Probabilistic Finite Element for Fracture Mechanics

This paper presents a probabilistic methodology for fracture mechanics analysis of cracked structures. The main focus is on probabilistic aspect which related the nature of crack in material. The methodology involves finite element analysis; statistical models for uncertainty in material properties, crack size, fracture toughness and loads; and standard reliability methods for evaluating probabilistic characteristics of fracture parameter. When a crack is observed, the problem is to know whether it is suitable to repair the structure as a priority or if it can be justified that an accident will not occur. Therefore, the probabilistic analysis can provide the failure probability knowing that there is a crack and that the load can reach accidental values defined in a particular range. The probability of failure caused by uncertainties related to loads and material properties of the structure is estimated using Monte Carlo simulation technique. Numerical examples are presented to show that probabilistic methodology based on Monte Carlo simulation provides accurate estimates of failure probability for use in fracture mechanics.

Keywords: *Probabilistic Fracture Mechanics, Finite Element Method, Probability of Failure, Monte Carlo*

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Introduction

The performance of an engineered system or product is often affected by unavoidable uncertainties (Apostolakis, 1990). It may be attributed to the inhomogeneous material properties. Probabilistic uncertainty analysis quantifies the effect of input random variables on model outputs. The uncertainties inherent in the loading and the properties of mechanical systems necessitate a probabilistic approach as a realistic and rational platform for both design and analysis. Probability theory determines how the uncertainties in crack size, loads, and material properties, when modelled accurately, affect the integrity of cracked structures. Probabilistic fracture mechanics (PFM) provides a more rational means to describe the actual behaviour and reliability of structures than traditional deterministic methods (Provan, 1987).

Several methods with various degrees of complexity that can be used to estimate the reliability or safety index or the probability of failure have been developed or implemented. Many of these methods are applicable when the limit state equations are explicit functions of the random variables involved in a problem. Most of these methods are based on a finite element method (FEM). Although FEM based methods are well developed, research in probabilistic analysis has not been widespread and is only currently gaining attention.

The originality of mean value first-order second moment (MVFOSM) method was introduced by Cornell (1969). The MVFOSM method based on a first-order Taylor series approximation of the performance function linearized at the mean values of the random variables. However MVFOSM method has obvious deficiencies such as it uses only the first two moments of random variables instead of the complete distribution information (Haldar and Mahadevan, 2001, Youn and Choi, 2004) and it assumes that the response is normally distributed.

Then, first-order reliability method (FORM) was introduced which produces more accurate solutions than MVFOSM method. However, FORM may not generate accurate results when transform the original random variables into standard normal variables and increases the nonlinearity of the performance function (Youn and Choi, 2004).

Grigoriu et al. (1990) applied first and second order reliability methods (FORM/SORM) to predict the probability of fracture initiation and a confidence interval of the direction of crack extension. The method can account for random loads, material properties, and crack geometry. However, the randomness in crack geometry was modelled by response surface approximations of stress intensity factor as explicit functions of crack geometry. Furthermore, the usefulness of response surface based methods is limited, since they cannot be applied for general fracture mechanics analysis (Guofeng Chen et al., 2001).

This paper presents a computational methodology for probabilistic characterization of fracture initiation in cracked structures. The methodology based on finite element method for deterministic stress analysis, statistical models for loads and material properties and Monte Carlo method for probabilistic analysis. Examples are presented to illustrate the proposed methodology lead to sufficiently close results for the cracked structures. The results from these examples show that the methodology is capable of predicting deterministic and probabilistic characteristic for use in fracture mechanics.

Finite Element Calculation

In order to perform probabilistic analysis, the finite element analysis needs to be well developed. In this study triangular mesh generation using the advancing front method was used. The mesh finally optimised by smoothing and associated boundary conditions are found by interpolation from the initial geometry conditions, then finally producing the output files. The remeshing algorithms place a rosette of quarter point elements around the crack tip, and then rebuild the mesh around the crack tip. A computer code has been developed using FORTRAN programming language for finite element analysis calculation processes, which is based on displacement control for cracked structure modelling.

The important parameter used in linear elastic fracture mechanics are the stress intensity factors in various modes. In this paper, the stress intensity factors during simulation steps were calculated by using the displacement extrapolation method, which shown to be highly accurate. In this paper, the displacement extrapolation method (Phongthanapanich and Dechaumphai, 2004) is used to calculate the stress intensity factors as follows:

$$K_I = \frac{E}{3(1+\upsilon)(1+\kappa)}\sqrt{\frac{2\pi}{L}}\left(4(v_b - v_d) - \frac{(v_c - v_e)}{2}\right) \quad (1)$$

$$K_{II} = \frac{E}{3(1+\upsilon)(1+\kappa)}\sqrt{\frac{2\pi}{L}}\left(4(u_b - u_d) - \frac{(u_c - u_e)}{2}\right) \quad (2)$$

where E is the modulus of elasticity, v is the Poisson's ratio, κ is the elastic parameter defined by

$$\kappa = \begin{cases} (3-4v) & \text{plane strain} \\ (3-v)/(1+v) & \text{plane stress} \end{cases}$$

and L is the quarter-point element length. The u′ and v′ are the displacement components in the x′ and y′ directions, respectively; the subscripts indicate their position as shown in Figure 1.

