analysis method with a multistage approach. The technique has several interesting features that are the auxiliary convergent parameter $\hbar$ which can enlarge the radius of convergence of the series and the solution from $[t_0, t)$ will be derived by subdividing this interval into $[t_0, t_1)$, $[t_1, t_2)$, ..., $[t_{n-1}, t]$ and applying the HAM solution on each subinterval. The initial approximation in each interval is taken from the solution in the previous interval. It is the advantage of using MHAM. This method is very accurate for long intervals and past in calculation. And it was proved by researchers investigate the method in recent years, such as Dinesha et al. (2018) and Rangkuti and Alomari (2021).

In this work, we present algorithms based on MHAM and MVIM to find approximate analytic solutions for hyperchaotic finance system. By selecting proper auxiliary linear operators and auxiliary convergent parameters, we can derive accurate solutions using MHAM. The generated analytic solution is compared with RK4 solution and demonstrates the solution's efficiency.

Another way to prove the powerful MHAM, we also compare the MHAM solution and Multistage variational iteration method (MVIM). The MVIM constructed a correction functional with a general Lagrangian multiplier running with a multistage technique. The MVIM successfully solved various applications (Goh et al. 2009). This paper is the first work that generates the approximate analytic solution for the hyperchaotic finance system to the best of our knowledge.

MATERIALS AND METHODS

In this section, we introduce the MHAM and MVIM for the hyperchaotic finance system.

MHAM METHOD

In this part, we construct the MHAM for the hyperchaotic finance system using the base function $\{t^n \mid n \geq 0\}$. So, it is convenient to pick out the following initial guesses of the solution

$$x_0(t) = c_1, \quad y_0(t) = c_2, \quad z_0(t) = c_3, \quad w_0(t) = c_4. \quad (5)$$

The based function leads to the linear operator

$$L[\phi(t;q)] = \frac{\partial \phi(t;q)}{\partial t}, \quad (6)$$

the initial condition can obtain the constant A for the equation $L[A] = 0$. Defining the embedded parameter $q \in [0,1]$, auxiliary parameters $\hbar \leq 0$, and the zeroth-order deformation problems

$$(1-q)L[\hat{x}(t;q) - x_0(t)] = q\hbar\, N_x[\hat{y}(t;q), \hat{z}(t;q), \widehat{w}(t;q)], \quad (7)$$

$$(1-q)L[\hat{y}(t;q) - y_0(t)] = q\hbar\, N_y[\hat{x}(t;q), \hat{y}(t;q)], \quad (8)$$

$$(1-q)L[\hat{z}(t;q) - z_0(t)] = q\hbar\, N_z[\hat{z}(t;q), \widehat{w}(t;q)], \quad (9)$$

$$(1-q)L[\widehat{w}(t;q) - w_0(t)] = q\hbar\, N_w[\hat{x}(t;q), \hat{y}(t;q), \widehat{w}(t;q)], \quad (10)$$

with the conditions

$$\hat{x}(t^*;q) = c_1, \quad \hat{y}(t^*;q) = c_2, \quad \hat{z}(t^*;q) = c_3, \quad \widehat{w}(t^*;q) = c_4 \quad (11)$$

and the non-linear operators labeled as N_x, N_y, N_z and N_w as

$$N_x[\hat{y}(t;q), \hat{z}(t;q), \widehat{w}(t;q)] = \frac{\partial \hat{x}(t;q)}{\partial t} - \hat{z}(t;q) - (\hat{y}(t;q) - \alpha)\hat{x}(t;q) - \widehat{w}(t;q), \quad (12)$$

$$N_y[\hat{x}(t;q), \hat{y}(t;q)] = \frac{\partial \hat{y}(t;q)}{\partial t} - 1 + \beta \hat{y}(t;q) + \hat{x}^2(t;q), \quad (13)$$

$$N_z[\hat{x}(t;q), \hat{z}(t;q)] = \frac{\partial \hat{z}(t;q)}{\partial t} + \hat{x}(t;q) + \gamma \hat{z}(t;q), \quad (14)$$

$$N_w[\hat{x}(t;q), \hat{y}(t;q), \widehat{w}(t;q)] = \frac{\partial \widehat{w}(t;q)}{\partial t} + \mu \hat{x}(t;q)\hat{y}(t;q) + \varphi \widehat{w}(t;q), \quad (15)$$

For $q = 0$, the Equations (7)-(10) have the solutions

$$\hat{x}(t;0) = x_0(t), \quad \hat{y}(t;0) = y_0(t), \quad \hat{z}(t;0) = z_0(t), \quad \widehat{w}(t;0) = w_0(t), \quad (16)$$

and for $q = 1$ we have

$$\hat{x}(t;1) = x(t), \quad \hat{y}(t;1) = y(t), \quad \hat{z}(t;1) = z(t), \quad \widehat{w}(t;1) = w(t). \quad (17)$$

