Splitting Strategies for Preconditioned Explicit Group Schemes
(Strategi Pembelahan bagi Kaedah Kumpulan Tak Tersirat Berprasyarat)

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ABSTRACT

Combining iterative methods with appropriate preconditioners is a worthwhile effort in improving the performances of the methods since the reliability of these methods have been shown to improve by the use of appropriate preconditioning techniques. However, the hardest issue is to find the suitable preconditioners which are computationally inexpensive and easy to solve for the group methods. The aim of this paper is to study the performance of the Explicit Group (EG) method preconditioned by a specific ‘splitting’ approach in solving the two dimensional elliptic partial differential equation. Our goal in this work is to investigate whether the performance of this group method is affected by this preconditioner. The experimental work performed is reported and discussed.

Keywords: preconditioner; explicit group methods; elliptic equation

INTRODUCTION

Explicit group (EG) iterative methods for the numerical solution of sparse linear systems derived from the discretisation of self-adjoint elliptic partial have been shown to be computationally superior than the existing line Successive Over-Relaxation (SOR) iterative methods. The study of numerical solutions by this novel approach of using a small fixed size group strategy by the SOR iterative method indicates that these group methods require less storage and are simpler to program than the classic one and two-line iterative methods (Evans & Yousif 1990, Yousif & Evans 1986). The construction of new grouping of the mesh points into smaller size groups of 2, 4, 6, 9, 12, 16, and 25 points were extensively investigated by Yousif and Evans (1986), where their analysis indicate that the 4-point grouping of mesh points is more efficient than the one line and two-line block methods and any other alternative grouping in this class of algorithms. One of the main reasons behind this is due to the inherent symmetry which the 4-point group produces which enables more efficient manipulation of the algorithms by reducing the multiplicative operations required in solving the problem. Since the reliability and robustness of iterative methods may now be improved by the use of preconditioning techniques, hence further efforts are being taken to combine the EG with appropriate preconditioners as a way to further improve the performance of the method.

The aim of this paper is to study the performance of the EG method preconditioned by a left-right ‘splitting’ approach introduced by Evans (1968). We will investigate whether this preconditioner is capable of improving the convergence rate of the original method without jeopardising the accuracy of the method. The paper is organised in five sections. Section 2 will elaborate on the EG iterative method in solving the boundary value problem consisting of an elliptic problem with Dirichlet boundary condition. This is followed by a description of the ‘splitting’ preconditioner in Section III. Section IV will discuss the numerical experiments carried out and the results obtained. Concluding remarks are given in Section V.

II. THE EXPLICIT GROUP ITERATIVE METHOD
Consider the finite difference discretization schemes for solving the two-dimensional elliptic equation with Dirichlet boundary conditions

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \quad (x, y) \in \Omega \]

Here \( \Omega \) is a continuous unit square solution domain. Assuming \( n \) to be odd, the mesh points are grouped in blocks of four points and the centred difference equation is applied to each of these points resulting in the following \((4 \times 4)\) system (Ali et al. 2003): (here, \( f \), and \( h = 1/n \))

This \((4 \times 4)\) system can be easily inverted to produce a four-point explicit group equation whose individual explicit equations are given by

\[ u_{i,j} = \frac{1}{4} [r_1 + 2r + 3r_4 + 4r_3] \]
\[ u_{i,j+1} = \frac{1}{3} [r_1 + 2r + 3r_4 + 4r_3] \]
\[ u_{i+1,j} = \frac{1}{3} [r_1 + 2r + 3r_4 + 4r_3] \]

where \( r_1 = u_{i,j} + u_{i+1,j} - h^2 f_{i,j} \)
\[ r_2 = u_{i+1,j} + u_{i+1,j+1} - h^2 f_{i+1,j+1} \]
\[ r_3 = u_{i,j+1} + u_{i+1,j+1} - h^2 f_{i+1,j+1} \]
\[ r_4 = u_{i+1,j+1} + u_{i+1,j} - h^2 f_{i+1,j} \]

We can establish the computational molecule at the point \((i,j)\) to be as shown in Fig. 1:
using formula (2.3) throughout the whole solution domain results in a linear system of the form $A\mathbf{u} = \mathbf{d}$, with $A = D - L - U$, where $L$ and $U$ are respectively lower and upper triangular matrices, and $D$ is a diagonal unit matrix. We transform this original system into the following preconditioned form:

$$M^{-1} A \mathbf{u} = M^{-1} \mathbf{d}$$

where $M$ is $(I - wL)$. Using an intermediate transformation vector and $w$, an acceleration factor to be defined later, the system (3.1) may be simplified to become

$$Bw \mathbf{u} = \mathbf{b}$$

or

$$\frac{1}{(1-wL)} \mathbf{x} = \mathbf{y}$$

(3.3)