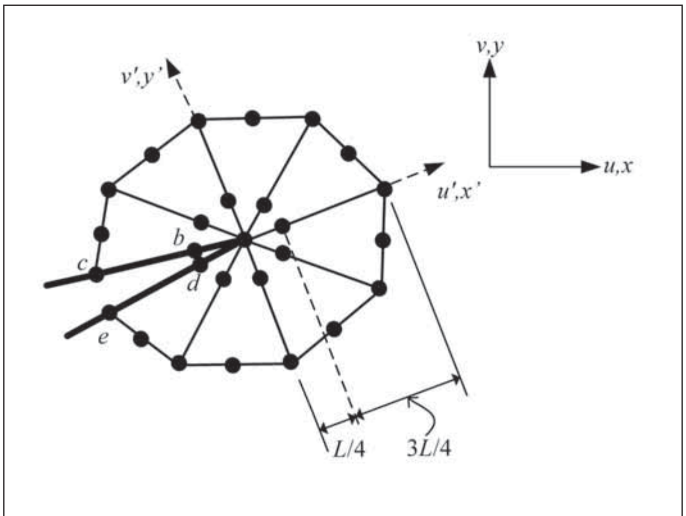

Figure1 – The Arrangements of the Natural Quarter Point Triangular Elements around the Crack Tip

For the elastic plastic materials, the crack tip is blunted by plasticity. Then the crack tip opening displacement (CTOD) introduced by A.A. Wells in 1961, used as a material crack parameter. Sutton et al. (2000) developed the CTOD criterion based on a detailed analysis of crack kinking, and assumed that the crack growth occurs when the current CTOD reaches a critical value.

The mesh refinement guided by a characteristic size of each element, predicted according to a given error rate and the degree of the element interpolation function. The error estimation for the simulation is based on stress smoothing. It was a point wise error in stress indicator (ESI) to evaluate the accuracy of the finite element solution.

In general, the smaller mesh sizes in a finite element mesh, give more accurate finite element approximate solution. However, reduction in the mesh size leads to greater computational effort.

The adaptive mesh refinement is based on a posteriori error estimator which is obtained from the solution from the previous mesh. The error estimator used in this paper is based on stress error norm. The strategy used to refine the mesh during analysis process is adopted from Alshoaibi et al. (2007) as follows:

(i) Determine the error norm for each element

$$\|e\|^e = \int_{\Omega^e} (\sigma - \sigma^*)^T (\sigma - \sigma^*) d\Omega \quad (3)$$

where σ is the stress field obtained from the finite element calculation and σ^* is the smoothed stress field.

(ii) Determine the average error norm over the whole domain

$$\|\hat{e}\| = \frac{1}{m}\sum_{e=1}^{m} \int_{\Omega^e} \sigma^T \sigma \, d\Omega \quad (4)$$

where m is the total number of elements in the whole domain.

(iii) Determine a variable, ε_e for each element as

$$\varepsilon_e = \frac{1}{\eta}\frac{(\|e\|^e)^{1/2}}{(\|\hat{e}\|)^{1/2}} \quad (5)$$

where η is a percentage that measures the permissible error for each element. If $\varepsilon_e > 1$ the size of the element is reduced and vice versa.

(iv) The new element size is determined as

$$\hat{h}_e = \frac{h_e}{(\varepsilon_e)^{1/p}} \quad (6)$$

where h_e is the old element size and p is the order of the interpolation shape function.

Monte Carlo Simulation Technique

The reliability or the probability of failure can be estimated by using several methods with various degrees of complexity such as FORM, first-order second-moment method (FOSM), Hasofer-Lind method and second-order reliability method (SORM). Many of these methods are applicable when the limit state equations are explicit functions of the random variables involved in a problem. But with a simulation technique, it is possible to calculate the probability of failure for both the explicit or implicit limit state functions. In fact, to evaluate the accuracy of these sophisticated technique, simulation is routinely used to independently evaluate the underlying probability of failure. The Monte Carlo simulation technique is the method commonly used for this purpose. This technique has evolved as a very powerful tool for engineers for evaluating the risk or reliability of complicated engineering systems.

The Monte Carlo simulation technique has five essential elements: (1) the problem in terms of all the random variables are defined; (2) the probabilistic characteristics of all the random variables in terms probability density functions (PDFs) and the corresponding parameters are quantified; (3) the values of these random variables are generated; (4) the problem evaluated deterministically for each set of realizations of all the random variables; (5) probabilistic information from number of simulations, such realization are extracted.

Formulation of the Problem

Consider a cracked structure under uncertain mechanical and geometric characteristics subject to random loads. Denote by X an N-dimensional random vector with components $X_1, X_2, \ldots, X_N$ characterizing uncertainties in the load, crack geometry, and material properties. For example, if the crack size a, elastic modulus E, far field applied stress magnitude σ^∞, and mode I fracture toughness at crack initiation K_{Ic}, are modelled as input random variables, then $X = g(a, E, \sigma^\infty, K_{Ic})$. Let stress intensity factor K, be a relevant crack driving force that can be calculated using standard finite element analysis. Suppose the structure fails when $K > K_{Ic}$. This requirement cannot be satisfied with certainty, since K is dependent on the input vector X which is random, and K_{Ic} itself to be a random variable. The K is evaluated by finite element method which can be expressed in Equation (1).

Quantifying the Probabilistic Characteristics of Random Variables

Mathematical modelling or representation of a random variable is thus a primary task in any probabilistic formulation, which needs to be conducted systematically. The double edged notched tension (DENT) specimen is considered to carry comprehensively evaluate the modelling of uncertainty by the developed program. Information on modulus of elasticity is presented in Table 1. Similar information can also be obtained for other random variables of interest.