When q escalates from 0 to 1, then $\hat{x}(t;q), \hat{y}(t;q), \hat{z}(t;q)$ and $\widehat{w}(t;q)$ vary from the initial conditions to the exact solutions. The Taylor expansion of $\hat{x}, \hat{y}, \hat{z}$ and $\widehat{w}$ with respect to q gives

$$\hat{x}(t;q) = x_0 + \sum_{m=0}^{\infty} x_m q^m, \quad (18)$$

$$\hat{y}(t;q) = y_0 + \sum_{m=0}^{\infty} y_m q^m, \quad (19)$$

$$\hat{z}(t;q) = z_0 + \sum_{m=0}^{\infty} z_m q^m, \quad (20)$$

$$\widehat{w}(t;q) = w_0 + \sum_{m=0}^{\infty} w_m q^m, \quad (21)$$

in which $x_m(t) = \frac{1}{m!}\frac{\partial^m \hat{x}(t;q)}{\partial q^m}$, $y_m(t) = \frac{1}{m!}\frac{\partial^m \hat{y}(t;q)}{\partial q^m}$, $z_m(t) = \frac{1}{m!}\frac{\partial^m \hat{z}(t;q)}{\partial q^m}$, $w_m(t) = \frac{1}{m!}\frac{\partial^m \widehat{w}(t;q)}{\partial q^m}$. Hence, employing equations (18)-(21) we obtain

$$x(t) = x_0 + \sum_{m=1}^{\infty} x_m(t), \quad (22)$$

$$y(t) = y_0 + \sum_{m=1}^{\infty} y_m(t), \quad (23)$$

$$z(t) = z_0 + \sum_{m=1}^{\infty} z_m(t), \quad (24)$$

$$w(t) = w_0 + \sum_{m=1}^{\infty} w_m(t). \quad (25)$$

Apply the operator

$$\frac{1}{m!}\frac{\partial^m[.]}{\partial q^m}\bigg|_{q-0}$$

on to the equations (7)-(10), then we have the m-th order deformations

$$L[x_m(t) - \chi_m x_{m-1}(t)] = \hbar \Re_m^x(t), \quad (26)$$

$$L[y_m(t) - \chi_m y_{m-1}(t)] = \hbar \Re_m^y(t), \quad (27)$$

$$L[z_m(t) - \chi_m z_{m-1}(t)] = \hbar \Re_m^z(t), \quad (28)$$

$$L[w_m(t) - \chi_m w_{m-1}(t)] = \hbar \Re_m^w(t), \quad (29)$$

and

$$x_m(t^*) = 0, \quad y_m(t^*) = 0, \quad z_m(t^*) = 0, \quad w_m(t^*) = 0 \quad (30)$$

are the initial conditions. The $\Re_m$ given by

$$\Re_m^x(t) = \dot{x}_{m-1} - z_{m-1} - \sum_{i=0}^{m-1} y_i x_{m-1-i} + \alpha x_{m-1}, \quad (31)$$

$$\Re_m^y(t) = \dot{y}_{m-1} - (1-\chi_m) + \beta y_{m-1} + \sum_{i=0}^{m-1} x_i x_{m-1-i}, \quad (32)$$

$$\Re_m^z(t) = \dot{z}_{m-1} + x_{m-1} + \gamma z_{m-1}, \quad (33)$$

$$\Re_m^w(t) = \dot{w}_{m-1} + \mu \sum_{i=0}^{m-1} x_i y_{m-1-i} + \varphi w_{m-1}. \quad (34)$$

The linear non-homogeneous equations (26)-(29) can easily solve subject to the conditions (30). And we have

$$x_1(t) = c_1 + 0.9\, c_1\hbar t - c_1c_2\hbar t - c_3\hbar t - c_4\hbar t,$$

$$y_1(t) = c_2 - \hbar t + {c_1}^2\hbar\, t + 0.2\, c_2\hbar\, t,$$

$$z_1(t) = c_3 + c_1\hbar t + 1.5c_3\hbar t,$$

$$w_1(t) = c_4 + 0.2c_1c_2\hbar t + c_4\hbar t\varphi,$$

$$\begin{aligned} x_2(t) = {} & c_1 + 1.8c_1\hbar t - 2c_1c_2\hbar t - 2c_3\hbar t - 2c_4\hbar t \\ & + 0.9c_1\hbar^2 t - c_1c_2\hbar^2 t \\ & - c_3\hbar^2 t - c_4\hbar^2 t + 0.405\, c_1\hbar^2t^2 - 0.5c_1^3\hbar^2t^2 \\ & - 1.1\, c_1c_2\hbar^2t^2 \\ & + 0.5c_1c_2^2\hbar^2t^2 - 1.2c_3\hbar^2t^2 + 0.5c_2c_3\hbar^2t^2 \\ & - 0.45c_4\hbar^2t^2 \\ & + 0.5c_2c_4\hbar^2t^2 - 0.5c_4\hbar^2t^2\varphi, \end{aligned}$$

