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FUZZY PROBABILITY ANALYSIS OF THE FATIGUE RESISTANCE OF STEEL STRUCTURAL MEMBERS UNDER BENDING

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Abstract. The paper is aimed at the fuzzy probabilistic analysis of fatigue resistance due to uncertainty of input parameters. The fatigue resistance of the steel member is evaluated by linear fracture mechanics as the number of cycles leading to the propagation of initial cracks into a critical crack resulting in brittle fracture. When the histogram of stress range is known, the fatigue resistance is a random variable. In the event that the histogram is unknown or was acquired from a small number of experiments, another source of uncertainty is of an epistemic origin. Two basic approaches, which make provision for uncertainty of input histograms of stress range, are illustrated in the paper. Uncertainty of histograms of stress range is taken into account by the variability of equivalent stress range in the first stochastic approach. Input histograms as considered as members of a fuzzy set in the second approach.

Keywords: fatigue, steel, crack propagation, failure, probability, fuzzy, simulation, sensitivity.

1. Introduction

A number of variables and phenomena present during design processes (or review) of steel structures are burdened by smaller and/or larger uncertainties. Problems in determining the service life of bridge structures are due to an insufficiency of information necessary for modelling the propagation of fatigue cracks. The basic epistemic uncertainties include the origin of crack initiation and corresponding histograms of stress range from vehicle passages and further repeated loading of the structure.

The Wöhlerian approach, which does not model the propagation of fatigue cracks, is used when designing the new structures according to standards. Linear fracture mechanics can be applied for the description of fatigue crack propagation (Kala 2006). Linear fracture mechanics principles and theorems have been known for some decades but in comparison with the Wöhlerian approach, they are not as elaborate as to where they could be applied within the framework of standard prescriptions.

In general, the character of the quantities working upon the fatigue crack propagation is random. They include material, geometrical characteristics and the loading effect. Random quantities can be obtained from measurements. Difficulties arise when the reliability of a new structure is to be predicted. Neither loading effect nor other random characteristics are known at the time of structure design. Uncertainties encountered in such a case are not of stochastic character but are of a fuzzy origin.

The term "fuzzy" was used by Prof Lofti Zadeh, for the first time, in 1962 (Zadeh 1962). In 1965 he published his pioneer, today still classic paper entitled "Fuzzy sets" Zadeh 1965). The fuzzy sets have since spread practically to all aspects of scientific disciplines. Problems of structural dynamics, problems of partial reliability factor in member buckling (Ferracuti *et al.* 2005), stability problems in geotechnics (Vageesha *et al.* 2005) and optimum design (Lu *et al.* 2004) can be solved utilising fuzzy sets.

Many typical problems of structural design are characterised by both fuzzy and random uncertainties. Utilisation of fuzzy random variables and fuzzy random functions enables the mathematical description of uncertainty characterised by fuzzy randomness. Basic terms and definitions related to fuzzy randomness have been introduced, inter alia, by (Puri, Ralescu 1986). Fuzzy randomness arises when random variables – eg as a result of changing boundary conditions, cannot be observed with precision. If the fuzzy random function is solely dependent on time, a fuzzy random process is obtained.

Two basic approaches, which can be used for describing the combined statistical and epistemic uncertainty in the form of histograms of stress range, are illustrated in the paper. The first approach is stochastic and considers the equivalent stress range as a random variable. The histograms of stress range are considered as members of a fuzzy set in the second approach.

2. Linear elastic fracture mechanics

The calculation algorithm applied in the presented paper is based on the generally most used and recognised model describing fatigue crack growth. According to the Paris–Erdogan's equation, the crack propagation rate is as follows:

$$\frac{da}{dN} = C \cdot \left(\Delta K\right)^m, \tag{1}$$

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where a – crack size, N – number of cycles, C, m – material constants.

C, m are material constants which can be determined by statistical processing from a set of experimentally determined data pairs $(da/dN, \Delta K)$. The range of the stress intensity coefficient ΔK is defined by the relation:

$$\Delta K = K_{\max} - K_{\min} = F(a) \cdot \Delta\sigma \cdot \sqrt{\pi \cdot a}\,, \tag{2}$$

where $\Delta\sigma$ quasi–constant stress range, $F(a)$ calibration function.