Test no.	Elastic modulus, E(GPa)	Test no.	Elastic modulus, E(GPa)
1	72.1	21	79.3
2	78.5	22	72.3
3	72.4	23	67.4
4	73.5	24	72.5
5	72.8	25	72.0
6	66.5	26	77.7
7	73.1	27	70.0
8	73.4	28	80.0
9	73.5	29	73.8
10	68.6	30	71.1
11	71.9	31	73.0
12	74.1	32	70.8
13	67.6	33	75.6
14	75.2	34	74.4
15	70.8	35	71.6
16	76.8	36	65.0
17	70.7	37	68.8
18	71.3	38	71.2
19	69.8	39	71.7
20	71.4	40	71.8

Table 1 – Elastic Modulus E for the DENT Specimen

The mean or expected value of X, a measure of central tendency in the data, also known as the first central moment and denoted as E(X) or μ_x, can be calculated for the n observations as

$$E(X) = \mu_x = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{7}$$

The variance of X, a measure of spread in the data about the mean, also known as the second central moment and denoted hereafter as Var(X), can be estimated as

$$Var(X) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mu_x)^2 \tag{8}$$

The standard deviation, denoted as σ_X can be calculated as

$$\sigma_X = \sqrt{Var(X)} \tag{9}$$

Since the mean and the standard deviation values are expressed in the same units, a nondimensional term can be introduced by taking the ratio of the standard deviation and the mean. This is called the coefficient of variation (COV) and denoted as COV(X) or δ_X. Thus,

$$COV(X) = \delta_X = \frac{\sigma_X}{\mu_x} \tag{10}$$

A smaller value of the COV indicates a smaller amount of uncertainty or randomness in the variable and larger amount indicates a larger amount of uncertainty. In many engineering problems, a COV of 0.1 to 0.3 is common for a random variable (Haldar and Mahadevan, 2000).

Determination of Probability Distribution

In practice, the choice of probability distribution may be dictated by mathematical convenience or by familiarity with a distribution. In some cases, the physical process may suggest a specific form of distribution. As an example, elastic modulus E is frequently modelled as a Gaussian random variable for DENT specimen. The task is to establish its validity, based on sample information such as that given in Table 1. The underlying distribution can be established by conducting some statistical tests known as Goodness-of-fit tests for distribution.

Two commonly used statistical tests for this purpose are the Chi-square (χ^2) and the Kolmogorov-Smirnov (K-S) tests. Firstly, the K-S test is used because it is not necessary to divide the data into intervals so the error or judgment associated with the number or size of the interval is avoided. The K-S test compares the observed cumulative frequency and the cumulative density function (CDF) of an assumed theoretical distribution. The data was arranged in increasing order for the first step. Then the maximum difference between the two CDFs of the ordered data estimated by using

$$D_n = \max|F_X(x_i) - S_n(x_i)| \tag{11}$$

where $F_X(x^i)$ is the theoretical CDF of the assumed distribution at the ith observation of the ordered samples x_i, and $S_n(x_i)$ is the corresponding stepwise CDF of the observed ordered samples. $S_n(x_i)$ can be estimated as

$$S_n(x_i) = \left\{\begin{array}{l} 0,\ x < x_1 \\ \dfrac{m}{n},\ x_m \le x \le x_{m+1} \\ 1,\ x \ge x_n \end{array}\right\} \tag{12}$$

The concept is shown in Figure 2. Mathematically, D_n is a random variable and its distribution depends on the sample size n. The CDF of D_n can be related to the signif-icance level α as

$$P(D_n \le D_n^{\alpha}) = 1 - \alpha \tag{13}$$

and the D_n^{α} values at various significance levels α can be obtained from a standard mathematical table as shown in Appendix 1. Then, according to the K-S test, if the maximum difference D_n is less than or equal to the tabulated value D_n^{α} , the assumed distribution is acceptable at the significance level α.

The elastic modulus E data given in Table 1 are considered. For Gaussian distribution, the two parameters are $\mu_E = 72.4$ GPa and $\sigma_E = 3.259$ GPa. The maximum differences D_n for the Gaussian distributions is calculated as shown in Table 2 and is found to be 0.1121. The result for the Gaussian distribution is plotted in Figure 2. For a 5% significance level and 40 sample points, $D_{40}^{0.05}$ is found to be 0.21 from Appendix 1. Thus, Gaussian distribution is acceptable with 5% significance level for the K-S test.

Generation of Random Numbers

The elastic modulus E is a Gaussian random variable with $\mu_E = 72.4$ GPa and $\sigma_E = 3.259$ GPa, and far field tensile stress σ^{∞} is a uniformly distributed random variable between 48.3 and 103.4 MPa. Both are statistically independent random variables. N random numbers generated for elastic modulus E according to its probabilistic characteristics and another N random numbers for σ^{∞}, which is uniformly distributed.