$$\begin{aligned} y_2(t) = {} & c_2 - \hbar t + c_1^2\hbar t + 0.2c_2\hbar t \\ & - \hbar(t - c_1^2 t - 0.2c_2 t + \hbar t - c_1^2\hbar t - 0.2c_2\hbar t + 0.1\hbar t^2 \\ & - c_1^2\hbar t^2 - 0.02c_2\hbar t^2 + c_1^2c_2\hbar t^2 + c_1c_3\hbar t^2 + c_1c_4\hbar t^2), \end{aligned}$$

$$\begin{aligned} z_2(t) = {} & c_3 + 2c_1\hbar t + 3c_3\hbar t + c_1\hbar^2 t + 1.5c_3\hbar^2 t + \\ & + 1.2c_1\hbar^2t^2 \\ & - 0.5c_1c_2\hbar^2t^2 + 0.625c_3\hbar^2t^2 - 0.5c_4\hbar^2t^2, \end{aligned}$$

$$\begin{aligned} w_2(t) = {} & c_4 + 0.2c_1c_2\hbar t + c_4\hbar t\varphi \\ & + 0.1\hbar(2c_1c_2t + 2c_1c_2\hbar t - c_1\hbar t^2 + c_1^3\hbar t^2 + 1.1c_1c_2\hbar t^2 \\ & + 10c_4\varphi + 10c_4\hbar t^2\varphi^2). \end{aligned}$$

Now the n-th order of the HAM solution given by $X = \sum_{i=0}^{n} x_i(t), Y = \sum_{i=0}^{n} y_i(t), Z = \sum_{i=0}^{n} z_i(t), W = \sum_{i=0}^{n} w_i(t)$. The HAM solution can be generated if $t^*= 0$. If the range of t is large, then HAM solution is not accurate enough. For that, we apply the MHAM. To do that, we select the range of $t \in [0,100]$, then we divide the interval into the subintervals $[0, t_1)$, ..., $[t_{i-1}, t_i)$, ..., $[t_{N-1}, 100]$, where $t_i - t_{i-1} = \Delta t = h$ is the step size, and we calculate the solution at each subinterval. So, t^* will go from $t_0 = 0$ to $t_{N-1} = 100$-h by step size Δt, the new initial conditions for each step interval can be determined by the continuity of the solution (i.e. the solution on the interval $[t_{i-1}, t_i)$ has initial value $x(t_{i-1}) = c_1^*$. Therefore, the error will be reduced, and the choice of the new initial conditions will influence the consistency of the solution. Consequently, it is obtaining a continuous solution for the hyperchaotic finance system.

MVIM METHOD

In this subsection, we introduce the solution by MVIM. For that, we build the following correction functional to the hyperchaotic Finance model:

$$x_{i+1}(t) = x_i + \int_{t^*}^{t} \lambda_1(s)\big[\dot{x}_i(s) - \tilde{z}_i(s) - \big(\tilde{y}_i(s)\tilde{z}_i(s) - \alpha x_i(s) - \tilde{w}_i(s)\big)\big]\, ds, \quad (35)$$

$$y_{i+1}(t) = y_i + \int_{t^*}^{t} \lambda_2(s)[\dot{y}_i(s) - 1 + \beta\tilde{y}_i(s) + \tilde{x}_i^2(s)]\, ds, \quad (36)$$

$$z_{i+1}(t) = z_i + \int_{t^*}^{t} \lambda_3(s)[\dot{z}_i(s) + \tilde{x}_i(s) + \gamma\tilde{z}_i(s)]\, ds \quad (37)$$

$$w_{i+1}(t) = w_i + \int_{t^*}^{t} \lambda_4(s)[\dot{w}_i(s) - \mu\tilde{x}_i(s)\tilde{y}_i(s) + \varphi w_i(s)]\, ds, \quad (38)$$

where $\lambda_i(t)$, i = 1,2,3,4 are the general Lagrange multipliers, and x_i, y_i, z_i and w_i. The restricted variations give $\delta\tilde{x}_i(0) = \delta\tilde{y}_i(0) = \delta\tilde{z}_i(0) = \delta\tilde{w}_i(0)$ (Inokuti et al. 1978). Taking the variation with respect to the independent variables x_i, y_i, z_i and w_i, we have

$$\delta x_{i+1}(t) = \delta x_i + \delta\int_{t^*}^{t} \lambda_1(s)\big[\dot{x}_i(s) - \tilde{z}_i(s) - \big(\tilde{y}_i(s)\tilde{z}_i(s) - \alpha x_i(s) - \tilde{w}_i(s)\big)\big]\, ds, \quad (39)$$

$$\delta y_{i+1}(t) = \delta y_i + \delta\int_{t^*}^{t} \lambda_2(s)[\dot{y}_i(s) - 1 + \beta\tilde{y}_i(s) + \tilde{x}_i^2(s)]\, ds, \quad (40)$$

$$\delta z_{i+1}(t) = \delta z_i + \delta\int_{t^*}^{t} \lambda_3(s)[\dot{z}_i(s) + \tilde{x}_i(s) + \gamma\tilde{z}_i(s)]\, ds, \quad (41)$$

$$\delta w_{i+1}(t) = \delta w_i + \delta\int_{t^*}^{t} \lambda_4(s)[\dot{w}_i(s) - \mu\tilde{x}_i(s)\tilde{y}_i(s) + \varphi w_i(s)]\, ds. \quad (42)$$

