1

SYSTEMS OF LINEAR EQUATIONS

- Introduction
- Elimination Methods
- Decomposition Methods
- Matrix Inverse and Determinant
- Errors, Residuals and Condition Number
- Iteration Methods
- Incomplete and Redundant Systems
1.1 Introduction

- The system of linear equations is formed by the addition of the products of a variable with a coefficient, which is also a constant.
- The system of linear equation can be solved via matrix approach.
- The general form of a set of a linear equation having \( n \) linear equations and \( n \) unknowns is

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    \vdots & \qquad \vdots & \ddots & \vdots \\
    a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{align*}
\]

(1.1)

where \( x_1, x_2, \ldots, x_n \) are variables or unknowns, \( a_{ij} \) and \( b_j \) are coefficient or constant (real or complex).

- Eq. (1.1) can be written in a more compact form:

\[
[a_y] \cdot \{x_j\} = \{b_i\} \quad A \cdot x = b
\]

(1.2)

where \( A \) is a matrix \([a_{ij}]\) of size \( n \times n \), \( x \) is a variable vector \( \{x_j\} \) and \( b \) is a right-hand side vector \( \{b_j\} \).

- The process of solving Eq. (1.2) yield three possible solutions:

1. **Unique solution** — e.g.:

\[
\begin{align*}
    3x_1 + x_2 &= 1 \\
    x_1 + 3x_2 &= 1
\end{align*}
\]

\[ x_1 = x_2 = \frac{1}{4} \]

2. **No solution** — e.g.:

\[
\begin{align*}
    -x_1 + x_2 &= 1 \\
    x_1 - x_2 &= 1
\end{align*}
\]

3. **Infinite solutions** — e.g.:

\[
\begin{align*}
    x_1 + x_2 &= 1 \\
    2x_1 + 2x_2 &= 2
\end{align*}
\]
1.2 Elimination Methods

The most popular method is the \textit{Gauss elimination} method, which comprises of two steps:

1. **Forward elimination** to form an upper triangular system via row-based transformation process,
2. **Back substitution** to produce the solution of \( x_j \).

Consider the following system:

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    &\vdots \\
    a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{align*}
\]

If \( a_{11} \neq 0 \), for \( i = 2,3,\ldots,n \), subtract the \( i \)-th equation with the product of \( a_{ij}/a_{11} \) with the first equation to produce the first transformed system:

\[
\begin{align*}
    a_{11}x_1 &+ a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
    a_{22}^{(1)}x_2 &+ \cdots + a_{2n}^{(1)}x_n = b_2^{(1)} \\
    &\vdots \\
    a_{n2}^{(1)}x_2 &+ \cdots + a_{nn}^{(1)}x_n = b_n^{(1)}
\end{align*}
\]

where

\[
\begin{align*}
    a_{ij}^{(1)} &= a_{ij} - \frac{a_{ij}}{a_{11}}a_{1j} \quad \text{for } i,j = 2,3,\ldots,n \\
    b_i^{(1)} &= b_i - \frac{a_{i1}}{a_{11}}b_1 \quad \text{for } i = 2,3,\ldots,n
\end{align*}
\]

The process can be repeated for \( (n-1) \) times until the \( (n-1) \)-th transformed system is formed as followed, which completes the forward eliminations:

\[
\begin{align*}
    a_{11}x_1 &+ a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\
    a_{22}^{(1)}x_2 &+ a_{23}^{(1)}x_3 + \cdots + a_{2n}^{(1)}x_n = b_2^{(1)} \\
    a_{33}^{(2)}x_3 &+ \cdots + a_{3n}^{(2)}x_n = b_2^{(2)} \\
    &\vdots \\
    a_{nn}^{(n-1)}x_n &= b_n^{(n-1)}
\end{align*}
\]
where

\[
\begin{align*}
   a_{ij}^{(k)} &= a_{ij}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} a_{kj}^{(k-1)} & \text{for } i, j = k+1, \ldots, n \quad (1.4a)
   \\
b_{i}^{(k)} &= b_{i}^{(k-1)} - \frac{a_{ik}^{(k-1)}}{a_{kk}^{(k-1)}} b_{k}^{(k-1)} & \text{for } i = k+1, \ldots, n \quad (1.4b)
\end{align*}
\]