The generation of random numbers according to a specific distribution is the heart of Monte Carlo simulation. In general, all modern computers

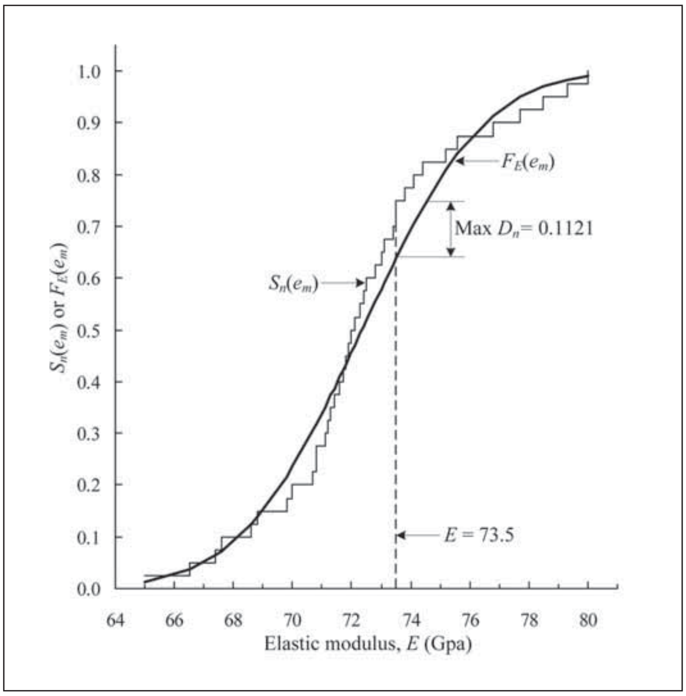

Figure 2 – K-S Test on Elastic Modulus for Gaussian Distribution

m	E(GPa)	$S_n(e_m) = m/n$	$F_E(e_m)$	$D_n = \lvert F_E(e_m) - S_n(e_m)\rvert$
1	65.0	0.0250	0.0121	0.0129
2	66.5	0.0500	0.0363	0.0137
3	67.4	0.0750	0.0644	0.0106
4	67.6	0.1000	0.0725	0.0275
5	68.6	0.1250	0.1249	0.0001
6	68.8	0.1500	0.1380	0.0120
7	69.8	0.1750	0.2170	0.0420
8	70.0	0.2000	0.2354	0.0354
9	70.7	0.2250	0.3063	0.0813
10	70.8	0.2500	0.3172	0.0672
11	70.8	0.2750	0.3172	0.0422
12	71.1	0.3000	0.3506	0.0506
13	71.2	0.3250	0.3621	0.0371
14	71.3	0.3500	0.3737	0.0237
15	71.4	0.3750	0.3853	0.0103
16	71.6	0.4000	0.4090	0.0090
17	71.7	0.4250	0.4210	0.0040
18	71.8	0.4500	0.4330	0.0170
19	71.9	0.4750	0.4451	0.0299
20	72.0	0.5000	0.4572	0.0428
21	72.1	0.5250	0.4694	0.0556
22	72.3	0.5500	0.4939	0.0561
23	72.4	0.5750	0.5061	0.0689
24	72.5	0.6000	0.5184	0.0816
25	72.8	0.6250	0.5549	0.0701
26	73.0	0.6500	0.5790	0.0710
27	73.1	0.6750	0.5910	0.0840
28	73.4	0.7000	0.6263	0.0737
29	73.5	0.7250	0.6379	0.0871
30	73.5	0.7500	0.6379	0.1121
31	73.8	0.7750	0.6718	0.1032
32	74.1	0.8000	0.7044	0.0956
33	74.4	0.8250	0.7353	0.0897
34	75.2	0.8500	0.8091	0.0409
35	75.6	0.8750	0.8407	0.0343
36	76.8	0.9000	0.9140	0.0140
37	77.7	0.9250	0.9497	0.0247
38	78.5	0.9500	0.9704	0.0204
39	79.3	0.9750	0.9835	0.0085
40	80.0	1.0000	0.9905	0.0095

Table 2 – K-S Test on Elastic Modulus for DENT Specimen

have the capability to generate uniformly distributed random numbers between 0 and l. Corresponding to an arbitrary seed value, the generators produced the required number of uniform random numbers between 0 and 1. By changing the seed value, different sets of random numbers can be generated. Depending upon the size of the computer, the random numbers may be repeated. Random numbers generated this way are called pseudo random numbers. Fifty random numbers for a uniform distribution between 0 and 1 are given in Table 3. These random numbers will be used in the subsequent discussion.

Then, the uniform random numbers u_i between 0 and 1, transformed to random numbers with the appropriate characteristics. The process is shown graphically in Figure 3. This is commonly known as the inverse transformation technique or inverse CDF method. In this method, the CDF of the random variable is equated to the generated random number u_i, that is, $F_X(x_i) = u_i$, and the equation is solved for x_i as

$$x_i = F_X^{-1}(u_i) \quad (14)$$

0.86061	0.15017	0.42171	0.48932	0.73958
0.92546	0.74098	0.95349	0.54707	0.51527
0.41806	0.58515	0.16119	0.64271	0.63765
0.28964	0.70074	0.58394	0.66930	0.52224
0.14225	0.09666	0.95626	0.27681	0.46079
0.44961	0.97948	0.20661	0.90451	0.17326
0.24653	0.65400	0.24566	0.79163	0.10593
0.21687	0.67980	0.94934	0.42397	0.72448
0.56503	0.46872	0.16118	0.68086	0.44245
0.40015	0.12846	0.01988	0.82174	0.37091