Making each of the above correction functional (39)-(42) stationary, and $\delta x_i(t^*) = 0$, $\delta y_i(t^*) = 0$, $\delta z_i(t^*) = 0$ and $\delta w_i(t^*) = 0$, then, we obtain the four sets of stationary conditions for $\lambda_i(t)$, i = 1,2,3,4. At this point, the Lagrange multipliers can be identified as $\lambda_1(s) = e^{\alpha(s-t)}$, $\lambda_2(s) = e^{\beta(s-t)}$, $\lambda_3(s) = e^{\gamma(s-t)}$ and $\lambda_4(s) = e^{\wedge\mu(s-t)}$. Thus, the iteration formulas are

$$x_{i+1}(t) = x_i + \int_{t^*}^{t} e^{\alpha(s-t)}\big[\dot{x}_i(s) - \tilde{z}_i(s) - \big(\tilde{y}_i(s)\tilde{z}_i(s) - \alpha x_i(s) - \tilde{w}_i(s)\big)\big]\, ds, \quad (43)$$

$$y_{i+1}(t) = y_i + \int_{t^*}^{t} e^{\beta(s-t)}[\dot{y}_i(s) - 1 + \beta\tilde{y}_i(s) + \tilde{x}_i^2(s)]\, ds, \quad (44)$$

$$z_{i+1}(t) = z_i + \int_{t^*}^{t} e^{\gamma(s-t)}[\dot{z}_i(s) + \tilde{x}_i(s) + \gamma\tilde{z}_i(s)]\, ds \quad (45)$$

$$w_{i+1}(t) = w_i + \int_{t^*}^{t} e^{\mu(s-t)}[\dot{w}_i(s) - \mu\tilde{x}_i(s)\tilde{y}_i(s) + \varphi w_i(s)]\, ds, \quad (46)$$

For i = 0, we have

$$x_1(t) = c_1 + \frac{\big(e^{\alpha(t-t^*)} - 1\big)[c_1(c_2 + \alpha) + c_3 + c_4]}{\alpha}, \quad (47)$$

$$y_1(t) = c_2 - \frac{(c_1^2 + \beta c_2 - 1)\big(e^{\beta(t-t^*)} - 1\big)}{\beta}, \quad (48)$$

$$z_1(t) = c_3 - \frac{(c_1 + c_3\delta)\big(e^{\delta(t-t^*)} - 1\big)}{\delta}, \quad (49)$$

$$w_1(t) = c_4 - \frac{\big(e^{\varphi(t-t^*)} - 1\big)(c_1c_2\theta + c_4\varphi)}{\varphi}. \quad (50)$$

In the same way, we can generate the n-th order of the approximation, which is defined as $X = x_n(t)$, $Y = y_n(t)$, $Z = z_n(t)$, $W = w_n(t)$. Follow the same strategies in the above section for dividing the interval and approximate the initial conditions from the previous interval. Then we have the multistage VIM.

ERROR ANALYSIS

Since the exact solution of the finance system is unknown yet, we present the following residual error

$$RE_x = \dot{X} - Z - (Y - \alpha)X - W, \quad (51)$$

$$RE_y = \dot{Y} \mp \beta Y + X^2, \quad (52)$$

$$RE_z = \dot{Z} + X + \gamma W, \quad (53)$$

$$RE_w = \dot{W} + \mu XY + \varphi W, \quad (54)$$

where X, Y, Z, W and $\dot{X}, \dot{Y}, \dot{Z}, \dot{W}$ are the MHAM or MVIM solution and its derivatives with respect to t of equations (1)--(4). We remarked that the magnitude of the MHAM and MVIM solution errors depends on the approximation order and the duration of the subintervals. Since the solution is analytical at each step, the residual error can be easily obtained at each time stage. We will find the residual error of every step of the time by applying MHAM and MVIM.

CONVERGENCE

In this section, we introduce the following convergence theorem of the proposed algorithm.

