PARTIAL DIFFERENTIAL EQUATIONS

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7.1 Introduction

- **Partial Differential Equations** (PDE) are differential equations which have at least two independent variables, e.g.

\[
\frac{\partial^2 u}{\partial x^2} + 4xy \frac{\partial^2 u}{\partial y^2} + u = 3 \quad \text{second order & linear}
\]

\[
\left( \frac{\partial^2 u}{\partial x^2} \right)^3 + 6 \frac{\partial^3 u}{\partial x \partial y^2} = x \quad \text{third order & nonlinear}
\]

- For a second order linear two-dimensional equation, a general equation is

\[
A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \left( x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) = 0 \quad (7.1)
\]

which can be divided into three types.

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<th>Types of second order linear PDEs</th>
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<tr>
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7.2 Elliptic Equations

- **Elliptic PDEs** are generally related to *steady-state problems with diffusivity* having boundary conditions, e.g. the Laplace equation.

- Consider a steady-state 2-D heat conduction problem:

\[
\Delta z \Delta x \Delta y \Delta t = q(x) \Delta y \Delta z \Delta t + q(y) \Delta x \Delta z \Delta t = q(x + \Delta x) \Delta y \Delta z \Delta t + q(y + \Delta y) \Delta x \Delta z \Delta t
\]

which can be summarised into

\[
\frac{(q(x) - q(x + \Delta x)) \Delta y + (q(y) - q(y + \Delta y)) \Delta x}{\Delta x} + \frac{q(x) - q(x + \Delta x)}{\Delta x} + \frac{q(y) - q(y + \Delta y)}{\Delta y} = 0
\]

For \( \Delta x, \Delta y \to 0 \):

\[
- \frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} = 0
\] (7.2)
• In conduction heat transfer, the relationship between heat flux and temperature is given by the Fourier Law:

\[ q_x = -k \rho C \frac{\partial T}{\partial x}, \quad q_y = -k \rho C \frac{\partial T}{\partial y} \] (7.3)

where \( k \) is heat diffusivity [m²/s], \( \rho \) is density [kg/m³] and \( C \) is heat capacity [J/kg°C]. Combining Eq. (7.2) with Eq. (7.3) gives the Laplace equation:

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \] (7.4)

• If a heat source or heat generation of \( Q(x, y) \) is present in the domain, a Poisson equation can be formed:

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + Q(x, y) = 0 \] (7.5)

• To solve Eq. (7.4), use the central differencing:

\[
\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} \\
\frac{\partial^2 T}{\partial y^2} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}
\]

Hence, Eq. (7.4) can be written in an algebraic form:

\[
\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0
\]

Taking \( \Delta x = \Delta y \) gives

\[
T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0 \] (7.6)
Eq. (7.6) needs boundary conditions (BC), which may be in the form of:

1. Fixed value — Dirichlet (see Fig. 7.3), e.g. Fixed temperature
2. First derivative, or gradient — Neumann, e.g. Fix heat flux or insulated

For node (1,1) — $T_{01} = 75^\circ$C and $T_{10} = 0^\circ$C:

\[
T_{21} + T_{01} + T_{12} + T_{10} - 4T_{11} = 0
\]

\[
4T_{11} - T_{12} - T_{21} = 75
\]
Hence, the complete system (can be assembled into a matrix equation) is:

\[
\begin{pmatrix}
4T_{11} & -T_{21} & -T_{12} & \quad & = & \quad 75 \\
-T_{11} & +4T_{21} & -T_{31} & -T_{22} & = & \quad 0 \\
-4T_{21} & -T_{22} & -T_{32} & = & \quad 50 \\
-4T_{31} & -T_{32} & -T_{33} & = & \quad 75 \\
-4T_{12} & -T_{22} & -T_{23} & = & \quad 0 \\
-4T_{22} & -T_{23} & -T_{23} & = & \quad 50 \\
-4T_{13} & -T_{23} & -T_{23} & = & \quad 175 \\
-4T_{23} & -T_{23} & -T_{23} & = & \quad 100 \\
-4T_{33} & -T_{23} & -T_{23} & = & \quad 150 \\
\end{pmatrix}
\]