Table 3 – Uniform random numbers between 0 and 1

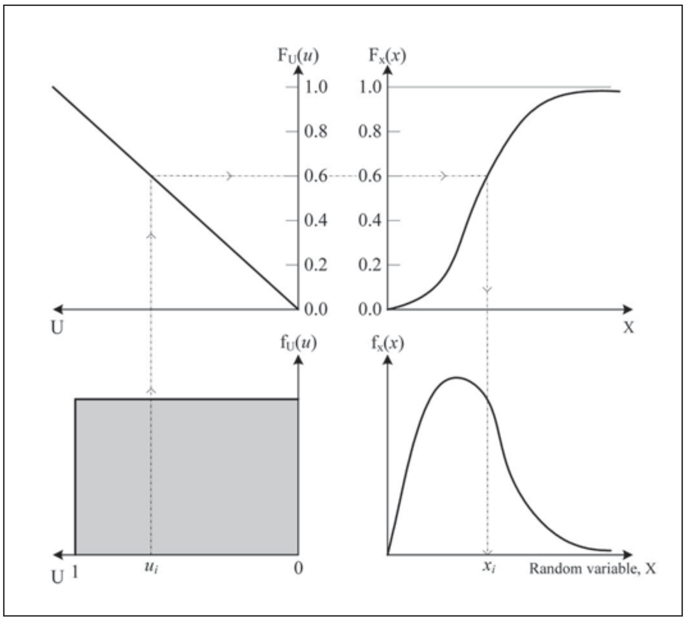

Figure 3 – Mapping for Simulation

A simple example to describe the technique is the transformation of a uniform random number U between 0 and 1, such as $u_1 = 0.86061$ (the first number in Table 3), to another uniform random number x_1 between two limits a and b. The CDF of U is u_i. Since X is uniform, its CDF will be $F_X(x) = (x - a)/(b - a)$. The transformation to obtain the corresponding x_i value can be accomplished by equating the two CDFs as

$$u_i = \frac{x_i - a}{b - a}$$

$$x_i = a + (b - a)u_i \quad (15)$$

When a = 0 and b = 1, $x_i = u_i$, which is obvious. If X is uniform between 10 and 20, the corresponding first random number is

$$x_i = 10 + (20 - 10)0.86061 = 18.6061$$

If X is normally distributed, that is, $N(u_X, \sigma_X)$, then $S = (X - \mu_X)/\sigma_X$ is a standard normal variate, that is, N(0, 1). It can be shown that

$$u_i = F_X(x_i) = \Phi(s_i) = \Phi\left(\frac{x_i - \mu_X}{\sigma_X}\right) \quad (16)$$

or

$$s_i = \frac{x_i - \mu_X}{\sigma_X}$$

Thus

$$x_i = \mu_X + \sigma_X s_i = \mu_X + \sigma_X \Phi^{-1}(u_i) \quad (17)$$

For Equation (17), the u_i values first need to be transformed to s_i, that is, $s_i = \Phi^{-1}(u_i)$, and Φ^{-1} is the inverse of the CDF of a standard normal variable.

Table 4 shows the set of 50 standard normal random numbers corresponding to the uniform random numbers between 0 and 1 given in Table 3. The x_i values can be calculated from the information on the s_i values. For $u_1 = 0.86061$, $s_1 = \Phi^{-1}(0.86061) = 1.08306$; with the information on μ_X and σ_X, the corresponding x_1 can be calculated.

If the random variable X is lognormally distributed with parameters λ_X and ζ_X, then the ith random number x_i according to the lognormal distribution can be generated as

$$u_i = \Phi\left(\frac{\ln x_i - \lambda_X}{\zeta_X}\right) \quad (18)$$

or

$$x_i = \exp[\lambda_X + \zeta_X \Phi^{-1}(u_i)] \quad (19)$$

1.08306	-1.03571	-0.19750	-0.02677	0.64205
1.44279	0.64637	1.67968	0.11826	0.03829
-0.20686	0.21509	-0.98958	0.36571	0.35218
-0.55444	0.52653	0.21198	0.43798	0.05578
-1.07027	-1.30081	1.70884	-0.59234	-0.09844
-0.12665	2.04313	-0.81824	1.30769	-0.94136
-0.68545	0.39614	-0.68821	0.81209	-1.24847
-0.78281	0.46714	1.63849	-0.19175	0.59620
0.16373	-0.07849	-0.98962	0.47011	-0.14476
-0.25296	-1.13370	-2.05621	0.92202	-0.32944

Table 4 – Standard Normal Random Numbers Corresponding to the Uniform Numbers in Table 3

Evaluation of the Problem

The N generated random numbers for each of the random variables in the problem gave N sets of random numbers, each set representing a realization of the problem. The generated sample points for the output or response, then used to calculate the probability of failure considering various performance criteria. The accuracy of the evaluation will increase as the number of simulations M, increases. The DENT and cracked pipe specimens are considered to carry comprehensively evaluate the modelling of uncertainty by the developed program in the forthcoming section.

Probability of failure

Consider the limit state represented by $X = g(a, E, \sigma^{\infty}, K_{Ic})$ corresponding to a failure mode for a structure. With all the random variables assumed to be statistically independent, the Monte Carlo simulation approach consists of drawing samples of the variables according to their PDFs and then feeding them into the mathematical model g( ). The samples thus obtained gave the probabilistic characteristics of the response random variable X. It is known that if the value of K is over than K_{Ic}, it indicates failure. Let M_f be the number of simulation cycles when K is over than K_{Ic} and let M be the total number of simulation cycles. Therefore, an estimate of the probability of failure P_f can be expressed as

$$P_f = \frac{M_f}{M} \quad (20)$$

Numerical Examples

In order to carry out comprehensive evaluation of the Monte Carlo simulation technique by the developed program, two well-known plate geometries namely, DENT and cracked pipe are considered.