Theorem 1 Let the function $\omega(t) \in C^{n+1}$ where $\omega(t):[c,d] \to \mathbb{R}$ with $W = span\{1, (t-t_0), (t-t_0)^2, \cdots, (t-t_0)^n\}$. If the convergent series $\tilde{\omega}_n(t) = \sum_{i=0}^{n} c_i (t - t_0)^i$ is the best approximation of $\omega(t)$ out of W with a radius of convergent at least 1, then we have the following mean error bound

$$\|\omega(t) - \tilde{\omega}_n\|_2 \le \frac{\alpha}{(n+1)!}\sqrt{\frac{h^{2n+3}}{2n+3}}, \quad (55)$$

where $\alpha = \max_{\tau \in (c,d)} \left|\omega^{(n+1)}(\tau)\right|$, *and* $h = (d - c) < 1$.

Proof 1 The n-th Taylor polynomial of $\omega(t)$ about $t_0 \in [c, d]$ is

$$\tilde{\omega}_n(t) = \sum_{i=0}^{n} \frac{(t - t_0)^i}{i!} \omega^{(i)}(t_0),$$

and its error bound given by

$$\left|\omega(t) - \sum_{i=0}^{n} \frac{(t - t_0)^i}{i!} \omega^{(i)}(t_0)\right| = \left|\omega^{(n+1)}(\tau) \frac{(t - t_0)^{n+1}}{(n+1)!}\right|,$$

for some $\tau \in (c, d)$.

Since $\tilde{\omega}_n(t) = \sum_{i=0}^{n} \omega_i(t)$, is the best approximation of $\omega(t)$ so we get

$$\|\omega(t) - \tilde{\omega}_n(t)\|_2^2 = \int_c^d \left(\omega^{(n+1)}(\tau) \frac{(t - t_0)^{n+1}}{(n+1)!}\right)^2 dt,$$

$$\le \frac{\alpha^2}{[(n+1)!]^2} \int_c^d (t - t_0)^{2n+2}\, dt,$$

$$= \frac{\alpha^2}{[(n+1)!]^2 (2n+3)} [(d - t_0)^{2n+3} - (c - t_0)^{2n+3}],$$

where $\alpha = \max_{\tau \in (c,d)} \left|\omega^{(n+1)}(\tau)\right|$. For $t_0 = c$, we have

$$\|\omega(t) - \tilde{\omega}_n(t)\|_2^2 \le \frac{\alpha^2 (d - c)^{2n+3}}{[(n+1)!]^2 (2n+3)},$$

$$\|\omega(t) - \tilde{\omega}_n(t)\|_2 \le \frac{\alpha}{(n+1)!}\sqrt{\frac{h^{2n+3}}{2n+3}},$$

where $h = d - c$. The above norm depends on the number of terms n and the step size $h \in (0, 1)$, if $n \to \infty$ we get,

$$\lim_{n \to \infty} \|\omega(t) - \tilde{\omega}_n(t)\|_2 = 0. \quad (56)$$

Similarly, this theorem is valued for for $x(t)$, $y(t)$, $z(t)$, and $w(t)$.

RESULTS AND DISCUSSION

We compute the MHAM and the MVIM solution for the hyperchaotic finance systems and compared the results by RK4, which is built-in Mathematica package. For that, we fix the values of the parameters $\alpha = 0.9$, $\beta = 0.2$, $\gamma = 1.5$, $\mu = 0.2$ and vary φ as $\varphi = 0.17, 0.5, 1.0, 2.0$. We also fixed the initial conditions $x(0) = 5$, $y(0) = 2$, $z(0) = -6$ and $w(0) = 4$, which were taken by Cao (2018). In this work, the range of t is chosen for $t \in [0,100]$. The calculations are made with the 3-terms MHAM and the 3rd iteration MVIM solutions.

MHAM is distinguished from other semi-analytical approaches by a nonzero auxiliary parameter $\hbar$, which allows the development of a family of solutions. As a result, depending on $\hbar$, the convergence region and the rate of solution series can be adjusted. By graphing the $\hbar$ -curves, which are horizontally parallel to the t axis, one can quickly determine the appropriate value of $\hbar$ to use. We decided to use 7-terms in the MHAM series solutions. The convergent region of $\hbar$ is presented by the $\hbar$ -curves in Figure 1,