It can be observed that system (3.4) reverts back to the original system (3.4) when $w = 0$. We need to solve the system (3.4) which is a function of $w$ with $w$ chosen so that the $p$-condition number of $B_w$ is smaller than that of $A$ if possible. Here $p$ is defined to be the ratio of the maximum eigenvalue to minimum eigenvalue of the coefficient matrix. The vector $\mathbf{x}$ may be evaluated by carrying out three basic processes (Evans 1983), i.e. the back substitution process,

the matrix vector multiplication,

and the process,

$$\mathbf{x} = (I - wL)^{-1} \mathbf{y}.$$  

(3.5)

To further improve the convergence rate, we apply the Simultaneous Displacement accelerated technique to the system (3.4) for $w = w_{opt}$ which is defined by

(3.7)

Assuming that the spectrum of real eigenvalues $\lambda_i$ of $B_w$ are bounded by the values $a$ and $b$ such that $0 < a < 2/b$. The fastest convergence rate is obtained by choosing $w$ so that the spectral radius of $(I - aB_w)$ is minimized which may be proven to occur when $w = w_{opt}$ with the rate of convergence $(Evans 1994)$. The iterations proceed in the $\mathbf{x}$ variable until convergence is achieved with a specified degree of accuracy.

IV. NUMERICAL EXPERIMENTATION AND RESULTS

To test the performance of the preconditioned iterative methods presented previously, we carried out experiments on the following elliptic problem which is the torsion problem,

over the unit square with the boundary conditions

$$u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0.$$  

(3.1)  

Throughout, computer programs were coded in Visual Studio C++ 6.0 programming language. The preconditioned methods were implemented for different mesh sizes $h^{-1} = 11, 21, 41, 61$ and 81. Tables 1, 2, 3 and 4 show the largest and smallest eigenvalues and $p$-condition numbers for the coefficient matrix $B_w$ versus the preconditioning factor $w$ with mesh sizes $h^{-1} = 21, 41, 61$ and 81 respectively, which were obtained by the Power method.

From Tables 1-4, it can be observed that a minimum

<p>| Table 1. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $Bw$ versus the preconditioning parameter $w$ for $h^{-1} = 21$ |  |  |</p>
<table>
<thead>
<tr>
<th>Preconditioning Parameter $w$</th>
<th>Maximum Eigenvalue</th>
<th>Minimum Eigenvalue</th>
<th>p-Condition Number</th>
<th>Iteration Number</th>
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<td>$1.666666$</td>
<td>326</td>
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<td>$1.666666$</td>
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</tbody>
</table>

TABLE 2. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 41$.

TABLE 3. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 61$.

TABLE 4. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 81$. 

TABLE 5. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 101$. 

TABLE 6. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 121$. 

TABLE 7. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 141$. 

TABLE 8. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 161$. 

TABLE 9. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 181$. 

TABLE 10. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 201$. 

TABLE 11. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 221$. 

TABLE 12. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 241$. 

TABLE 13. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 261$. 

TABLE 14. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 281$. 

TABLE 15. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 301$. 

TABLE 16. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 321$. 

TABLE 17. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 341$. 

TABLE 18. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 361$. 

TABLE 19. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 381$. 

TABLE 20. The maximum and minimum eigenvalues and p-condition numbers of the coefficient matrix $B_w$ versus the preconditioning parameter $w$ for $h^{-1} = 401$.
The $p$-condition number is achieved for the preconditioned coefficient matrix $B_w$ for each of the chosen mesh size. Table 5 depicts the results for the preconditioned EG accelerated with the Simultaneous Displacement technique for the chosen problem with and without the use of the preconditioning factor $w$. All the iterations were initiated from the same initial values and proceeded until the convergence criteria is achieved.

Fig. 3 which shows the $p$-condition numbers versus the preconditioning parameter $w$ confirms experimentally that a minimum $p$-condition number is achieved for the coefficient matrix $B_w$ for all the chosen mesh sizes.

**CONCLUSION**

In this work, we have numerically studied the performance of the explicit group scheme preconditioned by a highly efficient ‘splitting’ principle (Evans 1968) implemented in conjunction with the Simultaneous Displacement acceleration technique. The algebraic properties of the coefficient matrix which arise from this scheme were observed and shown to achieve minimum $p$-condition numbers for certain values of acceleration parameter $w$. The numerical results obtained revealed that a substantial number of iterations was reduced by the use of the preconditioner at these minimum $p$-condition numbers indicating that the fastest convergence rate has been obtained by choosing $a$ so that the spectral radius of $(I- a B_w)$ is minimized. From the numerical experiments, we can conclude that the application of the ‘splitting’ preconditioning strategy together
with the accelerated technique is capable of increasing the convergence rate of the group iterative methods by about 39-43%. It would be interesting to investigate the application of this preconditioning strategy in combination with other promising accelerated techniques such as the Richardson, Chebychev or Conjugate Gradient based methods. The findings will be reported soon.

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REFERENCES


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