The mathematical model describing fatigue crack growth of an element stressed by in–plane bending moment is defined, according to (Kunz 1991; Gocál 2000), by the calibration function:

$$F(a) = 1.12 - 1.39G + 7.32G^2 - 13.08G^3 + 14G^4\,, \tag{3}$$

where a – crack size, b – size of element.

The parameter G in (3) expresses the quotient $G = a/b$. The steel member under in–plane bending moment is shown on Fig. 1.

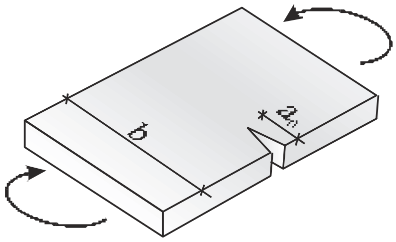

Fig. 1. Structural detail with initial crack

Rearrangement and integration of the Paris–Erdogan's equation (1) and consideration of the crack initiation propagation from a_1 to a_2, and corresponding number of cycles N_1 and N_2, provides the relation:

$$\int_{a1}^{a2} \frac{da}{\left[F(a)\cdot\sqrt{\pi\cdot a}\right]^m} = \int_{N1}^{N2} C \cdot \Delta\sigma^m dN\,. \tag{4}$$

When assembling a bridge structure (welding, cutting, drilling), the fatigue crack can initiate and propagate with the first loading cycle. For bridge structures, it is therefore justified to consider the number of cycles at the time of fatigue crack initiation by the value $N_0 = 0$. Provided that the initial crack size value is introduced as $a_1 = a_0$, and the final one, $a_2 = a_{cr}$, relation (4) can be rewritten in the form:

$$\int_{a_0}^{a_{cr}} \frac{da}{\left[F(a)\cdot\sqrt{\pi\cdot a}\right]^m} = C \cdot N \cdot \Delta\sigma^m\,, \tag{5}$$

where a_0 – initial crack size, a_{cr} – critical crack size, N – number of cycles, C, m – material constants.

N is the total number of cycles at crack growth from a_0 to a_{cr}. The quasi–constant stress range $\Delta\sigma$ is considered in equation (5). The real loading of a bridge structure is random in time; the character of stress ranges is thus random too.

The real stress change signal in time can be obtained by measurements on real bridges. The histogram of random stress ranges (Fig. 2) is evaluated by means of the methods rain flow, reservoir, range count etc. A bridge loaded by stress change with constant amplitude does not practically exist; it is to be taken into consideration in the calculation by the method of linear fracture mechanics as well. For the purpose of taking the stress range spectrum, the right side of equation (5) can be substituted by the sum:

$$C \cdot N \cdot \Delta\sigma^m \approx C \cdot \sum_{i=1}^{M} \left(n_i \cdot \Delta\sigma_i{}^m\right), \tag{6}$$

where $\Delta\sigma_i$ are stress amplitudes representing individual classes of the spectrum; n_i – the frequency of amplitudes in these spectrums, and M – the total number of classes of the spectrum. It holds that

$$\sum_{i=1}^{M} n_i = N\,. \tag{7}$$

In a limit case, one stress range and one frequency fall upon each class. Based on this, the expression (7) can be modified as follows:

$$C \cdot \sum_{i=1}^{M} \left(n_i \cdot \Delta\sigma_i^m\right) \approx C \sum_{j=1}^{N} \Delta\sigma_j{}^m\,, \tag{8}$$

where $\Delta\sigma_j$ are individual stress ranges. After substitution of Eq (8) into (5), a modified form of the Paris–Erdogan's Eq is obtained:

$$\int_{a_0}^{a_{cr}} \frac{da}{\left[F(a)\cdot\sqrt{\pi\cdot a}\right]^m} = C \sum_{j=1}^{N} \Delta\sigma_j{}^m\,. \tag{9}$$

2.1. Equivalent stress range

Now let us assume hypothetically that the histogram of the random quantity $\Delta\sigma_j$ in Eq (9) is known from an experimental research. For an illustration of the problem solved, let us consider the histogram in the form presented in Fig. 2.