Back substitutions can then be executed so that \(x_j\) are solved:

\[
x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}
\]

\[
x_k = \frac{1}{a_{kk}^{(k-1)}} \left[ b_k^{(k-1)} - \sum_{j=k+1}^{n} a_{kj}^{(k-1)} x_j \right] \quad \text{for } k = n-1, \ldots, 1 \quad (1.5b)
\]

- The above method can fail if \(a_{kk} \to 0\), the row has to be interchanged, which is referred to as pivoting:

\[
\begin{align*}
   x_2 &= 2 \\
   x_1 + x_2 &= 3 \quad \text{Pivoting} \quad x_1 + x_2 = 3 \\
   x_2 &= 2
\end{align*}
\]

where the new diagonal element \(a_{kk}^*\) is called a pivot, which can be selected among the maximum absolute value of \(a_{ik}\).

- The pivotal Gauss elimination gives a more accurate solutions, e.g. consider these systems (values to be rounded up to 3 significant figures):

Original Gauss elimination:

\[
\begin{bmatrix}
0.00126 & 0.417 & 0.418 \\
1.34 & -0.708 & 0.632
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0.00126 & 0.417 & 0.418 \\
0.00126 & 0.417 & 0.418
\end{bmatrix}
\rightarrow
\begin{bmatrix}
0.00126 & 0.417 & 0.418 \\
0.0 & -444.184 & -443.908
\end{bmatrix}
\]

\[
x_2 = 0.999 \quad x_1 = 1.125
\]

Pivotal Gauss elimination:

\[
\begin{bmatrix}
1.34 & -0.708 & 0.632 \\
0.00126 & 0.417 & 0.418
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1.34 & -0.708 & 0.632 \\
0.00126 & 0.417 & 0.418
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1.34 & -0.708 & 0.632 \\
0.0 & 0.418 & 0.417
\end{bmatrix}
\]

\[
x_2 = 0.998 \quad x_1 = 0.999
\]

Exact solution:

\[
x_1 = 1 \quad x_2 = 1
\]
Example 1.1

Solve the following system using the Gauss elimination method:

\[
\begin{align*}
2x_1 + x_2 + 3x_3 &= 1 \\
4x_1 + 4x_2 + 7x_3 &= 1 \\
2x_1 + 5x_2 + 9x_3 &= 3
\end{align*}
\]

Solution

The system can be rewritten in matrix form as:

\[
\begin{bmatrix}
2 & 1 & 3 \\
4 & 4 & 7 \\
2 & 5 & 9 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 \\
3 \\
\end{bmatrix}
\]

or

\[
\begin{bmatrix} A | b \end{bmatrix} = 
\begin{bmatrix}
2 & 1 & 3 & 1 \\
4 & 4 & 7 & 1 \\
2 & 5 & 9 & 3 \\
\end{bmatrix}
\]

First step of forward elimination:

\[
\begin{bmatrix}
2 & 1 & 3 & 1 \\
0 & 2 & 1 & -1 \\
0 & 4 & 6 & 2 \\
\end{bmatrix}
\]

Second step of forward elimination:

\[
\begin{bmatrix}
2 & 1 & 3 & 1 \\
0 & 2 & 1 & -1 \\
0 & 0 & 4 & 4 \\
\end{bmatrix}
\]

Hence, the transformed upper triangular system is:

\[
\begin{align*}
2x_1 + x_2 + 3x_3 &= 1 \\
2x_2 + x_3 &= -1 \\
4x_3 &= 4
\end{align*}
\]

Back substitutions are as follows

\[
\begin{align*}
x_3 &= \frac{4}{4} = 1 \\
x_2 &= \frac{-1 - x_3}{2} = -1 \\
x_1 &= \frac{1 - x_2 - 3x_3}{2} = -\frac{1}{2}
\end{align*}
\]
Example 1.2

Perform the pivotal Gauss elimination to the system given in Example 1.1.