- If the Gauss-Seidel method is used, at each node \((i,j)\):

\[
T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4} \quad (7.7)
\]

which can be solve iteratively via relaxation approach:

\[
T_{i,j}^{\text{new}} = \lambda T_{i,j}^{\text{new}} + (1 - \lambda)T_{i,j}^{\text{old}} \quad (7.8)
\]

where \(\lambda\) is the relaxation parameter \(1 \leq \lambda \leq 2\), and is subjected to the termination criterion as followed:

\[
\epsilon_{a,i,j} = \left| \frac{T_{i,j}^{\text{new}} - T_{i,j}^{\text{old}}}{T_{i,j}^{\text{new}}} \right| \times 100\%
\]

**Example 7.1**

Solve the problem in Fig. 7.3 using the Gauss-Seidel method using the termination criterion \(\epsilon_a \leq 1\%\) dan the relaxation parameter \(\lambda = 1.5\).

**Solution**

Let all the initial values be 0°C. For the first iteration:

\[
T_{11} = \frac{T_{21} + T_{01} + T_{12} + T_{10}}{4} = \frac{0 + 75 + 0 + 0}{4} = 18.75
\]

\[
T_{11}^{\text{new}} = 1.5(18.75) + (1 - 1.5)0 = 28.125
\]

\[
T_{21} = \frac{T_{31} + T_{11} + T_{22} + T_{20}}{4} = \frac{0 + 28.125 + 0 + 0}{4} = 7.03125
\]
\[ T_{21}^{\text{new}} = 1.5(7.03125) + (1 - 1.5)0 = 10.54688 \]
\[ T_{31} = \frac{T_{41} + T_{21} + T_{32} + T_{30}}{4} = \frac{50 + 10.54688 + 0 + 0}{4} = 15.13672 \]
\[ T_{31}^{\text{new}} = 1.5(15.13672) + (1 - 1.5)0 = 2.70508 \]

and, for other nodes:
\[ T_{12} = 38.67188 \quad T_{22} = 18.45703 \quad T_{23} = 34.18579 \]
\[ T_{13} = 80.12696 \quad T_{23} = 74.46900 \quad T_{33} = 96.99554 \]

Error for node (1,1) — for the first iteration, all errors are 100%:
\[ \varepsilon_{a_{11}} = \frac{|28.125 - 0|}{28.125} \times 100 = 100\% \]

For the second iteration:
\[ T_{11} = 32.51953 \quad T_{21} = 22.35718 \quad T_{31} = 28.60108 \]
\[ T_{12} = 57.95288 \quad T_{22} = 61.63333 \quad T_{32} = 71.86833 \]
\[ T_{13} = 75.21973 \quad T_{23} = 87.95872 \quad T_{33} = 67.68736 \]

The process is repeated until the ninth iteration in which the termination criterion is fulfilled (\( \varepsilon_a < 1\% \)):
\[ T_{11} = 43.00061 \quad T_{21} = 33.29755 \quad T_{31} = 33.88506 \]
\[ T_{12} = 63.21152 \quad T_{22} = 56.11238 \quad T_{32} = 52.33999 \]
\[ T_{13} = 78.58718 \quad T_{23} = 76.06402 \quad T_{33} = 69.71050 \]
7.3 Parabolic Equations

- **Parabolic PDEs** are generally related to *transient problems with diffusivity*, e.g. the 1-D heat conduction equation.

- For a transient problem, there are three approaches:
  1. **Explicit** method,
  2. **Implicit** method,
  3. **Semi-implicit** method — the **Crank-Nicolson** method.