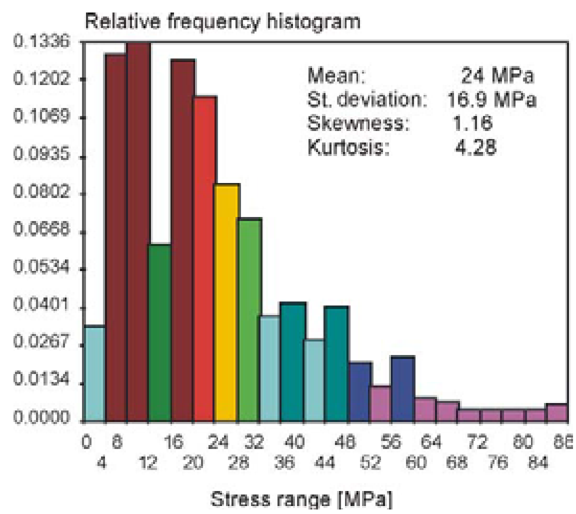

Fig. 2. Histogram of stress range

The character of quantities a_0, a_{cr}, b, m, C in the relation (9) is generally random. Random realizations of quantities a_0, a_{cr}, b, m can be simulated by the Monte Carlo (MC) simulation method. If a_0, a_{cr}, b, m, C are random quantities, it is necessary to simulate N realizations (also by means of the MC method) of $\Delta\sigma$ at each run of the MC method (one realization of a_0, a_{cr}, b, m, C), which is very demanding numerically. A problem arose when it was necessary to simulate $N = 10^6$ to 10^7 realisations of the histogram $\Delta\sigma_j$ in each run of MC method; it was numerically very demanding even on very quickly working computers.

When comparing relations (9) and (5), it is possible to substitute the histogram from Fig. 2 by one equivalent stress range $\Delta\sigma_E$ which can be determined from the relation (Tomica 2003):

$$\Delta\sigma_E = \left(\frac{1}{N} \sum_{j=1}^{N} \Delta\sigma_j{}^m \right)^{\frac{1}{m}}. \quad (10)$$

The computation relation (5) can be formally overwritten:

$$\int_{a_0}^{a_{cr}} \frac{da}{\left[F(a) \cdot \sqrt{\pi \cdot a} \right]^m} = C \cdot N \cdot \Delta\sigma_E{}^m. \quad (11)$$

The standard deviation $\Delta\sigma_E$ is equal to zero only under the assumption that, in the histogram determination, neither statistical nor epistemic uncertainty exists; it is practically impossible in real practice. For the histogram from Fig. 2, $N = 10^6$ MC runs were applied, and $m = 3$ (Gocál 2000). Based on (10), it has been calculated that the value of $\Delta\sigma_E = 34.3$ MPa.

3. Uncertainty of equivalent stress range

3.1. Statistical analysis of equivalent stress range

In the stochastic analysis, the uncertainties connected with the determination of equivalent range $\Delta\sigma_E$ are usually taken into consideration by the fact that an equivalent range is a random variable. In relation (11), $\Delta\sigma_E$ is usually considered to be a random quantity with Gaussian density function (Gocál 2000); by this, the uncertainties in the histogram shape should be taken into account. Problems occur particularly when determining the standard deviation $\Delta\sigma_E$.

The statistical analysis of the histogram shape uncertainty can be practically explained by two simple examples.

3.1.1. Example 1

Let us presume that there is no information on the histogram shape and that the random stress range lies within the interval $\langle 0; 100 \rangle$ MPa. Further, let us suppose that the histogram consists of 6 classes and that the total number of measurements is 24. Up to 24 measurements can be randomly carried out in each class, ie, the problem results in 118 755 combinations of histogram shapes.

The realisations of 118 755 histograms were computed by a program compiled in the language Delphi. The method of generating histograms is practically evident from the part of printout in programming language Pascal:

```
For i1:=0 to 24 do
For i2:=0 to 24 do
For i3:=0 to 24 do
For i4:=0 to 24 do
For i5:=0 to 24 do
For i6:=0 to 24 do
begin
if i1+i2+i3+i4+i5+i6=24
then begin
Writeln('Frequency: ',i1, ' ', i2, ' ', i3, ' ', i4, ' ', i5, ' ', i6);
sum:=sum+1;
end;
end;
Writeln('Combination = ',sum);
```

The realization example of the 102 037th histogram combination is presented in Fig. 3. For each histogram, 10 000 realisations were simulated by the Latin Hypercube Sampling (LHS) method. The LHS method is a method of MC type, giving for statistical analysis results better than the MC method (Kala 2006).