Solution

The pivotal Gauss elimination can be performed as followed:

\[
\begin{bmatrix}
2 & 1 & 3 & 1 \\
4 & 4 & 7 & 1 \\
2 & 5 & 9 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
4 & 4 & 7 & 1 \\
2 & 1 & 3 & 1 \\
2 & 5 & 9 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
4 & 4 & 7 & 1 \\
0 & 1 & \frac{1}{2} & \frac{1}{2} \\
0 & 1 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
4 & 4 & 7 & 1 \\
0 & 3 & \frac{11}{2} & \frac{5}{2} \\
0 & 0 & \frac{4}{3} & \frac{4}{3}
\end{bmatrix}
\]

Hence, the upper triangular system is:

\[
4x_1 + 4x_2 + 7x_3 = 1 \\
3x_2 + \frac{11}{2}x_3 = \frac{5}{2} \\
\frac{4}{3}x_3 = \frac{4}{3}
\]

Then, back substitution can be performed:

\[
x_3 = \frac{4}{3} / \frac{4}{3} = 1,
\]

\[
x_2 = \frac{5}{2} - \frac{11}{2}(1) = \frac{3}{2} = -1,
\]

\[
x_1 = 1 - 4(-1) - 7(1) = -\frac{1}{2}.
\]
1.3 Decomposition Methods

- In some cases, the left-hand side matrix $A$ is frequently used while the right-hand side vector $b$ is changed depending on the case.

- The overall system can be transformed to an upper triangular form so that it can be used repeatedly for different $b$, thus matrix $A$ has to be decomposed.

- For a general non-symmetric system, the popular method is the Doolittle or $LU$ decomposition:

$$ A = LU $$

where $L$ and $U$ are the lower and upper triangular matrices, respectively:

$$\begin{bmatrix}
ar_{11} & a_{12} & a_{13} \\
ar_{21} & a_{22} & a_{23} \\
ar_{31} & a_{32} & a_{33}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
l_{21} & 1 & 0 \\
l_{31} & l_{32} & 1
\end{bmatrix} \begin{bmatrix}
u_{11} & u_{12} & u_{13} \\
u_{21} & u_{22} & u_{23} \\
u_{31} & u_{32} & u_{33}
\end{bmatrix}

\equiv \begin{bmatrix}
u_{11} & u_{12} & u_{13} \\
l_{21} & u_{22} & u_{23} \\
l_{31} & l_{32} & u_{33}
\end{bmatrix} \text{ (in memory)}

The solution steps of the system are as followed:

$$ A \cdot x = b \quad \Rightarrow \quad LU \cdot x = b $$

By taking an intermediate vector $y$:

$$ U \cdot x = y \quad \text{(1.7)} $$

Hence,

$$ L \cdot y = b \quad \text{(1.8)} $$

- The elements for $L$ and $U$ can be obtained from the Gauss elimination:

$$ U = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
0 & a_{22}^{(1)} & a_{23}^{(1)} \\
0 & 0 & a_{33}^{(2)}
\end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix}
1 & 0 & 0 \\
\frac{a_{21}}{a_{11}} & 1 & 0 \\
\frac{a_{31}}{a_{11}} & \frac{a_{22}^{(1)}}{a_{22}} & 1
\end{bmatrix}$$
Another variation of the LU decomposition is the \textit{Crout decomposition}, which maintains $u_{ii} = 1$ for $i = 1, 2, \ldots, n$ in $U$ instead of $L$:

For the first row and column:

$$l_{i1} = a_{i1} \quad \text{for } i = 1, 2, \ldots, n \quad (1.9a)$$

$$u_{ij} = \frac{a_{ij}}{l_{11}} \quad \text{for } j = 2, 3, \ldots, n \quad (1.9b)$$

For $j = 2, 3, \ldots, n-1$:

$$l_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \quad \text{for } i = j, j+1, \ldots, n \quad (1.9c)$$

$$u_{jk} = \frac{a_{jk} - \sum_{i=1}^{j-1} l_{ji} u_{ik}}{l_{jj}} \quad \text{for } k = j+1, j+2, \ldots, n \quad (1.9d)$$

dan,

$$l_{nn} = a_{nn} - \sum_{k=1}^{n-1} l_{nk} u_{kn} \quad (1.9e)$$

If the system is \textit{symmetric}, the \textit{Cholesky decomposition} can be used, where matrix $A$ can be decomposed such that:

$$A = LL^T \quad (1.10)$$

For the $k$-th row:

$$l_{ki} = \frac{a_{ki} - \sum_{j=1}^{i-1} l_{kj} l_{ji}}{l_{ii}} \quad \text{for } i = 1, 2, \ldots, k-1 \quad (1.11a)$$

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2} \quad (1.11b)$$

This method optimises the use of computer memory in storing the decomposed form of $A$. 
**Example 1.5**

Decompose the following matrix using the Doolittle LU decomposition:

\[ A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 4 & 7 \\ 2 & 5 & 9 \end{bmatrix} \]