- Consider a transient 1-D heat conduction problem:

  \[
  \frac{\partial}{\partial t} - \text{output} = \text{storage} \\
  q(x) \Delta y \Delta z \Delta t + q(x + \Delta x) \Delta y \Delta z \Delta t = \Delta x \Delta y \Delta z \rho C \Delta T
  \]

  Dividing with the volume \( \Delta x \Delta y \Delta z \) and the time interval \( \Delta t \):

  \[
  \frac{q(x) - q(x + \Delta x)}{\Delta x} = \rho C \frac{\Delta T}{\Delta t}
  \]

  In the limits of \( \Delta x, \Delta t \to 0 \):

  \[
  - \frac{\partial q}{\partial x} = \rho C \frac{\partial T}{\partial t}
  \]

  By using the Fourier law, Eq. (7.3), Eq. (7.9) becomes

  \[
  k \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}
  \]
For the explicit method, the right-hand side of Eq. (7.9) can be discretised via central difference as:

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{(\Delta x)^2}$$

and, the left-hand side can be discretised via forward difference as:

$$\frac{\partial T}{\partial t} = \frac{T_{i}^{l+1} - T_i^l}{\Delta t}$$

Hence, Eq. (7.10) can be written in the algebraic form:

$$k \frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{(\Delta x)^2} = \frac{T_{i}^{l+1} - T_i^l}{\Delta t}$$

and

$$T_i^{l+1} = T_i^l + \lambda(T_{i+1}^l - 2T_i^l + T_{i-1}^l)$$

where $\lambda = k \Delta t/(\Delta x)^2$.

FIGURE 7.5  Finite difference grid for the heat conduction equation

FIGURE 7.6  Computational grid for the explicit method
Example 7.2

Use the explicit method to determine the temperature distribution for a slender rod having a length of 10 cm. At time \( t = 0 \), the temperature of the rod is 20°C and the boundary conditions are \( T(0) = 100°C \) and \( T(10\text{ cm}) = 50°C \). Use the conduction coefficient \( k = 0.835 \text{ cm}^2/\text{s} \), the time interval \( \Delta t = 0.5 \text{ s} \) and the step size \( \Delta x = 2 \text{ cm} \).

Solution

Calculate \( \lambda \):

\[
\lambda = \frac{k \Delta t}{(\Delta x)^2} = \frac{(0.835)(0.5)}{2^2} = 0.104375
\]

At \( t = 0 \):

\[
T^0_1 = T^0_2 = T^0_3 = T^0_4 = 20
\]

Use Eq. (7.12). At \( t = 0.5 \text{ s} \):

\[
\begin{align*}
T^1_1 &= 20 + 0.1044[20 - 2(20) + 100] = 28.35 \\
T^1_2 &= 20 + 0.1044[20 - 2(20) + 20] = 20 \\
T^1_3 &= 20 + 0.1044[20 - 2(20) + 20] = 20 \\
T^1_4 &= 20 + 0.1044[50 - 2(20) + 20] = 23.1313
\end{align*}
\]

At \( t = 1 \text{ s} \):

\[
\begin{align*}
T^2_1 &= 28.352 + 0.1044[20 - 2(28.352) + 100] = 34.9569 \\
T^2_2 &= 20 + 0.1044[20 - 2(20) + 28.352] = 20.8715 \\
T^2_3 &= 20 + 0.1044[23.132 - 2(20) + 20] = 20.3268 \\
T^2_4 &= 23.132 + 0.1044[50 - 2(23.132) + 20] = 25.6089
\end{align*}
\]

The calculation can be continued to produce the result as in Fig. 7.7.
For the explicit method, a more accurate result can be obtained if $\Delta x$ and $\Delta t$ approach zeros. However, stability of the results is only obtained if:

$$\lambda \leq \frac{1}{2}$$

or

$$\Delta t \leq \frac{1}{2} \frac{(\Delta x)^2}{k}$$

If the stability condition is not fulfilled, the results are contaminated by oscillation.

