Chapter 2

One Dimensional Problems

2.1 Introduction

In the last chapter, we have seen u,σ, ϵ,T vectors and f. For one dimensional problems, the vectors is a function of *x*. The vectors become:

$$\mathbf{u} = u(x), \sigma = \sigma(x), \epsilon = \epsilon(x), \mathbf{T} = T(x) \textbf{ and } \mathbf{f} = f(x)$$

Furthermore, the stress-strain and strain-displacement relations are:

$\sigma = E(\epsilon)$**and** $\epsilon = \frac{du}{dx}$

The loading consist of three types:

Body force, f is a distributed force acting on every elemental volume of the body and has units of force per unit volume. For example the self weight due to gravity.

$$f = \frac{force}{volume}$$

Traction force, T is a distributed load acting on the surface of the body. It is defined as force per unit length. For example frictional resistance, viscous drag and surface shear.

$$T = \frac{force}{area@length}$$

2.2 Finite Element Modeling

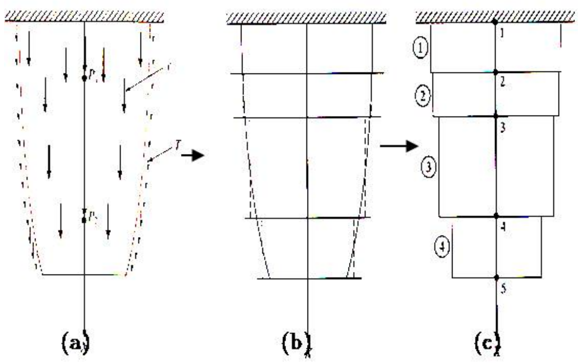
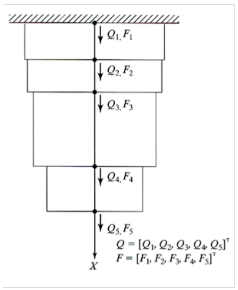
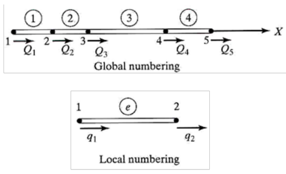


Figure 1:

In one dimensional problem, every node is permitted to displace only in the $\pm x$. Thus, each node has only one degree of freedom(dof). For five nodes, the displacements along each dof are denoted by$\mathbf{Q} = [Q_1,\ Q_2, \ldots,\ Q_5]^{\mathbf{T}}$and global load vector is denoted by $\mathbf{F} = [F_1,\ F_2, \ldots,\ F_5]^{\mathbf{T}}$.





Global Element	Local Element	
	Node 1	Node 2
1	1	2
2	2	3
3	3	4
4	4	5

Figure 2:

2.3 Coordinates and Shape Functions

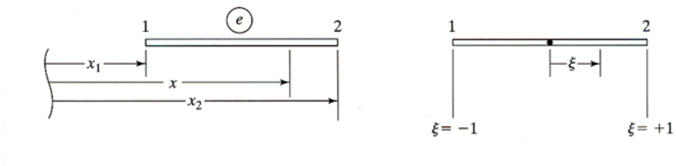


Figure 3:

Linear shape functions is given by

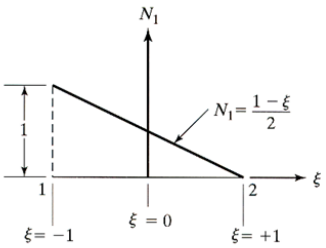


Figure 4:

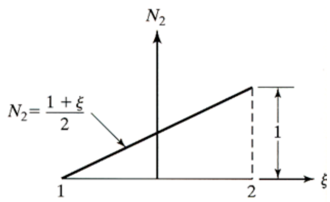


Figure 5:

$$N_1(\xi) = \tfrac{1-\xi}{2} \text{ and } N_2(\xi) = \tfrac{1+\xi}{2}$$

The graph of the shape function N_1 can be seen in Figure 4 where $N_1 = 1$ at $\xi =$ -1 and $N_1 =$0 at $\xi=$ 1. Similarly defined for the graph of N_2.

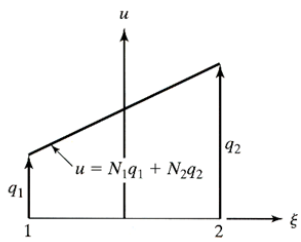


Figure 6:

$$u = N_1 q_1 + N_2 q_2 \text{ or } u = N\,q$$

$$u = \frac{1-\xi}{2} q_1 + \frac{1+\xi}{2} q_2$$

$$\frac{du}{d\xi} = \frac{-q_1 + q_2}{2}$$

$$\epsilon = \frac{du}{dx}$$

$$\epsilon = \frac{du}{d\xi}\frac{d\xi}{dx} \qquad \text{(chain rule)}$$

$$\frac{d\xi}{dx} = \frac{2}{x_2 - x_1}$$

$$\epsilon = \tfrac{1}{x_2 - x_1}(-q_1 + q_2) \text{ or } \epsilon = \mathrm{Bq}$$