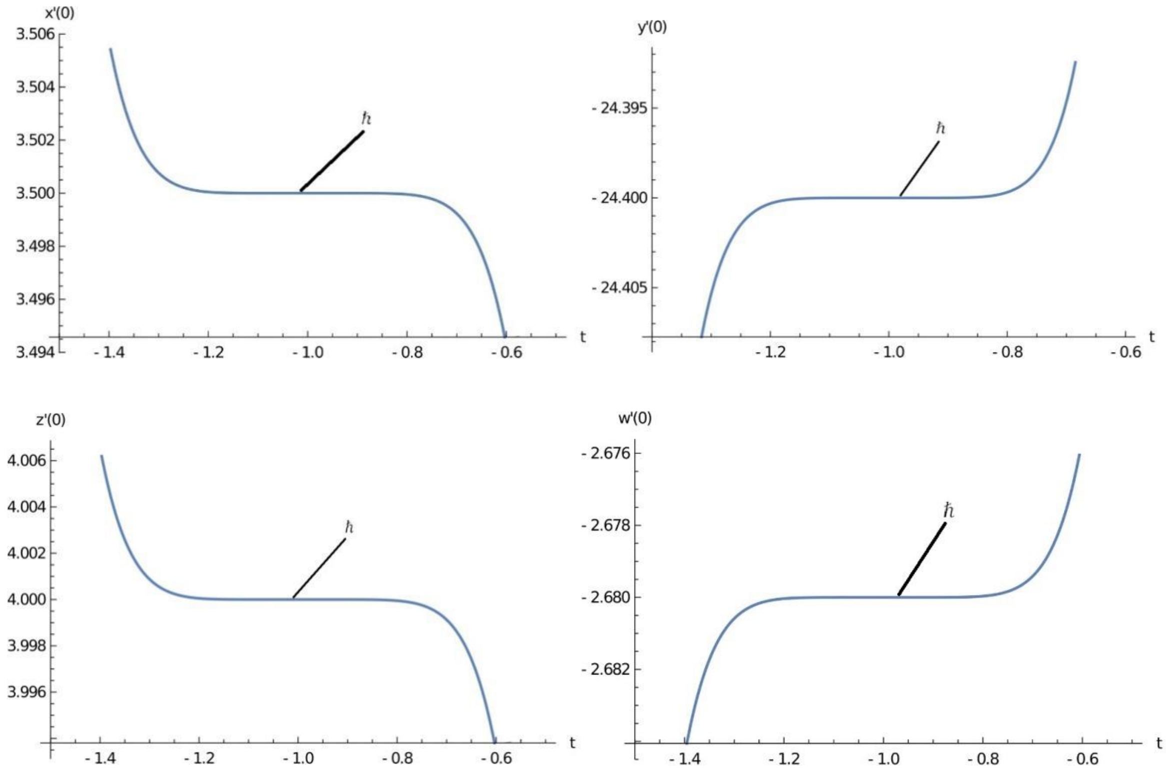

FIGURE 1. The $\hbar$-curves for 7-terms MSHAM for $\varphi = 0.17$

From Figure 1, the curves parallel to t axis in $\hbar \in$(-1.2,-0.8). For this work, we choose $\hbar = -1$.

Next, Figure 2 shows the phase portraits of the hyperchaotic finance system with different variables and given parameters. Here, $\alpha = 0.9$, $\beta = 0.2$, $\gamma = 1.5$, $\mu = 0.2$ and $\varphi = 0.17$.