To prevent oscillation, the **implicit method** can be used, where the second order spatial derivative is approximated at time $l+1$:

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}$$

**FIGURE 7.7** Result for Ex. 7.2

**FIGURE 7.8** Computational grid for the implicit method
Hence, Eq. (7.10) becomes

$$k \frac{T_{i}^{l+1} - 2T_{i}^{l+1} + T_{i}^{l+1}}{(\Delta x)^2} = \frac{T_{i}^{l+1} - T_{i}^{l}}{\Delta t}$$  (7.13)

$$- \lambda T_{i}^{l+1} + (1 + 2\lambda)T_{i}^{l+1} - \lambda T_{i+1}^{l+1} = T_{i}^{l}$$  (7.14)

Eq. (7.14) leads to $n$ simultaneous linear equations having $n$ unknowns.

**Example 7.3**

Repeat Ex. 7.2 using the implicit method.

**Solution**

From Ex. 7.2, $\lambda = 0.104375$. From Eq. (7.14):

$$- 0.1044(100) + [1 + 2(0.1044)]T_{1}^{l} - 0.1044T_{2}^{l} = 20$$

$$[1 + 2(0.1044)]T_{1}^{l} - 0.1044T_{2}^{l} = 30.44$$

Hence, the following system of linear equations can be formed:

$$\begin{bmatrix}
1.2088 & -0.1044 & 0 & 0 \\
-0.1044 & 1.2088 & -0.1044 & 0 \\
0 & -0.1044 & 1.2088 & -0.1044 \\
0 & 0 & -0.1044 & 1.2088 \\
\end{bmatrix}\begin{bmatrix}
T_{1}^{l} \\
T_{2}^{l} \\
T_{3}^{l} \\
T_{4}^{l} \\
\end{bmatrix} = \begin{bmatrix}
30.44 \\
20 \\
20 \\
25.22 \\
\end{bmatrix}$$


- The combination of the explicit and implicit approaches produces the *semi-implicit* method, and one of its kind is the **Crank-Nicolson method**:

![Crank-Nicolson method diagram](image)

**FIGURE 7.9** Computational grid for the Crank-Nicolson method
In this method, the finite difference term for the spatial derivative is
\[
\frac{\partial^2 T}{\partial x^2} = \frac{1}{2} \left[ \frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{(\Delta x)^2} + \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2} \right]
\]  
(7.15)

Hence, Eq. (7.10) becomes
\[
-\lambda T_{i-1}^{l+1} + 2(1 + \lambda)T_i^{l+1} - \lambda T_{i+1}^{l+1} = \lambda T_{i-1}^l + 2(1 - \lambda)T_i^l + \lambda T_{i+1}^l
\]  
(7.16)

**Example 7.4**

Repeat Ex. 7.2 using the Crank-Nicolson method.

**Solution**

From Ex. 7.2, \(\lambda = 0.104375\). From Eq. (7.16):
\[
-0.10437(100) + 2(1 + 0.10437)T_1^l - 0.10437T_2^l = 0.10437(100) + 2(1 - 0.10437)20 + 0.10437(20)
\]
\[
2.20874T_1^l - 0.10437T_2^l = 58.7866
\]

By considering other nodes, the following system of linear equations can be formed:
\[
\begin{bmatrix}
  2.2087 & -0.1044 & 0 & 0 \\
-0.1044 & 2.2087 & -0.1044 & 0 \\
0 & -0.1044 & 2.2087 & -0.1044 \\
0 & 0 & -0.1044 & 2.2087
\end{bmatrix}
\begin{bmatrix}
  T_1^l \\
  T_2^l \\
  T_3^l \\
  T_4^l
\end{bmatrix} =
\begin{bmatrix}
  58.7866 \\
  40 \\
  40 \\
  48.3496
\end{bmatrix}
\]

\(\mathbf{T} = [27.5778, 20.3652, 20.1517, 22.8424]^T\)
7.4 Hyperbolic Equations

- **Hyperbolic PDEs** are generally related to *transient problems with convection*, e.g. the 1-D wave equation.