$$\mathbf{B} = \frac{1}{x2 - x1}\left[\begin{array}{cc} -1 & 1 \end{array}\right]$$

$$\sigma = E\mathbf{Bq}$$

2.4 The Potential Energy Approach

$$U_e = \frac{1}{2}\int_e \sigma^{\mathrm{T}} \epsilon A \, dx$$

2.4.1 Element Stiffness Matrix

$$U_e = \frac{1}{2}\int_e \mathrm{q}^{\mathrm{T}} \mathrm{B}^{\mathrm{T}} E \mathrm{B} \mathrm{q} A \, dx$$

$$U_e = \frac{1}{2}\mathrm{q}^{\mathrm{T}} \left[A_e \frac{\ell_e}{2} E_e \mathrm{B}^{\mathrm{T}} \mathrm{B} \int_{-1}^{1} d\xi \right] \mathrm{q}$$

$$U_e = \frac{1}{2}\mathrm{q}^{\mathrm{T}} \frac{A_e E_e}{\ell_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathrm{q}$$

$$\mathrm{k}^e = \frac{A_e E_e}{\ell_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2.4.2 Force Terms

$$\int_e u^{\mathrm{T}} f \, A \, dx = \mathrm{q}^{\mathrm{T}} \begin{Bmatrix} A_e f \int_e N_1 \, dx \\ A_e f \int_e N_2 \, dx \end{Bmatrix}$$

$$\int_e u^{\mathrm{T}} f\, A\, dx = \mathrm{q}^{\mathrm{T}} \frac{A_e}{2}\, \ell_e\, f \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\}$$

$$\mathrm{f}^e = \frac{A_e \ell_e f}{2} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\}$$

$$\int_e u^{\mathrm{T}} T\, dx = q^{\mathrm{T}} \left\{ \begin{array}{c} T \int_e N_1\, dx \\ T \int_e N_2\, dx \end{array} \right\}$$

$$\int_e u^{\mathrm{T}} T\, dx = q^{\mathrm{T}} \frac{T \ell_e}{2} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\}$$

$$\mathrm{T}^e = \frac{T \ell_e}{2} \left\{ \begin{array}{c} 1 \\ 1 \end{array} \right\}$$

$$\prod = \frac{1}{2} \mathrm{Q}^{\mathrm{T}} \mathrm{K} \mathrm{Q} - \mathrm{Q}^{\mathrm{T}} \mathrm{F}$$

$$\mathrm{K} \leftarrow \sum_e \mathrm{k}^e$$

$$\mathrm{F} \leftarrow \sum_e (\mathrm{f}^e + \mathrm{T}^e) + \mathrm{P}$$

2.5 Temperature Effects

$$\theta e = E_e A_e \frac{\ell_e}{2}\epsilon_0 \int_{-1}^{1} B^{\mathrm{T}}\, d\xi$$

$$\theta_e = \frac{E_e A_e \ell_e \alpha \Delta T}{x_2 - x_1} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\mathrm{F} \leftarrow \sum_e (\mathrm{f}^e + \mathrm{T}^e + \theta^e) + \mathrm{P}$$

Example

A lamp pole is made from galvanized iron will be placed at the Kajang-Seremban Highway. Four lamps will be placed at the top and each weight 25 kg. The wall thickness of the pole is uniform and 10 mm thick. The mechanical properties of the galvanized iron is shown in Table 1 and the geometry is given in Figure 1. Temperature during day light is 39^{o}C while during the night is 20oC. Determine the length changes due to the temperature changes which include the weight of the lamps and pole by calculations.

Two linear element

One quadratic element

Table 1: Galvanized iron mechanical properties

Young Modulus, E	120 GPa
Poisson Ratio, v	0.35
Thermal Coefficient, α	12.0 x$10^{-6}/^{\mathrm{O}}$C
Weight per unit volume, f	30kN/m^3

Solution:

Global	Local Element	
Element	Nod 1	Nod 2
1	1	2
2	2	3

Cross section area element 1, $\mathrm{A}_1 = \pi/4(0.45^2 - 0.44^2)$

Cross section area element 2, $\mathrm{A}_2 = \pi/4(0.35^2 - 0.34^2)$

Element Stiffness Matrix Expresssion

$$\mathrm{k}_{(i)} = \frac{EA_i}{\ell_i}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathrm{K} = \frac{EA_1}{\ell_1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{EA_1}{\ell_1}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\mathrm{K} = 10^6\begin{bmatrix} 33.552 & -33.552 & 0 \\ -33.552 & 59.531 & -29.011 \\ 0 & -26.011 & 26.011 \end{bmatrix}$$