Double Edged Notched Tension (DENT)

Consider a two dimensional DENT specimen subjected to quasi-static far field tension stress σ^{∞}. The geometry of the DENT specimen, shown in Figure 4a, has width 2W, length 2L and crack length a. The load, crack size and material properties were treated as statistically independent random variables. The Poisson's ratio of v = 0.3 was assumed to be deterministic. Figure 4b depicts a finite element mesh of DENT specimen. A total of 1184 elements and 2451 nodes were used in the mesh. Both plane stress and plane strain conditions were studied. Focused elements were used in the vicinity of crack tip.

The probabilistic characteristics of the stress intensity factor for the DENT specimen represented by Equation (1) can now be generated using the Monte Carlo simulation technique. For the sake of brevity, only 10 simulation cycles results are shown here. Again, elastic modulus E is N(72.4 GPa, 3.259 GPa) and σ^{∞} is uniform between 48.3 and 103.4 MPa. Since σ^{∞} is uniformly distributed, its mean value and COV can be calculated. In this case, the mean value is $\mu_{\sigma^{\infty}} = (48.3 + 103.4) / 2 = 75.85$ MPa and the COV is

$$\delta_{\sigma^{\infty}} = \frac{2}{\sqrt{12}}\left[\frac{(103.4 - 48.3)}{(103.4 + 48.3)}\right] = 0.21$$

and the corresponding standard deviation is $\sigma_{\sigma^{\infty}} = 0.21\ (75.85) = 15.93$ MPa.

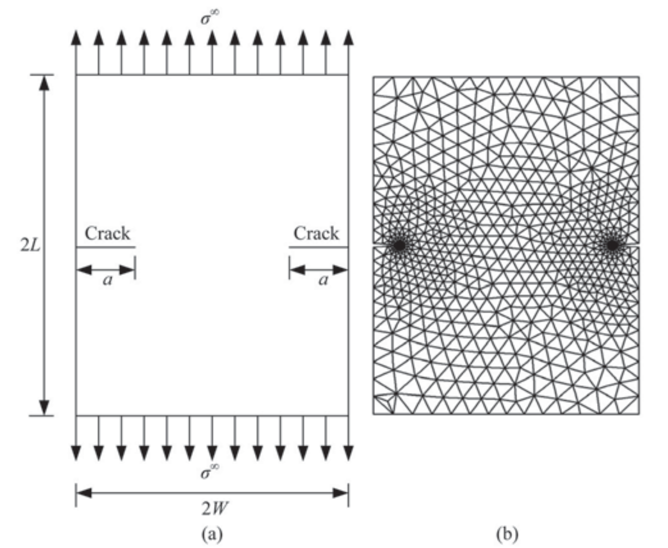

Figure 4 – A DENT Specimen Under Far-field Uniform Tension; (a) Geometry and Loads, (b) Finite Element Mesh

Suppose 10 uniform random numbers between 0 and 1 (the first 10 numbers in Table 3) are generated for elastic modulus E and another 10 (the next 10 numbers in Table 3) are generated for σ^∞. The steps involved in generating a set of random numbers for e_i and σ^∞_i according to their statistical characteristics are summarized in Table 5. For the elastic modulus E under consideration, the first random number according to the normal distribution is

$$e_1 = 72.4 + 3.259(1.08306) = 75.92969$$

The k_{Ii} values were calculated base on generated a set of random numbers for e_i and σ^∞_i.

	E~N(72.4,3.259)		σ^∞ Uniform between 48.3 and 103.4MPa	
u_i	s_i	e_i	u_i	σ^∞_i
0.86061	1.08306	75.92969	0.83771	94.45782
0.92546	1.44279	77.10205	0.73006	88.52631
0.41806	-0.20686	71.72584	0.56341	79.34389
0.28964	-0.55444	70.59308	0.82178	93.58008
0.14225	-1.07027	68.91199	0.32715	66.32597
0.44961	-0.12665	71.98725	0.68853	86.23800
0.24653	-0.68545	70.16612	0.74358	89.27126
0.21687	-0.78281	69.84882	0.24672	61.89427
0.56503	0.16373	72.93360	0.90324	98.06852
0.40015	-0.25296	71.57560	0.79263	91.97391

Table 5 – Monte Carlo Simulations

Using the sample points for the stress intensity factor thus generated, the mean and standard deviation are can be calculated, respectively. This example indicates the power and simplicity of the simulation technique. With the data given in Table 1 and the corresponding equations, the following information in Table 6 can be calculated.

Random variable	Mean	COV	Probability distribution
Normalised crack length, a/w	0.5	Variable[a]	Lognormal
Elastic modulus, E	72.4 GPa	0.05	Gaussian
Initiation fracture toughness, K_{Ic}	24.83 MPa.m$^{1/2}$	0.51	Lognormal
Far field tensile stress, σ^∞	75.85 MPa	0.21	Gaussian

[a] Arbitrarily varied.

Table 6 – Statistical Properties of Random Input for DENT Specimen

The probability of failure of the DENT specimen calculated by using Monte Carlo simulation technique in two different ways; by considering the statistics of the limit state equation, and by counting the failures in different cycles of simulations. In the first method, using the mean and the standard deviation of K_{Ic}, the safety indices calculated as shown in Table 7. Assuming K_{Ic} is a lognormal random variable, the corresponding probability of failure is shown Column 3.

A number of probabilistic analyses were performed to calculate the probability of failure P_f of the DENT specimen, as a function of mean far field tensile stress $E[\sigma^\infty]$, where E[.] is the expectation (mean) operator. Figure 5 plots the P_f versus $E[\sigma^\infty]$ results for $v_{a/W}$ = 20 percent and the plane stress condition, where $v_{a/W}$ is the COV of the normalised crack length a/W. As can be seen in Figure 2, the probability of failure by Guofeng Chen et al. (2001) and FORM are in good agreement with the present study results.

