KKKM4213

COMPUTATIONAL FLUID DYNAMICS

Chapter 3:

Basic Conservation Laws & Governing Equations

Presented by:

Dr. Wan Mohd Faizal Wan Mahmood

Ir. Dr. Zambri Bin Harun

Dept. of Mechanical & Materials Engineering
Universiti Kebangsaan Malaysia

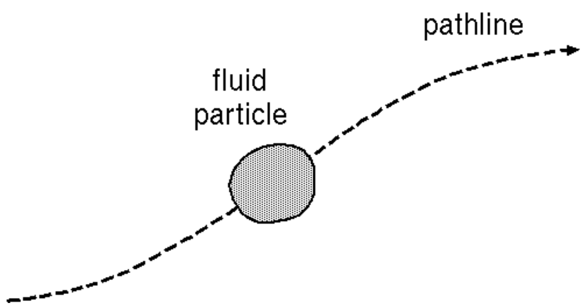
Basic Conservation Laws

- The basic conservation laws of physics include:
  - Conservation of mass: mass is neither created nor destroyed.
  - Newton's second law of motion: the change of momentum equals the sum of forces on a fluid particle.
  - First law of thermodynamics (conservation of energy): rate of change of energy equals the sum of rate of heat addition to and work done on fluid particle.

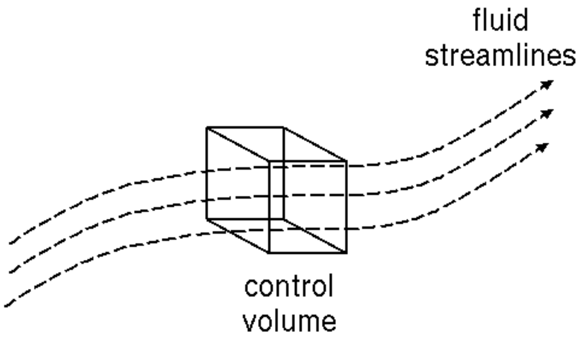
- The fluid is treated as a continuum.
  - For length scales of, say, 1μm and larger, the molecular structure and motions may be ignored.

Lagrangian vs. Eulerian Description

A fluid flow field can be thought of as being comprised of a large number of finite sized fluid particles which have mass, momentum, internal energy, and other properties. Mathematical laws can then be written for each fluid particle. This is the Lagrangian description of fluid motion.



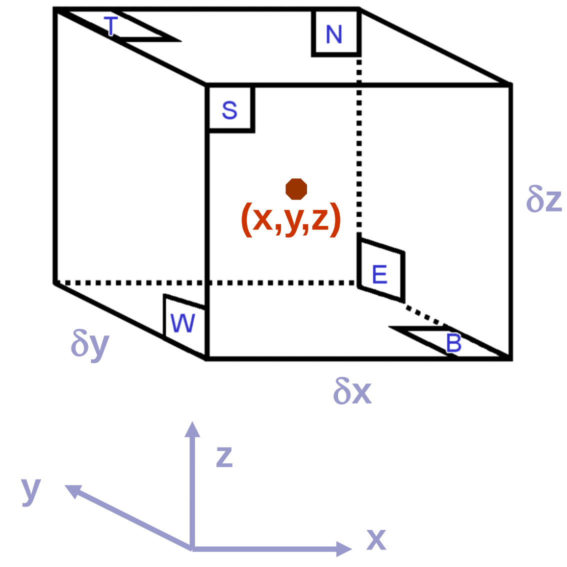
Another view of fluid motion is the Eulerian description. In the Eulerian description of fluid motion, we consider how flow properties change at a fluid element that is fixed in space and time (x,y,z,t), rather than following individual fluid particles.



Fluid Element and Properties

- The behaviour of the fluid is described in terms of macroscopic properties:
  - Velocity **u.**
  - Pressure *p*.
  - Density ρ.
  - Temperature *T*.
- Properties are averages of a sufficiently large number of molecules.
- A fluid element can be thought of as the smallest volume for which the continuum assumption is valid.