Force Term Calculation

Body Force

General equation element body force vectors, $f_{(i)} = \frac{A_i \ell_i f}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$

Global body force vectors, $\mathrm{f} = \frac{A_1 \ell_1 f}{2} \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} + \frac{A_2 \ell_2 f}{2} \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix}$

$$\mathrm{f} = 10^3 \begin{Bmatrix} 2.621 \\ 4.653 \\ 2.032 \end{Bmatrix}$$

General equation for element thermal load, $\theta_{(i)} = \frac{E_i A_i \ell_i \alpha \Delta T}{x_2 - x_1} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$**, where** $\ell_i = (x_1 - x_2)_i$

Global thermal load, $\theta = E A_1 \alpha \Delta T \begin{Bmatrix} -1 \\ 1 \\ 0 \end{Bmatrix} + E A_2 \alpha \Delta T \begin{Bmatrix} 0 \\ -1 \\ 1 \end{Bmatrix}$

Substituting E, α, $A1$, $A2$, and ΔT values gives

$$\theta = 10^3 \begin{Bmatrix} -191.625 \\ 47.635 \\ 151.277 \end{Bmatrix}$$

Point load

Load at a point is given by $\mathrm{P} = \begin{Bmatrix} 0 \\ 0 \\ P_3 \end{Bmatrix} = 10^3 \begin{Bmatrix} 0 \\ 0 \\ 0.981 \end{Bmatrix}$

Global load vectors becomes $\mathrm{F} = f + \theta + \mathrm{P} = 10^3 \begin{Bmatrix} -188.625 \\ 47.635 \\ 151.277 \end{Bmatrix}$

Elimination Approach

In this approach, global stiffness matrix K, is a reduced stiffness matrix from the original K matrix obtained by eliminating rows and colums corresponding to the specified or 'support' degree of freedom.

$$\mathrm{K} = 10^6 \begin{bmatrix} 33.552 & 33.552 & 0 \\ -33.552 & 59.531 & -26.011 \\ 0 & -26.011 & 26.011 \end{bmatrix} = 10^6 \begin{bmatrix} 59.531 & -26.011 \\ -26.011 & 26.011 \end{bmatrix}$$

Similarly, this elimination will also produce the new global load vector, F.

$$\mathrm{F} = 10^3 \begin{Bmatrix} 188.625 \\ 47.635 \\ 151.277 \end{Bmatrix} = 10^3 \begin{Bmatrix} 47.635 \\ 151.277 \end{Bmatrix}$$

Solution of the equation $\mathrm{KQ} = \mathrm{F}$ **will give**

$$10^6 \begin{bmatrix} 59.531 & -26.011 \\ -26.011 & 26.011 \end{bmatrix} \begin{Bmatrix} Q_2 \\ Q_3 \end{Bmatrix} = 10^3 \begin{Bmatrix} 47.635 \\ 151.277 \end{Bmatrix}$$

Matrix above can be easily solved. If the matrix is large, it is much easier to use Gaussian Elimination (Refer to the GAUSS program) to solve the matrix.

$$Q2 = 0.005934\, m, \quad Q3 = 0.01175\, m$$

$$Q = \begin{bmatrix} 0, & 0.005934 & 0.01175 \end{bmatrix}^{\mathrm{T}} m$$

Penalty Approach

In this approach, global stiffness matrix, K is modified by adding a large number *C* to the first diagonal element which has specified boundary condition. Similarly, global load vector is also modified by adding *C* and boundary condition number. Consider a specified boundary condition $Q_1 = a_1$, then, the modified stiffness matrix and modified load vector is given by :

$$\begin{bmatrix} (K_{11} + C) & K_{12} & \cdots & K_{1N} \\ K_{12} & K_{22} & \cdots & K_{2N} \\ \vdots & \vdots & & \vdots \\ K_{N1} & K_{N2} & \vdots & K_{NN} \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_N \end{Bmatrix} = \begin{Bmatrix} F_1 + C_{a1} \\ F_2 \\ \vdots \\ F_N \end{Bmatrix}$$

It is suggested that the value of *C* is

$$C = max\,|K_{ij}|\,\mathrm{x}\,10^n$$

Where *n* is satisfied by taking the value of 4, 5 or 6 depending on the user's experience and computer capability.

Proceed to our problem above, by using penalty approach, the value of max $|K_{ij}|$ is at the K_{22} element. Value $C = 59.531(10^6)\,\mathrm{x}\,10^4$ and $a_1 = 0$. Solutions of the equation $\mathrm{KQ} = \mathrm{F}$ give,

$$
10^6 \begin{bmatrix} 33.552 + 59.531(10^4) & 33.552 & 0 \\ -33.552 & 59.531 & -26.011 \\ 0 & -26.011 & 26.011 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = 10^3 \begin{Bmatrix} -188.625 \\ 47.635 \\ 151.277 \end{Bmatrix}
$$

Like the elimination approach, equation above can also be solve by using Gaussian Elimination.