**Solution**

With reference to the matrix elements derived in Example 1.1:

\[ U = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 4 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 4/2 & 1 & 0 \\ 2/2 & 4/2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \]

**Example 1.7**

Decompose the following matrix using the Cholesky decomposition:

\[ A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 5 \\ 3 & 5 & 9 \end{bmatrix} \]

**Solution**

By using Eq. 1.11:

\[
\begin{align*}
l_{11} &= \sqrt{a_{11}} = \sqrt{2}, & l_{21} &= a_{21}/l_{11} = 1/\sqrt{2}, \\
l_{31} &= a_{31}/l_{11} = 3/\sqrt{2}, & l_{22} &= \sqrt{a_{22} - l_{21}^2} = \sqrt{7/2}, \\
l_{32} &= \frac{a_{32} - l_{21}l_{31}}{l_{22}} = \sqrt{7/2}, & l_{33} &= \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = 1.
\end{align*}
\]

Maka,

\[
L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & \sqrt{7/2} & 0 \\ 3/\sqrt{2} & \sqrt{7/2} & 1 \end{bmatrix} = \begin{bmatrix} 1.41421 & 0 & 0 \\ 0.70712 & 1.87083 & 0 \\ 0.70712 & 1.87083 & 1 \end{bmatrix}
\]
1.4 Matrix Inverse and Determinant

- The Gauss elimination can be used to generate the inverse of a square matrix $A$ by replacing the left-hand side vector $b$ with an identity matrix $I$.

- By using the following identity:

$$A \cdot A^{-1} = I \quad (1.12)$$

If all columns of $A^{-1}$ are written as $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ and the columns of the $I$ as $e^{(1)}, e^{(2)}, \ldots, e^{(n)}$, respectively, thus Eq. (1.12) can be rewritten as:

$$A \cdot (x^{(1)}, x^{(2)}, \ldots, x^{(n)}) = (e^{(1)}, e^{(2)}, \ldots, e^{(n)})$$

Then, a set of $n$ linear systems can be assembled:

$$A \cdot x^{(1)} = e^{(1)}$$
$$A \cdot x^{(2)} = e^{(2)}$$
$$\vdots$$
$$A \cdot x^{(n)} = e^{(n)} \quad (1.13)$$

- Consequently, the determinant of matrix $A$ can simply be calculated using:

$$\text{det}(A) \equiv |A| = (-1)^{p} a_{11} a_{22} a_{33} \ldots a_{nn} = (-1)^{p} \prod_{i=1}^{n} a_{ii}^{(i-1)} \quad (1.14)$$

where $p$ is the number of row interchange operation during pivoting.
**Example 1.8**

Determine the inverse of the following matrix using the Gauss elimination:

\[
A = \begin{bmatrix}
4 & 2 & -1 \\
1 & 1 & 1 \\
2 & -1 & -1
\end{bmatrix}
\]

**Solution**

The combination of \(A\) and \(I\) can be represented in an augmented form:

\[
\begin{bmatrix}
4 & 2 & -1 & | & 1 & 0 & 0 \\
1 & 1 & 1 & | & 0 & 1 & 0 \\
2 & -1 & -1 & | & 0 & 0 & 1
\end{bmatrix}
\]

Upon back substitution:

\[
x^{(1)} = \begin{bmatrix} 0 \\ \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix},
\]

\[
x^{(2)} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{9} \\ \frac{8}{9} \end{bmatrix},
\]

\[
x^{(3)} = \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{9} \\ \frac{5}{9} \end{bmatrix}
\]

Hence, the inverse of \(A\) is

\[
A^{-1} = \begin{bmatrix}
0 & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & -\frac{2}{9} & -\frac{5}{9} \\
-\frac{1}{3} & \frac{8}{9} & \frac{2}{9}
\end{bmatrix}
\]

**Example 1.9**

Calculate the determinant of the matrix given in Example 1.8.

**Penyelesaian**

In Example 1.0, there is no row interchange performed, thus \(p = 0\). Hence,

\[
\det(A) = \begin{vmatrix}
4 & 2 & -1 \\
1 & 1 & 1 \\
2 & -1 & -1
\end{vmatrix} = (-1)^0 \times 0 = (4 \cdot \frac{1}{2} \cdot \frac{5}{4}) = 9
\]
1.5 Errors, Residuals and Condition Number

- If \( x^* \) is an approximate solution of a linear system \( A \cdot x = b \), then the system error is defined as
  \[
e = x - x^* \quad (1.16)
  \]

- On the other hand, the system residue \( r \) is defined as
  \[
r = A \cdot e \quad (1.17)
  \]
  or,
  \[
r = A \cdot x - A \cdot x^* = b - A \cdot x^*
  \]

- For a well-conditioned system, the residue can represent the error.