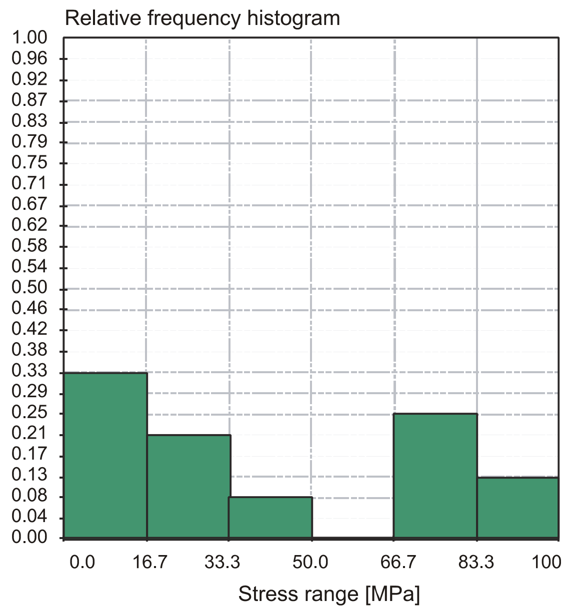

Fig. 3. One histogram realisation from 118 755 combinations

The processing of 118 755 histograms provided 118 755 files (each file contains $N = 10\,000$ simulated realisations). According to (10), 118 755 random realisations of $\Delta\sigma_E$ were the computed; one realization $\Delta\sigma_E$ was computed from (10) with used $N = 10\,000$ simulated runs.

The computation by using the LHS method described in this a way is numerically highly requiring, and it would be necessary for a higher number of histograms to optimise it; the equivalent stress range (10) can be considered as a measure of centre of a sample, as well.

Coefficient m was considered by the deterministic value $m = 3$. The output of statistical evaluation of 118 755 values $\Delta\sigma_E$ is presented in Fig. 4.

It is evident from Fig. 4 that the equivalent stress range $\Delta\sigma_E$ can be approximated by Gaussian density function very well. The negative value of skewness, 0.2495, is very interesting. If there is no information on the stress range histogram (each histogram shape is probably identical in the same way), Gaussian density function with mean value 61.36 MPa and standard deviation 10.22 MPa can be applied for equivalent stress range.

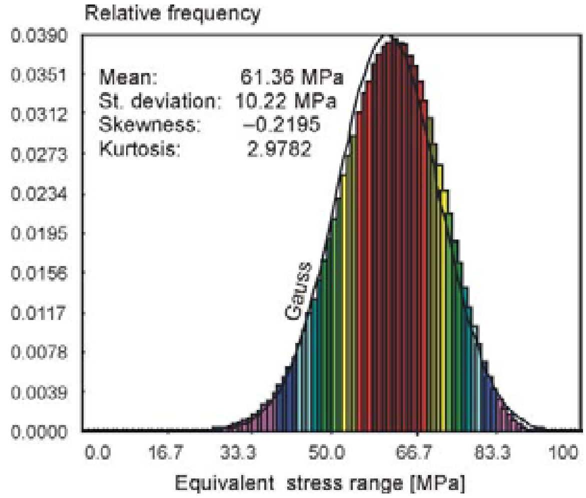

Fig. 4. Density function of equivalent stress range

The value $N = 10\ 000$ was chosen with regard to the computation time and laboriousness of input ordering for 118 755 random realizations $\Delta\sigma_E$. Accuracy of statistical characteristics depends on the number of the LHS method simulation runs.

3.1.2. Example 2

Lognormal density functions of stress range are considered (Fig. 5). The lognormal density function mean value is supposed to lie within the interval $\langle 30; 70 \rangle$ MPa. A set of 1 000 mean values $\{30, 30.04, 30.08,\dots,70\}$ MPa was considered; the difference between neighbouring values is 0.04 MPa. The standard deviation has been derived based on the assumption that 95 % realisations $\Delta\sigma_E$ lie within the interval $\langle 0; 100 \rangle$ MPa. An example of 5 lognormal probability density functions for 5 mean values 30, 40, 50, 60 and 70 MPa is presented in Fig. 5.