Fluid element for conservation laws

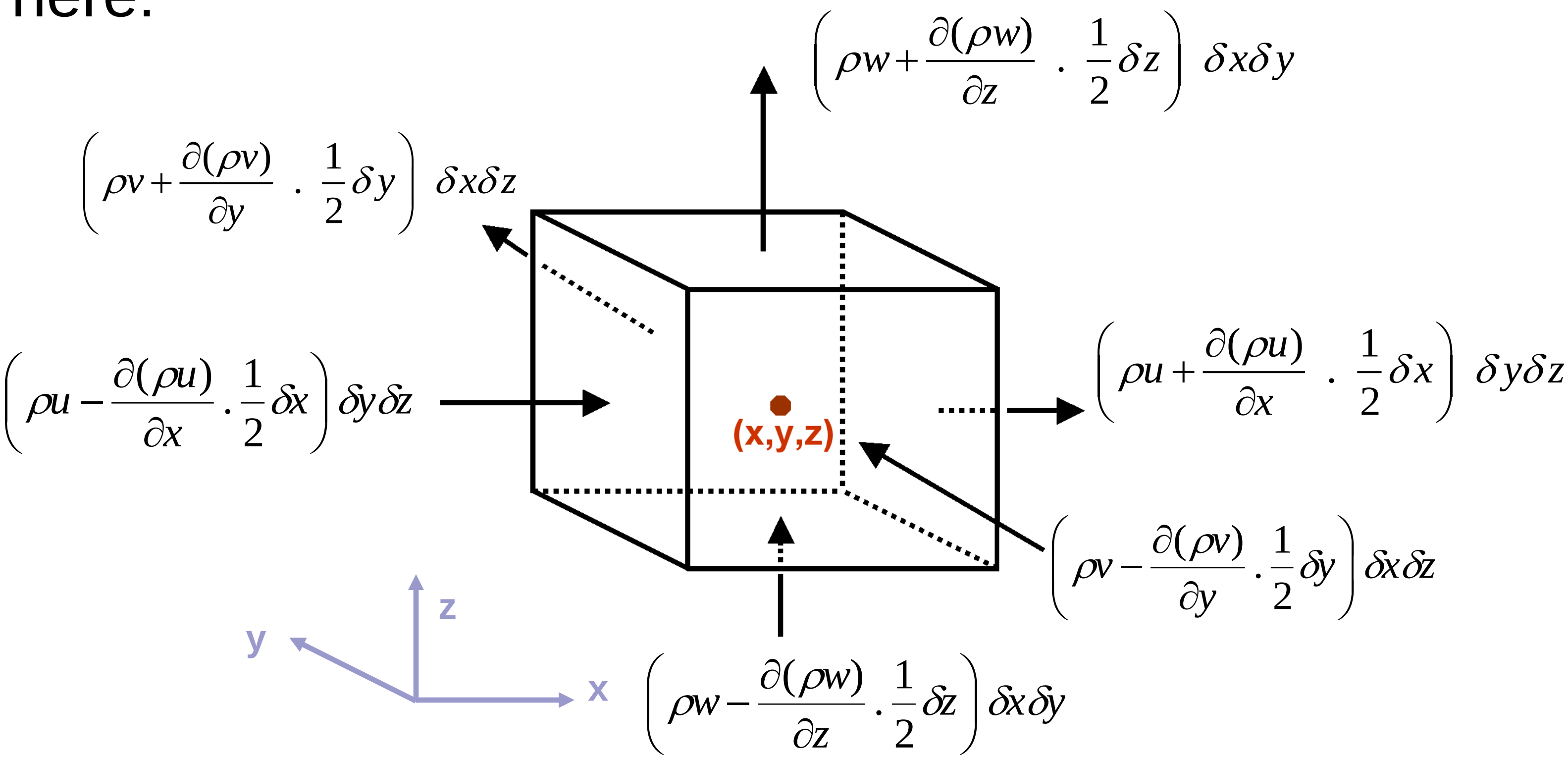


Faces are labeled North, East, West, South, Top and Bottom

Properties at faces are expressed as first two terms of a Taylor series expansion, e.g. for p: $p_W = p - \frac{\partial p}{\partial x}\frac{1}{2}\delta x$ and $p_E = p + \frac{\partial p}{\partial x}\frac{1}{2}\delta x$