Figure 6a and 6b indicates the plots of P_f versus $E[\sigma^\infty]$ using FORM and present study methodology for plane stress and plane strain conditions, for both deterministic ($v_{a/W}$ = 0) and random ($v_{a/W}$ = 10, $v_{a/W}$ = 20, $v_{a/W}$ = 40 percent) crack sizes. The results indicate that the failure probability increases with the COV of a/W, and can be much larger than the probabilities calculated for a deterministic crack size, particularly when the uncertainty of a/W is large. The predicted finite element results from this study well matched with the FORM results. This indicates that the Monte Carlo method provides accurate estimates of failure probability for use in fracture mechanics.

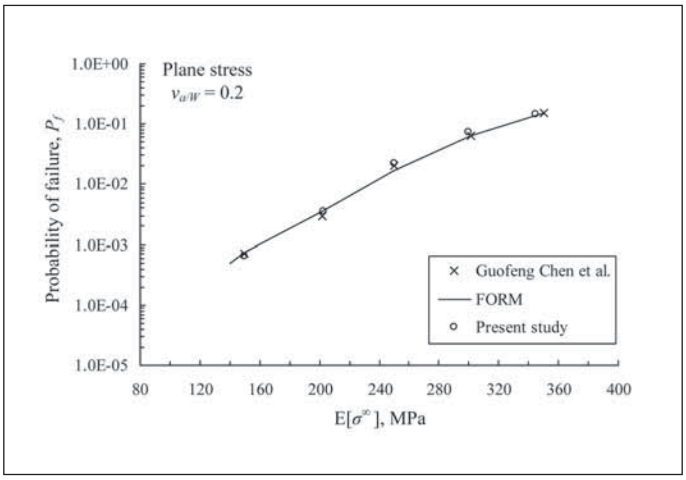

Figure 5 – Failure Probability of DENT Specimen by Guofeng Chen et al. (2001), FORM and Present Study

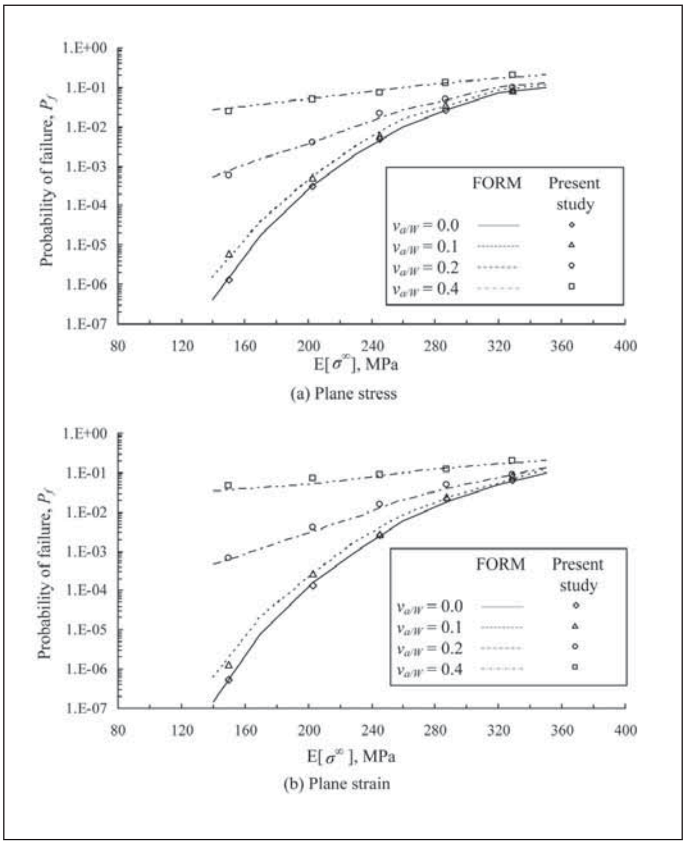

Figure 6 – Failure Probability of DENT Specimen by FORM and Present Study for Various Uncertainties in Crack Size; (a) Plane Stress, (b) Plain Strain

Cracked Pipe

The pipes of nuclear plants undergo great thermal and mechanical cycles which can lead to initiation and propagation of cracks. When a crack is observed, the problem is to know whether it is suitable to repair the structure as a priority or if it can be justified that an accident will not occur. Therefore, the probabilistic analysis can provide the failure probability knowing that there is a crack and that the load can reach accidental values defined in a particular range.

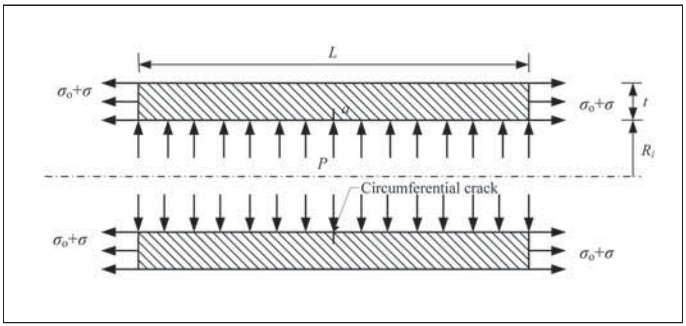

Figure 7 – Axisymmetrically Cracked Pipe

Figure 7 shows an axisymmetrically cracked pipe under internal pressure and axial tension. Due to the boundary conditions at the pipe ends, the applied hydraulic pressure induces, beside the radial pressure, longitudinal tension forces.