- Consider a 1-D wave equation, which is a hyperbolic PDE:

\[
\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0
\]  

(7.17)

- One of the methods is the **MacCormack’s technique**, which is an *explicit* finite-difference technique and is second-order-accurate in both space and time. By using the Taylor series:

\[
u_{i}^{t+\Delta t} = u_{i}^{t} + \left( \frac{\partial u}{\partial t} \right)_{\text{avg}} \Delta t
\]  

(7.18)

- This method consists of two steps: **predictor** and **corrector**. In **predictor step**, use forward difference in the right-hand side:

\[
\left( \frac{\partial u}{\partial t} \right)_{i}^{t} = -a \left( \frac{u_{i+1}^{t} - u_{i}^{t}}{\Delta x} \right)
\]  

(7.19)

Thus, from the Taylor series, the predicted value of *u* is:

\[
\bar{u}_{i}^{t+\Delta t} = u_{i}^{t} + \left( \frac{\partial u}{\partial t} \right)_{i}^{t} \Delta t
\]  

(7.20)

- In **corrector step**, by replacing the spatial derivatives with rearward differences:

\[
\left( \frac{\partial u}{\partial t} \right)_{i}^{t+\Delta t} = -a \left( \frac{\bar{u}_{i}^{t+\Delta t} - \bar{u}_{i-1}^{t+\Delta t}}{\Delta x} \right)
\]  

(7.21)

The average of time derivative can be obtained using

\[
\left( \frac{\partial u}{\partial t} \right)_{\text{avg}} = \frac{1}{2} \left[ \left( \frac{\partial u}{\partial t} \right)_{i}^{t} + \left( \frac{\partial u}{\partial t} \right)_{i}^{t+\Delta t} \right]
\]  

(7.22)
• Hence the final, “corrected” value at time $t + \Delta t$ is:

$$u_i^{t+\Delta t} = u_i^{t} + \left( \frac{\partial u}{\partial t} \right)_{\text{avg}} \Delta t$$

• The accuracy of the solution for a hyperbolic PDE is dependent on truncation and round off errors, and the term representing it is called artificial viscosity $\frac{1}{2} a \Delta x (1 - \nu)$.

• The effect of artificial viscosity leads to numerical dissipation, which is originated by the even-order derivatives in the truncated term, but it improves stability.

![FIGURE 7.10 Numerical dissipation: (a) $t = 0$, (b) $t > 0$](image)

• Another opposite effect is known as numerical dispersion, which is originated by the odd-order derivatives in the truncated term and causes ‘wiggles’.

![FIGURE 7.11 Numerical dispersion: (a) $t = 0$, (b) $t > 0$](image)
7.5 Finite Element Method — An Introduction

- The **Finite Element Method** (FEM) is a computer aided mathematical technique used to obtain an approximate numerical solution of a response of a physical system which is subjected to an external loading.
- By using this technique, the computational domain which is theoretically a continuum, is being discretised in form of simple geometries.
- The *mesh* is the computational domain which is an assembly of discrete elemental blocks known as *finite elements*, and the vertices defining the elements are called *nodes*.
- Governing equation is employed at each element to form a set of algebraic equations — *local* system.
- Local equations assembled to form a *global* system which is the solved to yield a vector of variables.