Stress and reaction force determination

Stresses on each element is given by $\sigma = E\mathrm{bq}$

$$
\sigma_1 = 120(10^9)\,\mathrm{x}\,\frac{1}{25}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.005934 \end{Bmatrix} = 28.483\,\mathrm{MPa}
$$

and

$$
\sigma_2 = 120(10^9)\,\mathrm{x}\,\frac{1}{25}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} 0.005934 \\ 0.011750 \end{Bmatrix} = 27.917\,\mathrm{MPa}
$$

Reaction force, R using $\mathrm{R} = \mathrm{KQ} - \mathrm{F}$

$$
R1 = \frac{120(10^9)}{25}\,\mathrm{x}\,6.990(10^{-3})\,\mathrm{x}\begin{bmatrix} 1 & -1 & 0 \end{bmatrix}\begin{Bmatrix} 0 \\ 0.005934 \\ 0.011750 \end{Bmatrix} - (188.625) = -394.047\,\mathrm{kN}
$$

Solution using computer program

```
NN NE NM NDIM NEN NDN
 3  2  1   1   2   1
ND NL NCH NPR NMPC
 1  1  2   2   0
Node#   X-Coordinate
 1       0
 2       25
 3       50
Elem#  N1  N2  Mat#  Area TempRise (NCH=2 Elem Char: Area, TempRise)
 1      1   2    1    6.990e-3   0
 2      2   3    1    5.419e-3   0
DOF#  Displacement
 1      0
DOF#  Load
3    981
MAT#  E   Alpha
 1   120E9  0
B1  i  B2 j  B3  (Multi-point constr. B1*Qi+B2*Qj=B3)
```

```
NODE NO.      DISPLACEMENT
 1             1.64699E-09
 2             2.923984E-05
 3             6.695436E-05
ELEM NO.      STRESS
 1             140343.3
 2             181029.7
NODE NO.      REACTION
 1            -980.9999
```

2.6 Quadratic Shape Functions

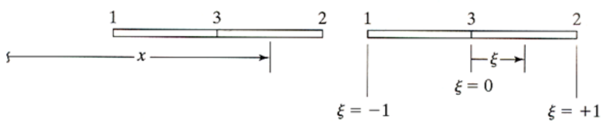


Figure 7:

$$\xi = \frac{2(x - x_3)}{x_2 - x_1}$$

$$N_1(\xi) = -\frac{1}{2}\xi(1-\xi)$$

$$N_2(\xi) = \frac{1}{2}\xi(1+\xi)$$

$$N_3(\xi) = (1+\xi)(1-\xi)$$

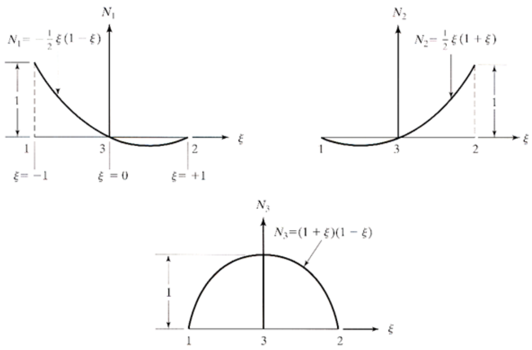


Figure 8:

$$u = N_1 q_1 + N_2 q_2 + N_3 q_3 \qquad or \qquad = \mathrm{Nq}$$

where $\mathrm{N} = [N_1,\ N_2,\ N_3]$**and** $\mathrm{q} = [q_1,\ q_2,\ q_3]^{\mathrm{T}}$

The strain is given by

$$\epsilon = \frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx}$$

here $\frac{du}{d\xi} = \left[\frac{dN_1}{d\xi}, \frac{dN_2}{d\xi}, \frac{dN_3}{d\xi}\right]$**and** $\frac{d\xi}{dx} = \frac{2}{x_2 - x_1}$

Then, $\epsilon = \frac{2}{x_2 - x_1}\left[-\frac{1-2\xi}{2}, \frac{1+2\xi}{2}, -2\xi\right] \mathrm{q}$ **or** $\epsilon = \mathrm{Bq}$

Like usual, $\sigma = E\mathrm{Bq}$

Element stifness matrix k^e **is given by**

$$\mathrm{k}^e = \frac{E_e A_e}{3\ell_e}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & 8 & 16 \end{bmatrix}$$

Element body force vectors f^e **is given by**

$$\mathrm{f}^e = A_e \ell_e f \begin{Bmatrix} 1/6 \\ 1/6 \\ 2/3 \end{Bmatrix}$$

Element body force vectors $\mathbf{T}^e$ **is given by**