The system variables are described as follows:

- a, the crack length (15 mm)
- L, the pipe length (1000 mm)
- P, the internal pressure (15.5 MPa)
- Ri, the inner radius (393.5 mm)
- t, the thickness (62.5 mm)
- σ, the applied tensile stress (varying from 100 up to 200 MPa). It represents the load effect which could accidentally increase, knowing that the nominal value is around 100 MPa.
- σ_0, the stress due to the end effects, given by

$$\sigma_0 = P \frac{R_i^2}{(R_i + t)^2 - R_i^2}$$

Figure 8 depicts an adaptive finite element mesh of cracked pipe. A half model was used to take advantage of the symmetry. Table 8 lists the means, COV and probability distributions of elastic modulus, crack tip opening angle, applied tensile stress and yield strength. The Poisson's ratio of v = 0.3 was assumed to be deterministic.

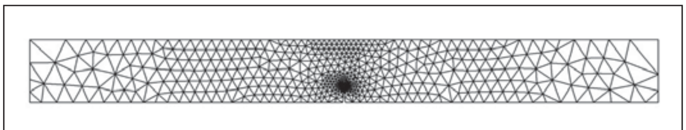

Figure 8 – An Adaptive Mesh for Cracked Pipe

Random variable	Mean	COV	Probability distribution
Elastic modulus, E	175.5GPa	0.05	Lognormal
Crack tip opening angle, CTOA	5.25°	0.15	Gaussian
Applied tensile stress, σ	150MPa	0.19	Gaussian
Yield strength, σ_y	260.5MPa	0.05	Lognormal

Table 8 – Statistical Properties of Random Input for Cracked Pipe

Figure 9 shows the comparisons of the probability of failure, P_f using present study method and published results done by Pendola et al. (2000) for the cracked pipe. The continuous lines in Figure 9 represent the values of P_f obtained from combinations of ANSYS-RYFES software. The circle points in Figure 10 indicate the P_f from this study involving elastic-plastic analysis. The P_f values from this study are comparatively closer to the Pendola et al. (2000) solution. The calculation of P_f is much simpler than the analysis presented in this paper.

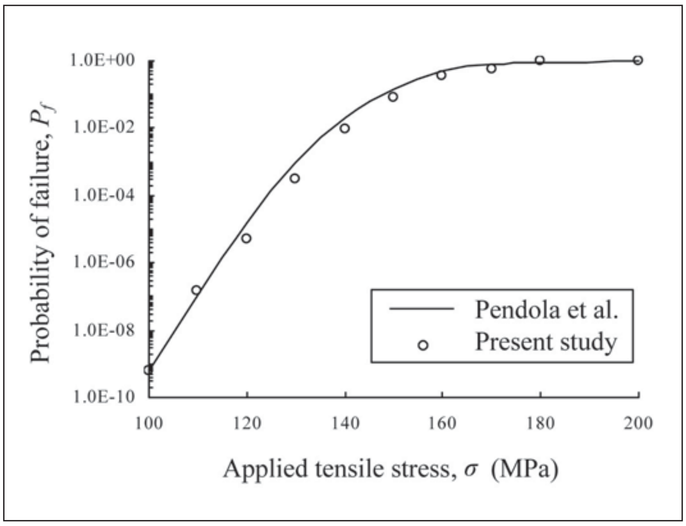

Figure 9 – Failure Probability of Cracked Pipe by Pendola et al. (2000) and Present Study

Conclusions

The probabilistic method has been presented for fracture mechanics analysis of cracked structures. The numerical examples presented in this paper are derived on linear-elastic and elastic-plastic fracture mechanics based failure criterion. The methodology involves development of finite element analysis codes, statistical models for uncertainty and probabilistic analyses using Monte Carlo simulation. The numerical implementations lead to sufficiently close results and attest the quality of the solution of the cracked model. The calculation of P_f is much simpler than the analysis presented in this paper. The results from these examples indicate that the methodology is capable of determining accurate probabilistic analyses in fracture mechanics.

Appendix 1

n	$D_n^{0.2}$	$D_n^{0.15}$	$D_n^{0.1}$	$D_n^{0.05}$	$D_n^{0.01}$
5	0.446	0.474	0.510	0.563	0.669
6	0.410	0.436	0.470	0.521	0.618
7	0.381	0.405	0.438	0.486	0.577
8	0.358	0.381	0.411	0.457	0.543
9	0.339	0.360	0.388	0.432	0.514
10	0.322	0.342	0.368	0.409	0.486
11	0.307	0.326	0.352	0.391	0.468
12	0.295	0.313	0.338	0.375	0.450
13	0.284	0.302	0.325	0.361	0.433
14	0.274	0.292	0.314	0.349	0.418
15	0.266	0.283	0.304	0.338	0.404
20	0.231	0.246	0.264	0.294	0.352
25	0.210	0.220	0.240	0.264	0.320
30	0.190	0.200	0.220	0.242	0.290
35	0.180	0.190	0.210	0.230	0.270
40	0.170	0.180	0.190	0.210	0.250
45	0.160	0.170	0.180	0.200	0.240
50	0.150	0.160	0.170	0.190	0.230
>50	$\frac{1.07}{\sqrt{n}}$	$\frac{1.14}{\sqrt{n}}$	$\frac{1.22}{\sqrt{n}}$	$\frac{1.36}{\sqrt{n}}$	$\frac{1.63}{\sqrt{n}}$

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