- Moreover, for comparison, a matrix or vector can be expressed in form of a scalar known as norm.

- For a vector \( x = (x_1, x_2, \ldots, x_n)^T \), the \( p \)-norm is defined as
  \[
  \|x\|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \quad (1.20)
  \]
  If \( p = 1 \), it is known as 1-norm:
  \[
  \|x\|_1 = |x_1| + |x_2| + \cdots + |x_n| = \sum_{i=1}^{n} |x_i| \quad (1.19)
  \]
  If \( p = 2 \), it is known as Euclidean norm:
  \[
  \|x\|_e = \|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2} \quad (1.18)
  \]
  If \( p \to \infty \), it is known as a maximum norm:
  \[
  \|x\|_\infty = \max_{1 \leq i \leq n} \{ |x_1|, |x_2|, \ldots, |x_n| \} = \max_{1 \leq i \leq n} |x_i|\]
  \[
  \|x\|_\infty = \max_{1 \leq i \leq n} \{ |x_1|, |x_2|, \ldots, |x_n| \} = \max_{1 \leq i \leq n} |x_i| \quad (1.21)
  \]

- For a matrix \( A = [a_{ij}] \) of size \( m \times n \), the Frobenius norm, which is equivalent to the Euclidean norm for vectors, is defined as
  \[
  \|A\|_e = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2} \quad (1.22)
  \]
and, the equivalent 1-norm and maximum norm for a matrix are defined as

\[
\|A\|_1 = \max_{1\leq j \leq n} \sum_{i=1}^n |a_{ij}| = \text{maximum sum of columns}
\]

(1.23)

\[
\|A\|_\infty = \max_{1\leq i \leq n} \sum_{j=1}^n |a_{ij}| = \text{maximum sum of rows}
\]

(1.24)

- The properties of norms of a vector or matrix \(A\) are as followed:
  
  1. \(\|A\| \geq 0\) and \(\|A\| = 0\) if, and only if, \(A = 0\).
  
  2. \(\|cA\| = |c| \cdot \|A\|\) where \(c\) is a scalar quantity.
  
  3. \(\|A + B\| \leq \|A\| + \|B\|\) \(\text{Triangular inequality}\), where \(B\) is a vector or matrix of the same dimension of \(A\).
  
  4. \(\|A \cdot B\| \leq \|A\| \cdot \|B\|\) \(\text{Schwarz inequality}\), where \(B\) is a vector or matrix which forms a valid product with \(A\).

- The concept of norms can be used to calculate the \textit{condition number} represents the ‘health’ of a linear system, either ill- or well-conditioned.

- If \(e\) is the error for the system \(A \cdot x = b\), from the relations \(A \cdot e = r\) and \(e = A^{-1} \cdot r\), the following inequality can be established:

\[
\|A\| \cdot |e| \geq |r| \quad \text{and} \quad |e| \leq \|A^{-1}\| \cdot |r| \quad \Rightarrow \quad \frac{|r|}{\|A\|} \leq |e| \leq \|A^{-1}\| \cdot |r|
\]

Also, from \(A \cdot x = b\) and \(x = A^{-1} \cdot b:\n\]

\[
\|A\| \cdot |x| \geq |b| \quad \text{and} \quad |x| \leq \|A^{-1}\| \cdot |b| \quad \Rightarrow \quad \frac{|b|}{\|A\|} \leq |x| \leq \|A^{-1}\| \cdot |b|
\]

Thus, the combination of both inequality relations yields the range of the relative error \(\|e\|/\|x\|\), i.e.

\[
\frac{1}{\|A\| \cdot \|A^{-1}\|} \cdot \frac{|r|}{|b|} \leq \frac{|e|}{|x|} \leq \left(\|A\| \cdot \|A^{-1}\|\right) \cdot \frac{|r|}{|b|}
\]

- Hence, the \textit{condition number} is defined as

\[
\kappa(A) = \|A\| \cdot \|A^{-1}\|
\]

(1.25)
where the range of the relative error is.

\[
\frac{1}{\kappa(A)} \frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x\|} \leq \kappa(A) \frac{\|r\|}{\|b\|}
\] (1.26)

- The characteristics of the condition number are that:

1. \(\kappa(A) \geq 1\) — the smaller the better, and otherwise.
2. If \(\kappa(A) \to 1\), the relative residual \(\frac{\|r\|}{\|b\|}\) can represent the relative errors \(\frac{\|e\|}{\|x\|}\).