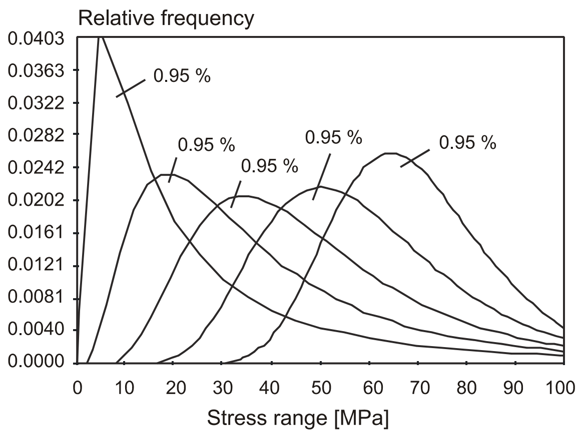

Fig. 5. Lognormal density function of stress range

Analogously as in the previous problem, 4 000 random realisations $\Delta\sigma_E$ were calculated from the relation (10), see Fig. 6. The coefficient m was considered by value $m = 3$; number of simulated realizations $N = 10\ 000$.

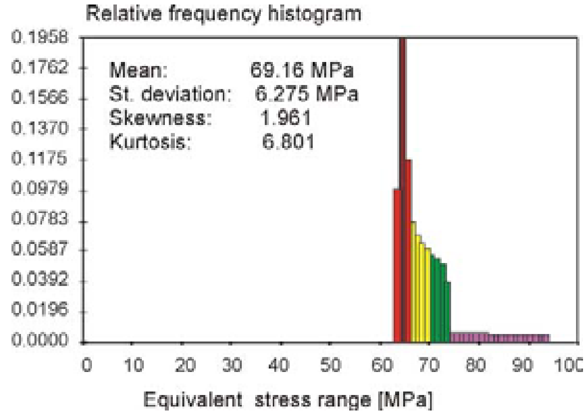

Fig. 6. Density function of equivalent stress range

In both cases the stress range was represented by a set of density functions; due to this, the standard deviation of $\Delta\sigma_E$ is not zero. For a purely stochastic uncertainty, the experiments should by evaluated by one histogram. The uncertainty of the value $\Delta\sigma_E$ is not a typical random uncertainty but a vague (fuzzy) uncertainty following from uncertain experiment conditions (vague histograms) or from the absence of experiments.

3.2. Fuzzy analysis of equivalent stress range

A fuzzy random quantity can be understood as a random quantity measured under uncertain (vague) conditions. Membership functions in Fig. 7 express the fuzzy uncertainty of equivalent stress range values for 20, 40, 60, 80 and 100 %. Supports of fuzzy numbers are 13.7 MPa; 27.4 MPa; 41.1 MPa; 54.8 MPa and 68.5 MPa. The kernel of all the fuzzy numbers is the common value $\Delta\sigma_E = 34.3$ MPa.

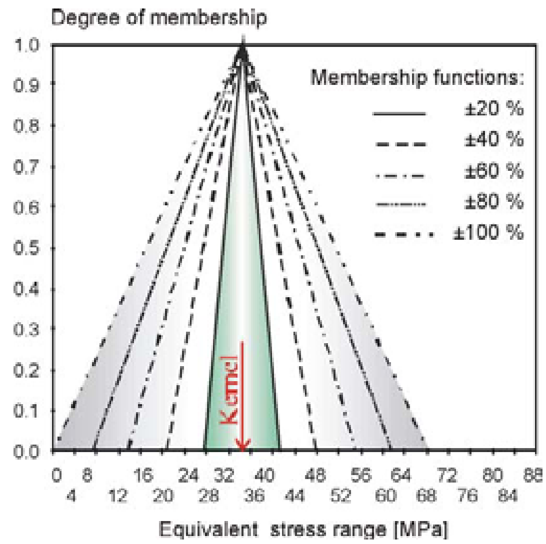

Fig. 7. Fuzzy numbers of equivalent stress range

The membership function has nothing in common with probability. In case of probability, the frequency of the occurrence of a phenomenon which had occurred (was experimentally found) would have to be studied.

4. Fuzzy random analysis of fatigue resistance

The fatigue resistance was defined as the number of cycles causing the initial crack size propagation up to the critical size. The fuzzy random uncertainty of fatigue resistance due to fuzzy equivalent stress range ± 20 % (green area in Fig. 7) and input random quantities were analysed.