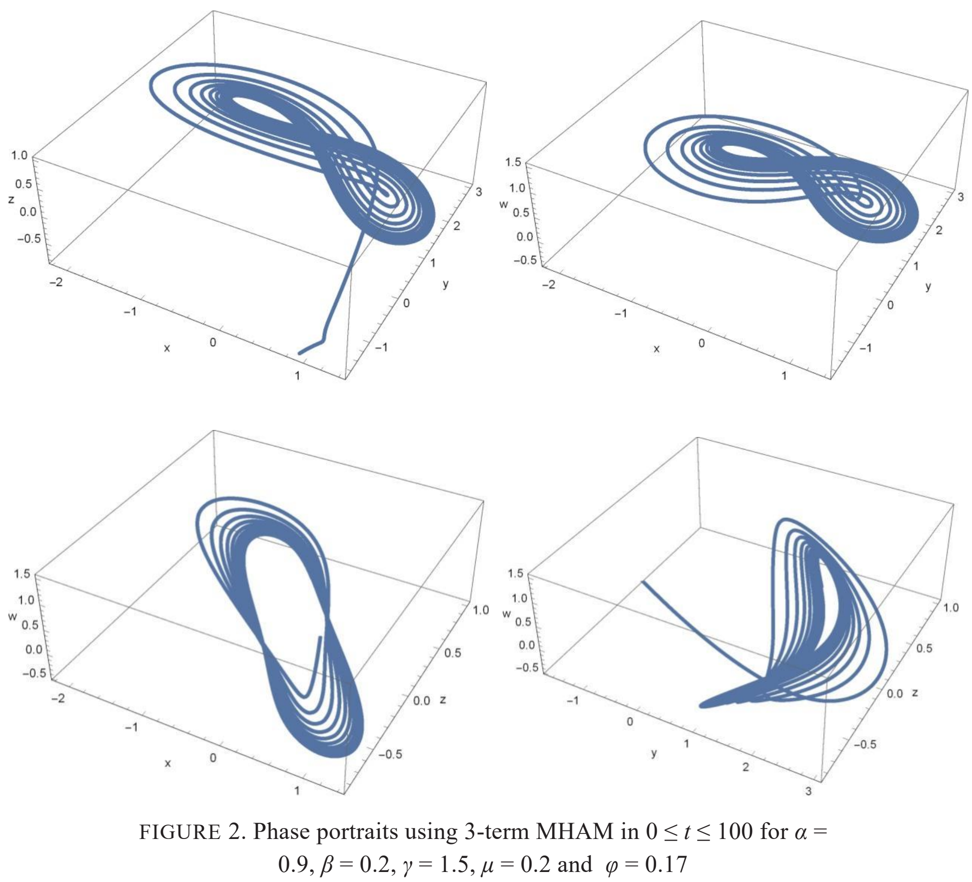

FIGURE 2. Phase portraits using 3-term MHAM in $0 \leq t \leq 100$ for $\alpha = 0.9$, $\beta = 0.2$, $\gamma = 1.5$, $\mu = 0.2$ and $\varphi = 0.17$

Figure 2 shows the strange attractor trajectory of the hyperchaotic finance system in (x, y, z)-space, (x, y, w)-space, (y, z, w)-space, and (x, w, z)-space, respectively. It is clear that the attractor is chaotic. Next, Figure 3 presents a residual error of the average profit margin $w(t)$ for various values of φ along the period $0 \leq t \leq 15$.

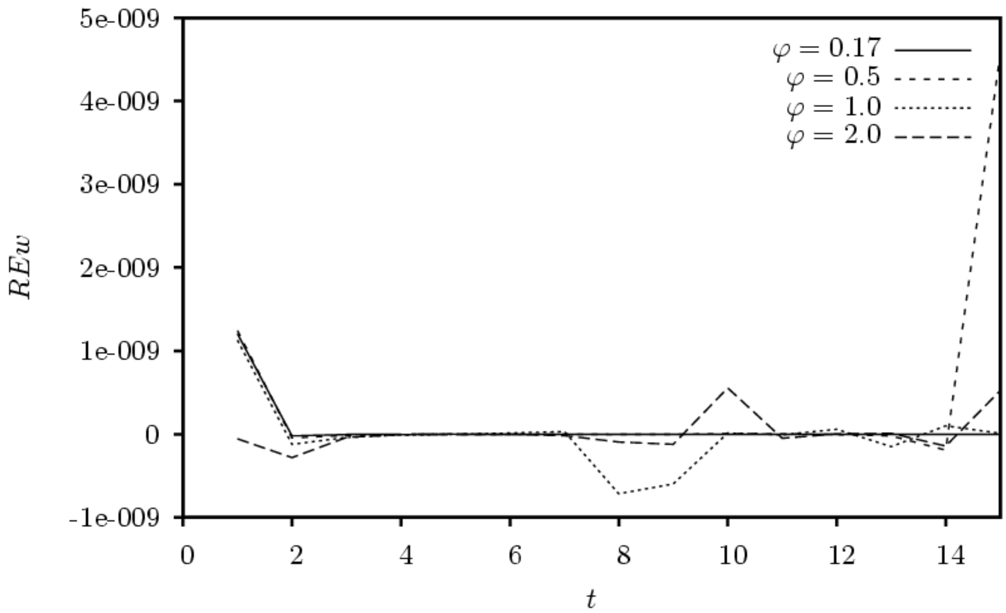

FIGURE 3. Residual error of the average profit margin using 3-terms of MHAM for various values of φ, i.e., φ = 0.17, 0.5, 1.0 and φ = 2.0

From Figure 3, by decreasing the value of φ, the residual error of average profit margin also reaches zero, which means MHAM is accurate to solve hyperchaotic finance system. Furthermore, the accuracy of MHAM can also be seen in Table 1. Table 1 presents the residual error of x, y, z and w using 3-terms MHAM and 3rd iteration MVIM. From Table 1, the residual error of MHAM is smaller than the residual error of MVIM. It means that only using 3-terms of MHAM; the good approximate solutions have been obtained.

TABLE 1. Residual error of 3-terms MHAM and 3rd iteration MVIM

	MHAM				MVIM			
t	RE_x	RE_y	RE_z	RE_w	RE_x	RE_y	RE_z	RE_w
5	-1.514E-12	-5.022E-12	-3.192E-12	-1.821E-12	-1.057E-07	3.056E-08	-3.171E-08	2.908E-07
10	2.798E-12	1.091E-12	9.215E-12	-3.164E-15	-1.278E-07	1.296E-08	-2.521E-08	2.926E-07
15	-7.674E-12	3.025E-12	-1.871E-12	2.603E-12	-1.210E-07	1.614E-08	-3.670E-07	2.787E-07
20	5.820E-11	3.839E-12	-1.628E-11	-2.036E-11	-8.331E-08	5.002E-09	-1.071E-07	2.022E-07
25	1.045E-10	-2.683E-11	-2.416E-11	-1.262E-11	-4.317E-07	-3.265E-07	9.333E-07	4.663E-08
30	-5.969E-09	2.908E-10	8.346E-10	1.322E-09	-1.582E-06	-1.025E-06	-3.218E-07	3.113E-07
35	-7.054E-11	4.825E-10	1.611E-10	5.715E-11	-6.281E-07	-8.088E-09	4.371E-07	3.933E-07
40	1.634E-10	-3.371E-10	2.514E-10	-6.569E-12	-8.045E-07	-8.461E-08	8.962E-07	-1.731E-07
45	-1.091E-09	7.401E-10	1.028E-10	2.041E-10	-2.137E-07	-3.549E-07	-5.746E-07	9.3E-08
50	-2.219E-10	1.271E-09	5.444E-10	1.012E-10	-1.380E-06	-4.015E-07	8.446E-07	5.384E-07