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<tr>
<th>1-D</th>
<th>Sample problem</th>
<th>Mesh</th>
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<td><img src="image2.png" alt="Sample problem" /></td>
<td><img src="image3.png" alt="Mesh" /></td>
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<table>
<thead>
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<th>Sample problem</th>
<th>Mesh</th>
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<table>
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<th>Sample problem</th>
<th>Mesh</th>
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<td><img src="image8.png" alt="Sample problem" /></td>
<td><img src="image9.png" alt="Mesh" /></td>
</tr>
</tbody>
</table>

**FIGURE 7.10** Examples of elements and their applications
Consider a 1-D steady state heat conduction:

\[ k \frac{\partial^2 T}{\partial x^2} + Q(x) = 0 \]  \hspace{1cm} (7.17)

Eq. (7.17) needs appropriate boundary conditions such that:

\[ T_{x=0} = T_0 \]
\[ q_{x=L} = h(T_L - T_\infty) \]  \hspace{1cm} (7.18)
FIGURE 7.12  Boundary condition for a 1D steady state heat conduction

For the first element:

The transformation of the coordinate system to a local system is given by an isoparametric coordinate $\xi$, i.e.

$$\xi = \frac{2}{x_2 - x_1} (x - x_1) - 1 \quad (7.19)$$

Thus, in order to calculate the temperature at the middle section, a linear interpolation function or a linear shape function can be used:

$$N_1(\xi) = \frac{1 - \xi}{2}$$
$$N_2(\xi) = \frac{1 + \xi}{2} \quad (7.20)$$

Hence, the temperature can be interpolated using the shape function as followed:

$$T(\xi) = N_1T_1 + N_2T_2 = NT^c \quad (7.21)$$
Differentiation of Eq. (7.21) gives

\[ d\xi = \frac{2}{x_2 - x_1} \, dx \]

Using a chain rule:

\[ \frac{dT}{dx} = \frac{dT}{d\xi} \frac{d\xi}{dx} \]

\[ = \frac{1}{x_2 - x_1} \left[ -1, \ 1 \right] \mathbf{T}^e \]

\[ = \mathbf{B} \mathbf{T}^e \]

where

\[ B = \frac{1}{x_2 - x_1} \left[ -1, \ 1 \right] \]

- In energy form, the heat conduction problem can be represented by

\[
\int_{\Omega} T \left\{ k \frac{\partial^2 T}{\partial x^2} + Q(x) \right\} \, d\Omega = 0
\]

\[
\Pi_T = \int_0^L \frac{1}{2} k \left( \frac{dT}{dx} \right)^2 \, dx - \int_0^L T Q(x) \, dx + \frac{1}{2} h(T_L - T_\infty)^2
\]

(7.22)

Discretisation of Eq. (7.22) gives

\[
\Pi = \sum_e \frac{1}{2} \mathbf{T}^e \left[ k \mathbf{I}_{ee} \int_{-1}^{1} \mathbf{B}^T \mathbf{B} \, d\xi \right] \mathbf{T}^e - \sum_e \left[ \mathbf{Q}_{ee} \int_{-1}^{1} \mathbf{N} \, d\xi \right] \mathbf{T}^e - \frac{1}{2} h(T_L - T_\infty)^2
\]

(7.23)
Hence, the stiffness matrix for the element is

$$k = \frac{k_e l_e}{2} \int_{-1}^{1} B^T B d\xi = \frac{k_e}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$  \hspace{1cm} (7.24)$$

and, the heat rate vector is

$$r = \frac{Q l_e}{2} \int_{-1}^{1} N d\xi = \frac{Q l_e}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$  \hspace{1cm} (7.25)$$

The global stiffness matrix can be produced via an assembly of all stiffness matrices for all elements, i.e.

$$K \gets \sum_e k_e$$  \hspace{1cm} (7.26)$$

Likewise, the global heat rate vector is an assembly by local heat rate vectors:

$$R \gets \sum_e r_e$$  \hspace{1cm} (7.27)$$

To combine with the boundary condition $T_1 = T_0$, a penalty method can be used:

$$\begin{bmatrix} (K_{11} + C) & K_{12} & \cdots & K_{1L} \\ K_{12} & K_{22} & \cdots & K_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ K_{L1} & K_{L2} & \cdots & K_{LL} + h \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_L \end{bmatrix} = \begin{bmatrix} R_1 + CT_0 \\ R_2 \\ \vdots \\ R_L + h T^\infty \end{bmatrix}$$  \hspace{1cm} (7.28)$$