4.1. Input random quantities

In compliance with the results of stochastic sensitivity analysis (Kala 2006), the initial crack size presents a dominant random quantity for which the statistical characteristics and the density function type have to be determined with a maximum precision. The initial crack size was modelled by a lognormal distribution (Tomica 2003) (Fig. 8). The problem is how to define the mean value and the standard deviation. According to the published experimental results (Tomica 2003; Hudák *et al.* 1999), based on crack propagating from the surface of weld joints, it is possible, for an initiating crack, to consider the lognormal distribution with mean $m_{a0} = 0.526$ mm and standard deviation $S_{a0} = 0.504$ mm.

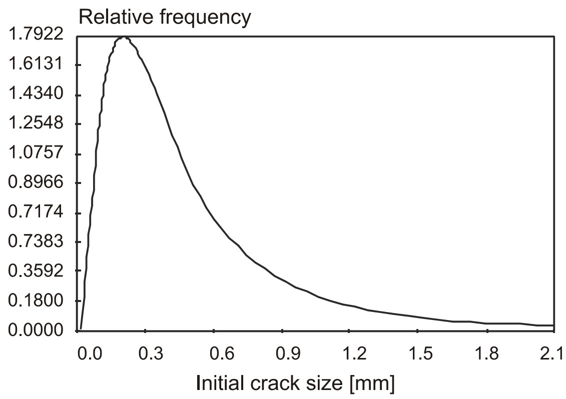

Fig. 8. Lognormal density function of initial crack size

The plate width b and critical length a_{cr} to which the crack propagates without the rise of macroplastic instability were considered as the other random quantities.

The coefficient m which is the function of many factors (Kunz 1991) was introduced randomly as well. The exponent m increases with decreasing fracture toughness. In our study, the parameter m was supposed, in a simplified way, with the Gaussian distribution to have the mean value of 3 and the variation coefficient of 0.01.

The strong correlation between the parameters C and m (Kunz 1991) was confirmed experimentally. Provided that the exponent m is not any universal constant, it follows from the dimensional analysis of the Paris–Erdogan's equation (1), that also the physical dimension of the constant C gets changed in general. According to (Kunz 1991), the mutual relation between C and m can be expressed as follows: $\log(C) = c_1 + c_2 m$, where c_1<0 a c_2<0 are the parameters for the given material grade. In our problem, we considered in compliance with (Kunz 1991), $c_1 = 11.141$, $c_2 = 0.507$ for the steel grade S235. The input random quantities are clearly given in Table.

Input random quantities

	Distribution	Mean	St. deviation
Initial crack size	Lognormal	0.526	0.504
Parameter m	Normal	3	0.03
Thickness	Normal	400	20
Critical crack size	Normal	200	15
Equivalent stress r.	Fuzzy number ± 20 % on Fig. 7		

4.2. Fuzzy random analysis of fatigue resistance

Fatigue resistance was analysed according to (11). The fuzzy analysis was evaluated based on the general extension principle for 11 α–cuts. The statistical analysis was evaluated by applying the LHS method.

The fuzzy analysis procedure may be explained by cut with degree of membership zero (uncertainty ±20 %). The first histogram with mean value $100 \cdot 10^6$ of fatigue resistance is evaluated for equivalent stress range 27.44 MPa (–20 %), and the last histogram with mean value $30 \cdot 10^6$ of fatigue resistance is evaluated for equivalent stress range 41.16 MPa (+20 %). The calculation procedure is analogical for ±18 %, ±16 %, ±14 % and the other 11 α–cuts.