Mass Balance

- Rate of increase of mass in fluid element equals the net rate of flow of mass into element.
- Rate of increase is: $\frac{\partial}{\partial t}(\rho \delta x \delta y \delta z) = \frac{\partial \rho}{\partial t} \delta x \delta y \delta z$
- The outflows (positive) and inflows (negative) are shown here:



Continuity Equation

- Summing all terms in the previous slide and dividing by the volume $\delta x \delta y \delta z$ results in:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

- In vector notation: $\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0}$

Change in density

Net flow of mass across boundaries
Convective term

- For incompressible fluids $\partial \rho / \partial t = 0$, and the equation becomes:

$$\nabla \cdot \mathbf{u} = 0$$

- Alternative ways to write this: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ and $\frac{\partial u_i}{\partial x_i} = 0$

Rate of Change for a Fluid Particle

- Fluid element is a volume stationary in space, and a fluid particle is a volume of fluid moving with the flow.
- A moving fluid particle experiences two rates of changes:
  - Change due to changes in the fluid as a function of time.
  - Change due to the fact that it moves to a different location in the fluid with different conditions.
- The sum of these two rates of changes for a property per unit mass ϕ is called the total or substantive derivative:

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt}$$

$$\boxed{\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + (\mathbf{u}.\nabla)\phi}$$

Rate of Change for a Fluid Element

- In most cases we are interested in the changes of a flow property for a fluid element, or fluid volume, that is stationary in space.
- However, some equations are easier derived for fluid particles. For a moving fluid particle, the total derivative per unit volume of this property ϕ is given by:

(for moving fluid particle)

$$\rho \frac{D\phi}{Dt} = \rho\left(\frac{\partial \phi}{\partial t} + (\mathbf{u}.\nabla)\phi\right)$$

(for given location in space)

- For a fluid element, for an arbitrary conserved property ϕ:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

Continuity equation

$$\frac{\partial (\rho\phi)}{\partial t} + \nabla \cdot (\rho\phi\mathbf{u}) = 0$$

Arbitrary property

Fluid Particle and Fluid Element

- The relationship between the equations for a fluid particle (Lagrangian) and a fluid element (Eulerian) can be derived as follows:

$$\frac{\partial(\rho\phi)}{\partial t}+\nabla\cdot(\rho\phi\mathbf{u})=\rho\left[\frac{\partial\phi}{\partial t}+(\mathbf{u}.\nabla)\phi\right]+\phi\underbrace{\left[\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})\right]}_{\text{zero because of continuity}}=\rho\frac{D\phi}{Dt}$$

$$\underbrace{\frac{\partial(\rho\phi)}{\partial t}}_{\text{Rate of increase of } \phi \text{ of fluid element}}+\underbrace{\nabla(\rho\phi\mathbf{u})}_{\text{Net rate of flow of } \phi \text{ out of fluid element}}=\underbrace{\rho\frac{D\phi}{Dt}}_{\text{Rate of increase of } \phi \text{ for a fluid particle}}$$

Relevant Entries for *Φ*

x-momentum	u	$\rho \frac{Du}{Dt}$	$\frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{u})$
y-momentum	v	$\rho \frac{Dv}{Dt}$	$\frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v \mathbf{u})$
z-momentum	w	$\rho \frac{Dw}{Dt}$	$\frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho w \mathbf{u})$
Energy	E	$\rho \frac{DE}{Dt}$	$\frac{\partial(\rho E)}{\partial t} + \nabla \cdot (\rho E \mathbf{u})$

Momentum Equations

- The conservation equations for momentum and energy for fluid particles can then be derived for an Eulerian frame (for fluid elements).
- Newton's second law:

  Rate of change of momentum equals sum of forces.