or, in a matrix form

$$K \cdot T = R$$  \hspace{1cm} (7.29)$$

where the penalty parameter $C$ can be estimated as:

$$C = \max |K_{ij}| \times 10^4$$  \hspace{1cm} (7.30)$$
**Example 7.5**

Fig. 7.15 shows plate of three composites, where the temperature at the right-hand side is $T_0 = 20^\circ C$ and the left-hand side is subjected to convection with $T_\infty = 800^\circ C$ and $h = 25 \text{ W/m}^2\text{C}$. Obtain a temperature distribution across the plate using the finite element method.

![Diagram of plate with three composites](image)

**Solution**

Divide the domain into three elements and construct local stiffness matrices for all elements:

$$k_1 = \frac{20}{0.3} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$k_2 = \frac{30}{0.15} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$k_3 = \frac{50}{0.15} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Combine all local stiffness matrices to form a global stiffness matrix:

$$K = 66.7 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$
whereas the heat rate vector is

\[ \mathbf{R} = [25 \times 800, \ 0, \ 0, \ 0]^T \]

The penalty parameter \( C \) can be estimated by

\[ C = \max \left| K_{ij} \right| \times 10^4 = 66.7 \times 8 \times 10^4 \]

Hence, the finite element system can be solved as followed:

\[
\begin{bmatrix}
1.375 & -1 & 0 & 0 \\
-1 & 4 & -3 & 0 \\
0 & -3 & 8 & -5 \\
0 & 0 & -5 & 80,005
\end{bmatrix}
\begin{bmatrix}
T_1 \\
T_2 \\
T_3 \\
T_4
\end{bmatrix}
= 
\begin{bmatrix}
25 \times 800 \\
0 \\
0 \\
10,672 \times 10^4
\end{bmatrix}
\]

\[ \mathbf{T} = [304.6, \ 119.0, \ 57.1, \ 20.0]^T \]
7.6 Boundary Element Method — An Introduction

- The **Boundary Element Method** (BEM) is a relatively new for PDE, where it consists only *boundary elements*, either line (2-D) or surface (3-D) elements.

- Consider a Laplace equation for a steady-state potential flow problems ($\psi$ is the stream function):

$$\nabla^2 \psi = 0$$

- For a multi-dimensional case, this equation leads to an analytical solution:

$$\psi^* = \frac{1}{2\pi} \ln \frac{1}{r}$$  \hspace{1cm} (7.31)

- Eq. (7.31) can be discretised to yield:

$$\frac{1}{2} \psi_i + \sum_{j=1}^{N} \psi_j \left( \int \frac{\partial \psi^*}{\partial n} ds \right) = \sum_{j=1}^{N} \left( \frac{\partial \psi}{\partial n} \right)_j \left( \int \psi^* ds \right)$$  \hspace{1cm} (7.32)
Exercises

1. A quarter of disc having a radius of 8 cm as shown below has a variation of temperature at the boundary aligned with the principal axes, while the temperature is fixed at 100°C at its outer radius of 8 cm. If the temperature distribution follows the Laplace equation:

\[ \nabla^2 T = 0 \]

By using the grid as shown:

a. Obtain a system of algebraic equations using the finite difference method,

b. Solve the system to obtain \( T_i \).

2. By using the implicit technique, solve the following heat conduction problem:

\[ \frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2}, \quad 0 < x < 10, \quad t > 0 \]

using the following boundary conditions:

\[ T(0,t) = 0, \quad T(10,t) = 100, \quad T(x,0) = 0. \]

Use the constant \( \alpha = 10 \), the time step \( \Delta t = 0.1 \), and a model of 10 grid including the grid at the boundary. Compare this solution with the solution obtained by using \( \Delta t = 0.3 \).