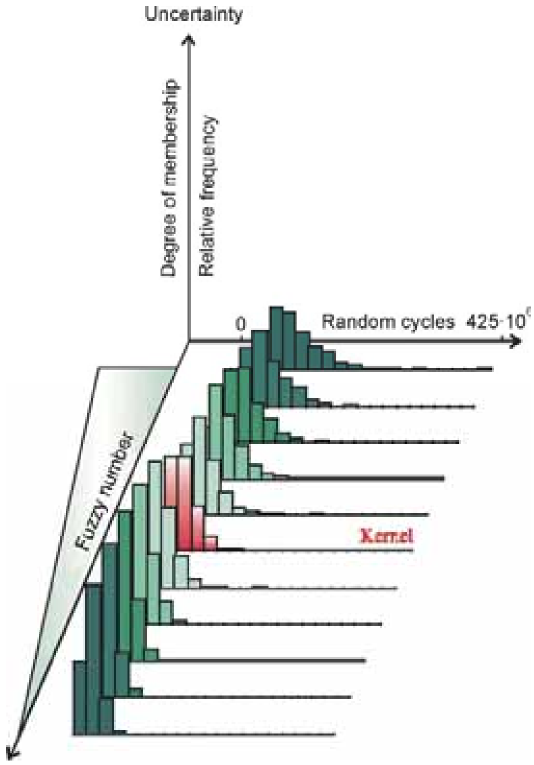

Fig. 9. Fuzzy set of fatigue resistance histogram

The fuzzy set of the fatigue resistance histograms is the output (Fig. 9). It is evident from Fig. 9 that with increasing value of equivalent stress range the average fatigue resistance value decreases. The membership functions of the mean value and standard deviation of the fatigue resistance are illustrated in Fig. 10. Both membership functions are non-linear.

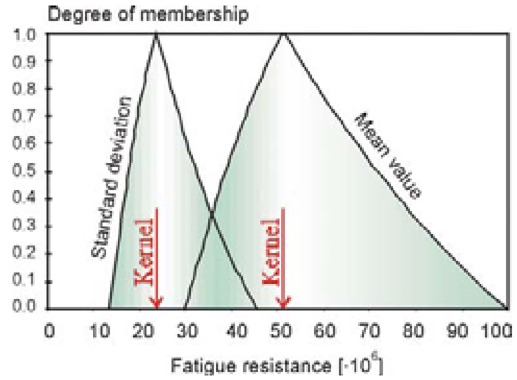

Fig. 10. Fuzzy numbers of mean and standard deviation

5. Conclusion

The output asymmetric non–linear membership functions of mean value and standard deviation of fatigue resistance vs. triangular symmetric membership function of equivalent stress range are obtained. This information is very valuable because it quantifies the non–linear dependence between the equivalent stress range and the theoretical fatigue resistance statistical characteristic.

The epistemic uncertainty of histogram of a stress range is taken into account by taking into account the equivalent stress range as fuzzy numbers. This approach is an alternative to a purely stochastic approach which considers the equivalent stress range as a random quantity, the information necessary for determination of its density function usually being absent. The fuzzy approach considers the input numbers and the output histograms as elements of sets. It extenders further possible analyses by operations with sets (intersection, union, etc.).

The higher level analysis considering the history of loading could be outlined as fuzzy random processes. Specific fuzzy random processes and specific optimization methods applicable to solution of the problem presented here are given, e.g. in (Möller, Reuter 2007; Karkauskas, Norkus 2006).

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FUZI TIKIMYBINĖ ANALIZĖ VERTINANT LENKIAMŲ PLIENINIŲ ELEMENTŲ ATSPARĮ NUOVARGIUI

Z. Kala

Santrauka

Straipsnyje nagrinėjamas atspario nuovargio esant neapibrėžtiems pradiniams duomenims vertinimas naudojant *fuzi* tikimybinę analizę. Plieninių elementų atsparis nuovargiui pagal tiesinę irimo mechaniką apibūdinamas ciklų skaičiumi, kai pradiniai plyšiai perauga į kritinį plyšį, sukeliantį trapų suirimą. Kai įtempimų kitimo histograma yra žinoma, atsparis nuovargiui yra atsitiktinis dydis. Kai histograma yra nežinoma arba ji atitinka mažą eksperimentų skaičių, atsiranda kitas episteminės kilmės neapibrėžtumas. Pateikiami du pagrindiniai būdai, rodantys histogramos neapibrėžtumą. Pirmuoju, stochastiniu būdu, įtempių kitimo diapazono histograma yra modeliuojama ekvivalentinio įtempio kitimu. Antruoju būdu pradinės histogramos nagrinėjamos kaip *fuzi* aibės elementai.

Reikšminiai žodžiai: nuovargis, plienas, plyšio plėtojimasis, suirimas, tikimybė, *fuzi*, modeliavimas, jautrumas.

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