- If the error is solely contributed by matrix \(A\), the inequality becomes:

\[
\frac{\|e\|}{\|x\|} \leq \kappa(A) \frac{\|E_A\|}{\|A\|}
\] (1.27)

- On the other hand, if the error is solely contributed by vector \(b\), the inequality becomes:

\[
\frac{\|e\|}{\|x\|} \leq \kappa(A) \frac{\|e_b\|}{\|b\|}
\] (1.28)

- Therefore, from Eqs. (1.26-8), it can be seen that the condition number can determine the range of error and thus the health of a system.
1.7 Iteration Methods

- For large systems (size > 200), the elimination and decomposition methods are not efficient due to increasing number of arithmetic operations.

- The number of arithmetic operations can be reduced via iteration methods, such as the Jacobi iteration and the Gauss-Seidel iteration methods.

- In the Jacobi iteration, Eq. (1.1) can be written for $x_i$ from the $i$-th equation:

\[
    x_1 = -\frac{1}{a_{11}} (a_{12} x_2 + a_{13} x_3 + \cdots + a_{1n} x_n - b_1)
\]

\[
    x_2 = -\frac{1}{a_{22}} (a_{21} x_1 + a_{23} x_3 + \cdots + a_{2n} x_n - b_2)
\]

\[\vdots\]

\[
    x_n = -\frac{1}{a_{nn}} (a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{n,n-1} x_{n-1} - b_n)
\]

Eq. (1.29) needs initial values $x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})^T$, which yield $x^{(1)} = (x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)})^T$, and the computation continues as followed:

\[
    x_1^{(k+1)} = -\frac{1}{a_{11}} (a_{12} x_2^{(k)} + a_{13} x_3^{(k)} + \cdots + a_{1n} x_n^{(k)} - b_1)
\]

\[
    x_2^{(k+1)} = -\frac{1}{a_{22}} (a_{21} x_1^{(k)} + a_{23} x_3^{(k)} + \cdots + a_{2n} x_n^{(k)} - b_2)
\]

\[\vdots\]

\[
    x_n^{(k+1)} = -\frac{1}{a_{nn}} (a_{n1} x_1^{(k)} + a_{n2} x_2^{(k)} + \cdots + a_{n,n-1} x_{n-1}^{(k)} - b_n)
\]

For $k \to \infty$, vector $x^{(k)}$ converges to its exact solution if the diagonal domain condition is followed, i.e.

\[
|a_{ii}| > \sum_{j=1 \atop j \neq i}^n |a_{ij}| \quad \text{for } i = 1, 2, \ldots, n
\]

and the matrix which follows this condition is called a diagonal domain matrix.
• To terminate the iteration process, a *convergence* or *termination criterion* can be specified, i.e.

\[ \| x^{(k+1)} - x^{(k)} \| < \varepsilon \]  

(1.32)

• The Gauss-Siedel iteration method uses the most current known solution after each arithmetic operation in order to speed up convergence:

\[ x_1^{(k+1)} = -\frac{1}{a_{11}} \left( a_{12} x_2^{(k)} + a_{13} x_3^{(k)} + \cdots + a_{1n} x_n^{(k)} - b_1 \right) \]

\[ x_2^{(k+1)} = -\frac{1}{a_{22}} \left( a_{21} x_1^{(k+1)} + a_{23} x_3^{(k)} + \cdots + a_{2n} x_n^{(k)} - b_2 \right) \]

\[ \vdots \]

\[ x_n^{(k+1)} = -\frac{1}{a_{nn}} \left( a_{n1} x_1^{(k+1)} + a_{n3} x_3^{(k)} + \cdots + a_{nn-1} x_{n-1}^{(k+1)} - b_n \right) \]  

(1.33)

• As of the Jacobi method, the Gauss-Siedel method must also observe the diagonal domain condition for convergence to be possible (see Fig. 1.1).