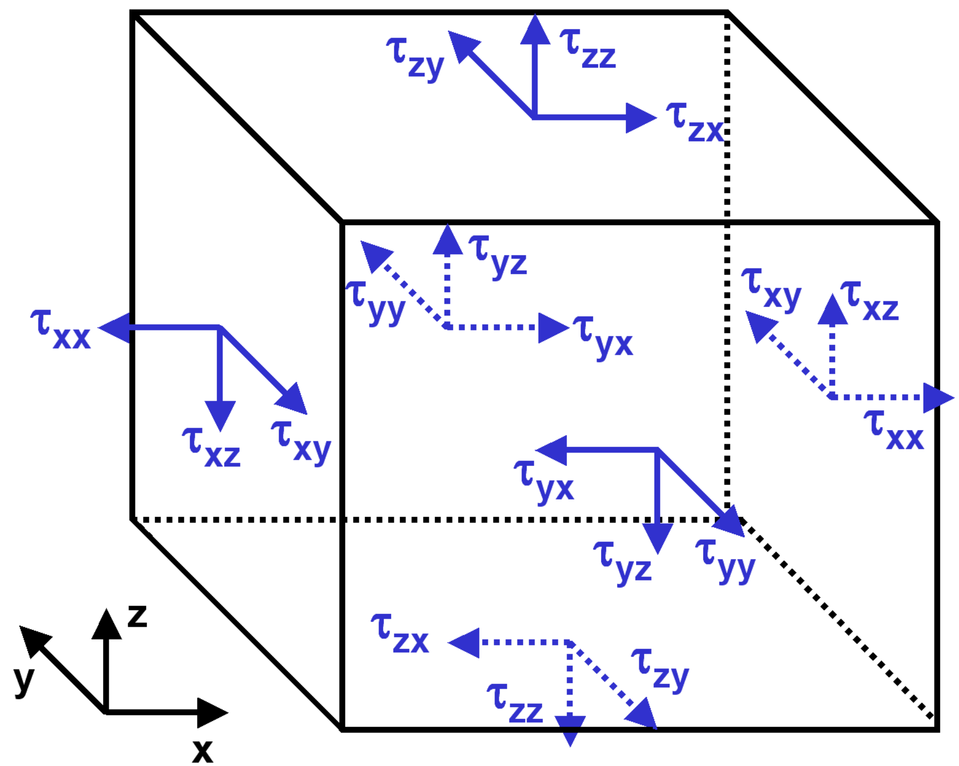
- Rate of increase of *x*-, *y*-, and *z*-momentum:

$$\rho \frac{Du}{Dt} \qquad \rho \frac{Dv}{Dt} \qquad \rho \frac{Dw}{Dt}$$

- Forces on fluid particles are:
  - Surface forces such as pressure and viscous forces.
  - Body forces, which act on a volume, such as gravity, centrifugal, Coriolis, and electromagnetic forces.

Viscous Stresses

- Stresses are forces per area. (unit: N/m^2 or Pa)
- Viscous stresses denoted by τ.
- Suffix notation τ_{ij} is used to indicate direction.
- Nine stress components.
  - τ_{xx}, τ_{yy}, τ_{zz} are normal stresses. e.g. τ_{zz} is the stress in the z-direction on a z-plane.
  - Other stresses are shear stresses. E.g. τ_{zy} is the stress in the y-direction on a z-plane.
- Forces aligned with the direction of a coordinate axis are positive. Opposite direction is negative.



Forces in the x-direction

$$(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}.\frac{1}{2}\delta z)\delta y \delta z$$

$$(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}.\frac{1}{2}\delta y)\delta x \delta z$$