The other way to show the accuracy of MHAM, we compare the approximate solution using MHAM and MVIM with the numerical solution using the fourth-order Runge Kutta (RK4) with a step length of $h = 0.001$. Table 2 presents the absolute error between 3-terms MHAM, 3rd iteration MVIM and RK4 for $\varphi = 0.17$ between MHAM and RK4 methods.

TABLE 2. The absolute error of 3-terms of MHAM and RK4, and 3rd iteration MVIM and RK4 for $\varphi = 0.17$

	MHAM				MVIM			
t	Δx	Δy	Δz	Δw	Δx	Δy	Δz	Δw
5	1.099E-09	5.729E-09	3.551E-09	2.077E-09	1.270E-07	5.128E-07	2.582E-07	7.373E-08
10	2.482E-09	2.732E-09	1.440E-11	1.615E-09	6.009E-08	1.222E-07	4.004E-08	2.751E-07
15	1.148E-08	1.684E-08	7.672E-08	2.332E-08	1.420E-07	5.937E-07	1.0563E-07	5.831E-07
20	3.559E-08	8.708E-08	1.328E-09	1.288E-07	8.670E-07	1.432E-06	4.413E-07	3.013E-07
25	1.446E-07	4.328E-07	9.801E-08	1.608E-07	2.372E-06	3.588E-07	2.129E-06	1.414E-07
30	1.495E-06	7.196E-07	5.776E-07	3.659E-07	1.843E-06	9.935E-07	9.461E-07	1.199E-06
35	1.395E-07	6.768E-08	3.556E-08	2.496E-07	2.114E-06	9.458E-07	3.219E-07	8.087E-07
40	5.149E-07	8.084E-07	5.073E-07	1.622E-07	1.086E-06	1.399E-06	3.561E-07	1.6E-06
45	1.302E-06	2.960E-07	5.597E-07	2.508E-07	1.299E-06	1.065E-06	1.216E-07	1.317E-06
50	6.544E-08	2.478E-07	1.888E-07	3.757E-07	1.435E-06	5.058E-06	8.951E-07	1.333E-06

From Table 2, the absolute errors of MHAM are less than 10^{-7} while the absolute errors of MVIM are less then 10^{-6} in interval $t\in[5,50]$. It means that MHAM is more accurate than MVIM for solving hyperchaotic finance system. To study the effect of the numbers of terms in MHAM, we compare the infinite norm of the residual error in the interval [1.10], using different number of terms in Table 3. It is clear that when the number of terms increases, the magnitude of error decreases. Moreover, the effect of the h on the solution can appear when we calculate the $\{\|RE_x\|_\infty, \|RE_y\|_\infty, \|RE_z\|_\infty, \|RE_w\|_\infty\}$, using 3 terms of MHAM and $h = 0.01$ which gives 1.10E-03, 2.11E-03, 1.720E-04, 5.15E-04, respectively. Finally, we noted that the standard HAM is convergent for a small interval, then we can start MHAM by choosing h from this interval.

TABLE 3. Infinite norm of the residual function for MHAM solution using different number of terms in the interval [0,10] with $\varphi = 0.17$ and $h = 0.001$

Number of terms	$\|\|RE_x\|\|_\infty$	$\|RE_y\|\|_\infty$	$\|\|RE_z\|\|_\infty$	$\|\|RE_w\|\|_\infty$
2	2.166E-04	2.1280E-04	1.8978E-05	8.8570E-06
3	1.0978E-06	2.1295E-06	1.7211E-07	5.1596E-07
4	8.6578E-09	1.0367E-08	6.4061E-10	1.1447E-09

CONCLUSIONS

MHAM obtains a continuous solution in this function for the hyperchaotic finance system. The updated approach has the benefit of having an analytical model of the solution within each time interval that is not feasible in other numerical techniques such as MVIM. The residual error and absolute error are defined and calculated for the solution of subintervals. We should also mention that the MHAM solutions were computed using a straightforward algorithm that did not require any perturbation or special transformations, φ, i.e. $\varphi = 0.17$, 0.5, 1.0, and $\varphi = 2.0$. The MHAM is more accurate than MVIM and also Interpretation via Mathematica package for solving hyperchaotic finance system. The MHAM can be an alternative method for other complex systems.

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