\[
\begin{align*}
x_1 - 2x_2 &= -2 \\
2x_1 + x_2 &= 2
\end{align*}
\]

\[
\begin{align*}
2x_1 + x_2 &= 2 \\
x_1 - 2x_2 &= -2
\end{align*}
\]

(a) The off-diagonal domain system  
(b) The diagonal domain system

**FIG. 1.1** Divergence and convergence in the Gauss-Seidel method
Example 1.10

Use the Jacobi iteration method to solve the following system up to 5 decimal points:

\[
\begin{align*}
64x_1 - 3x_2 - x_3 &= 14 \\
x_1 + x_2 + 40x_3 &= 20 \\
2x_1 - 90x_2 + x_3 &= -5
\end{align*}
\]

**Solution**

First of all, form a diagonal domain system:

\[
\begin{align*}
64x_1 - 3x_2 - x_3 &= 14 \\
2x_1 - 90x_2 + x_3 &= -5 \\
x_1 + x_2 + 40x_3 &= 20
\end{align*}
\]

Then, rewrite the system according to Eq. (1.30):

\[
\begin{align*}
x_1^{(k+1)} &= -\frac{1}{64}(-3x_2^{(k)} - x_3^{(k)} - 14) \\
x_2^{(k+1)} &= +\frac{1}{90}(2x_1^{(k)} + x_3^{(k)} + 5) \\
x_3^{(k+1)} &= -\frac{1}{40}(x_1^{(k)} + x_2^{(k)} - 20)
\end{align*}
\]

By taking an initial values \(x^{(0)} = (0, 0, 0)^T\), thus the method converges within 5 iterations:

Iteration no. 1: \(x^{(1)} = (0.21875, 0.05556, 0.50000)^T\),

Iteration no. 2: \(x^{(2)} = (0.22917, 0.06597, 0.49592)^T\),

Iteration no. 3: \(x^{(3)} = (0.22955, 0.06613, 0.49262)^T\),

Iteration no. 4: \(x^{(4)} = (0.22955, 0.06613, 0.49261)^T\),

Iteration no. 5: \(x^{(5)} = (0.22955, 0.06613, 0.49261)^T\).
Example 1.11

Repeat problem given in Example 1.10 using the Gauss-Seidel iteration method.

Solution

First of all, form a diagonal domain system:

\[
\begin{align*}
64x_1 - 3x_2 - x_3 &= 14 \\
2x_1 - 90x_2 + x_3 &= -5 \\
x_1 + x_2 + 40x_3 &= 20
\end{align*}
\]

By taking an initial values \( x^{(0)} = (0, 0, 0)^T \), the first solution in the first iteration:

\[
x_1^{(1)} = -\frac{1}{64} \left[ -3(0) - 0 - 14 \right] = 0.21875
\]

Use \( x_1^{(1)} \) to calculate \( x_2^{(1)} \) and so on, i.e.

\[
x_2^{(1)} = \frac{1}{90} \left[ 2(0.21875) + 0 + 5 \right] = 0.06042
\]

\[
x_3^{(1)} = -\frac{1}{40} \left( 0.21875 + 0.06042 - 20 \right) = 0.49302
\]

Hence, the method converges within 4 iterations:

\[
\begin{align*}
x^{(1)} &= (0.21875, 0.06042, 0.49302)^T, \\
x^{(2)} &= (0.22929, 0.06613, 0.49262)^T, \\
x^{(3)} &= (0.22955, 0.06613, 0.49261)^T, \\
x^{(4)} &= (0.22955, 0.06613, 0.49261)^T.
\end{align*}
\]
1.8 Incomplete and Redundant Systems

- If $m \neq n$, there will be two situations:
  
  1. $m < n$ — incomplete system.
  2. $m > n$ — redundant system.

- For incomplete system, no solution is possible since additional $(n - m)$ equations from other independent sources are required until $m = n$.