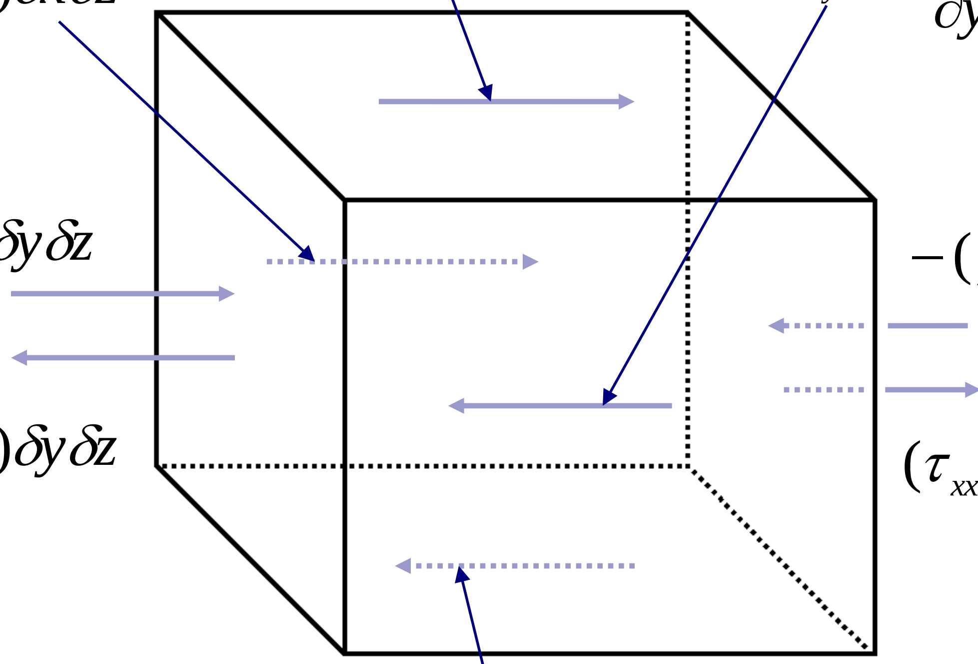
$$-(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}.\frac{1}{2}\delta y)\delta x \delta z$$

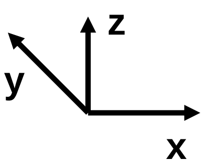
$$(p - \frac{\partial p}{\partial x}.\frac{1}{2}\delta x)\delta y \delta z$$

$$-(p + \frac{\partial p}{\partial x}.\frac{1}{2}\delta x)\delta y \delta z$$

$$-(\tau_{xx} - \frac{\partial \tau_{xx}}{\partial x}.\frac{1}{2}\delta x)\delta y \delta z$$

$$(\tau_{xx} + \frac{\partial \tau_{xx}}{\partial x}.\frac{1}{2}\delta x)\delta y \delta z$$





$$-(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}.\frac{1}{2}\delta z)\delta x \delta y$$

Net force in the x-direction is the sum of all the force components in that direction.

Momentum Equations

- Set the rate of change of x-momentum for a fluid particle Du/Dt equal to:
  - the sum of the forces due to surface stresses shown in the previous slide, plus
  - the body forces. These are usually lumped together into a source term S_M:

$$\rho \frac{Du}{Dt} = \frac{\partial(-p+\tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + S_{Mx}$$

  - p is a compressive stress and τ_{xx} is a tensile stress.
- Similarly for y- and z-momentum:

$$\rho \frac{Dv}{Dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial(-p+\tau_{yy})}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + S_{My}$$

$$\rho \frac{Dw}{Dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial(-p+\tau_{zz})}{\partial z} + S_{Mz}$$

Energy Equation

- First law of thermodynamics: rate of change of energy of a fluid particle is equal to the rate of heat addition plus the rate of work done.
- Rate of increase of energy is $\rho DE/Dt$.
  - Energy $E = i + ½\,(u^2+v^2+w^2)$.
  - Here, i is the internal (thermal energy).
  - $½\,(u^2+v^2+w^2)$ is the kinetic energy.
- Potential energy (gravitation) is usually treated separately and included as a source term.
- The energy equation is derived by setting the total derivative equal to the change in energy as a result of work done by viscous stresses and the net heat conduction.
- The kinetic energy equation is substracted to arrive at a conservation equation for the internal energy.

Work Done by Surface Stresses in x-direction

$$(u\tau_{zx} + \frac{\partial(u\tau_{zx})}{\partial z}.\frac{1}{2}\delta z)\delta y\delta z$$

$$(u\tau_{yx} + \frac{\partial(u\tau_{yx})}{\partial y}.\frac{1}{2}\delta y)\delta x\delta z$$