- For redundant system, a unique solution is not possible, and the system has to be optimised via least square method (also known as linear regression):

  $$ S = \|e\|^2_e = e^T e, $$

  $$ = (b - Ax)^T (b - Ax), $$

  $$ = b^T b - (Ax)^T b - b^T (Ax) + (Ax)^T (Ax), $$

  $$ = (Ax)^T (Ax) - (Ax)^T b. $$

Using the identity $(AB)^T = B^T A^T$:

$$ S = x^T A^T A x - x^T A^T b $$

Minimising $S$:

$$ \frac{\partial S}{\partial x^T} = 0 = A^T A x - A^T b $$

forms an approximate system of $n$ equations, i.e.

$$ A^T A x = A^T b \quad (1.34) $$

where the left-hand side matrix $A^T A$ is symmetry and the standard deviation $\sigma$ can be calculated from the Euclidean norm of $e$, i.e.:

$$ \sigma = \sqrt{\frac{S}{m-n}} = \frac{\|e\|_e}{\sqrt{m-n}} \quad (1.35) $$
Example 1.12

Calculate the best approximate solution for the following system:

\[
\begin{align*}
2x_1 &+ x_2 + 3x_3 = 1 \\
4x_1 &+ 4x_2 + 7x_3 = 1 \\
2x_1 &+ 5x_2 + 9x_3 = 3 \\
5x_1 &+ 5x_2 + 9x_3 = 2 \\
7x_1 &+ 10x_2 + 15x_3 = 4
\end{align*}
\]

Also, calculate the resulting standard deviation.

Solution

The above system can be rewritten in form of \( A \cdot x = b \) as:

\[
\begin{bmatrix}
2 & 1 & 3 \\
4 & 4 & 7 \\
2 & 5 & 9 \\
5 & 5 & 9 \\
7 & 10 & 15
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
1 \\
1 \\
3 \\
2 \\
4
\end{bmatrix}
\]

By using Eq. (1.34):

\[
\begin{bmatrix}
2 & 4 & 2 & 5 & 7 \\
1 & 4 & 5 & 10 \\
3 & 7 & 9 & 15 \\
7 & 10 & 15
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
2 & 4 & 2 & 5 & 7 \\
1 & 4 & 5 & 10 \\
3 & 7 & 9 & 15 \\
7 & 10 & 15
\end{bmatrix}
\begin{bmatrix}
1 \\
3 \\
2 \\
4
\end{bmatrix}
\]

where its solutions are

\[x_1 = -0.34930, \quad x_2 = -0.01996, \quad x_3 = -0.42914.\]
The standard deviation can be obtained from the Euclidean norm of the error $e$:

$$
e = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 \\ 4 & 4 & 7 \\ 2 & 5 & 9 \\ 5 & 5 & 9 \\ 7 & 10 & 15 \end{bmatrix} \begin{bmatrix} -0.34930 \\ -0.01996 \\ 0.42914 \end{bmatrix} = \begin{bmatrix} 0.43114 \\ -0.52695 \\ -0.06387 \\ -0.01597 \\ 0.20758 \end{bmatrix}$$

$$\|e\|_e = \sqrt{0.43114^2 + (-0.52695)^2 + (-0.06387)^2 + (-0.01597)^2 + 0.20758^2},$$

$$= 0.71483.$$

Therefore,

$$\sigma = \frac{0.71483}{\sqrt{5 - 3}} = 0.50546$$
Exercises

1. Consider the following system:

\[
\begin{bmatrix}
1 & 2 & 4 & 8 \\
0 & 1 & 2 & 3 \\
0 & 1 & 4 & 12 \\
1 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
=
\begin{bmatrix}
1.2 \\
-0.2 \\
0.8 \\
1.5
\end{bmatrix}
\]

a. Use the Gauss elimination method to obtain the solution of \(x_i\).

b. Calculate the determinant for the left-hand side matrix.

c. Generate the lower and upper triangular matrices using the Doolittle factorisation.

2. Consider the following system of 2 complex equations:

\[
\begin{bmatrix}
2 + 2i & -1 + 2i \\
-3i & 3 - 2i
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2
\end{bmatrix}
=
\begin{bmatrix}
1 - 4i \\
2 + 4i
\end{bmatrix}
\]

By writing \(z_k = x_k + y_ki\), solve the equation using the Gauss-Siedel iteration method using Microsoft Excel until it converges up to 5 decimal points.

3. Consider the following set of redundant equations:

\[
\begin{align*}
3x_1 - 2x_2 + x_3 &= 2 \\
x_1 - 3x_2 + x_3 &= 5 \\
x_1 + x_2 - x_3 &= -5 \\
2x_1 + x_2 &= -2 \\
2x_1 - x_2 + x_3 &= 2
\end{align*}
\]

a. Derive an approximate system of linear equations and solve it via the Gauss elimination.

b. Calculate the corresponding standard deviation.