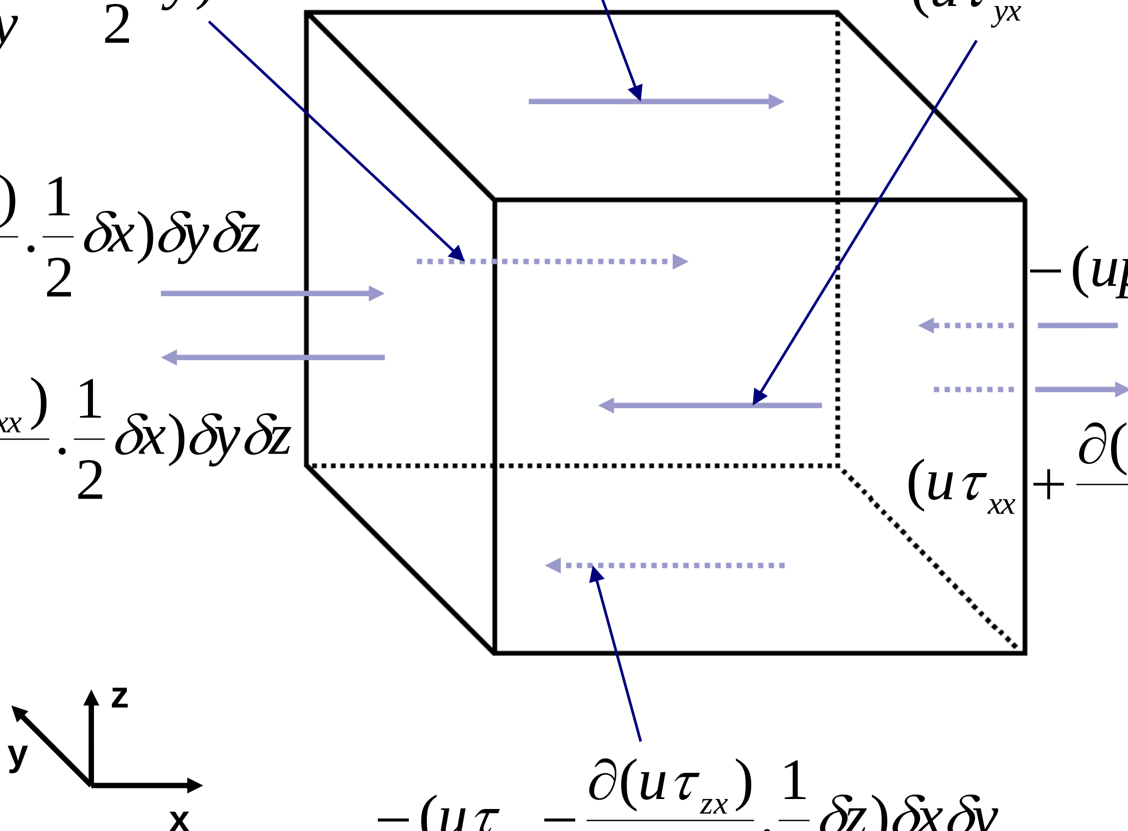
$$-(u\tau_{yx} - \frac{\partial(u\tau_{yx})}{\partial y}.\frac{1}{2}\delta y)\delta x\delta z$$

$$(up - \frac{\partial(up)}{\partial x}.\frac{1}{2}\delta x)\delta y\delta z$$

$$-(up + \frac{\partial(up)}{\partial x}.\frac{1}{2}\delta x)\delta y\delta z$$

$$-(u\tau_{xx} - \frac{\partial(u\tau_{xx})}{\partial x}.\frac{1}{2}\delta x)\delta y\delta z$$

$$(u\tau_{xx} + \frac{\partial(u\tau_{xx})}{\partial x}.\frac{1}{2}\delta x)\delta y\delta z$$



$$-(u\tau_{zx} - \frac{\partial(u\tau_{zx})}{\partial z}.\frac{1}{2}\delta z)\delta x\delta y$$

Work done is force times velocity.

Work Done by Surface Stresses

- The total rate of work done by surface stresses is calculated as follows:
  - For work done by x-components of stresses add all terms in the previous slide.
  - Do the same for the y- and z-components.
- Add all and divide by $\delta x \delta y \delta z$ to get the work done per unit volume by the surface stresses:

$$-\nabla \cdot (p\mathbf{u}) + \frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x}$$

$$+ \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(u\tau_{zz})}{\partial z}$$

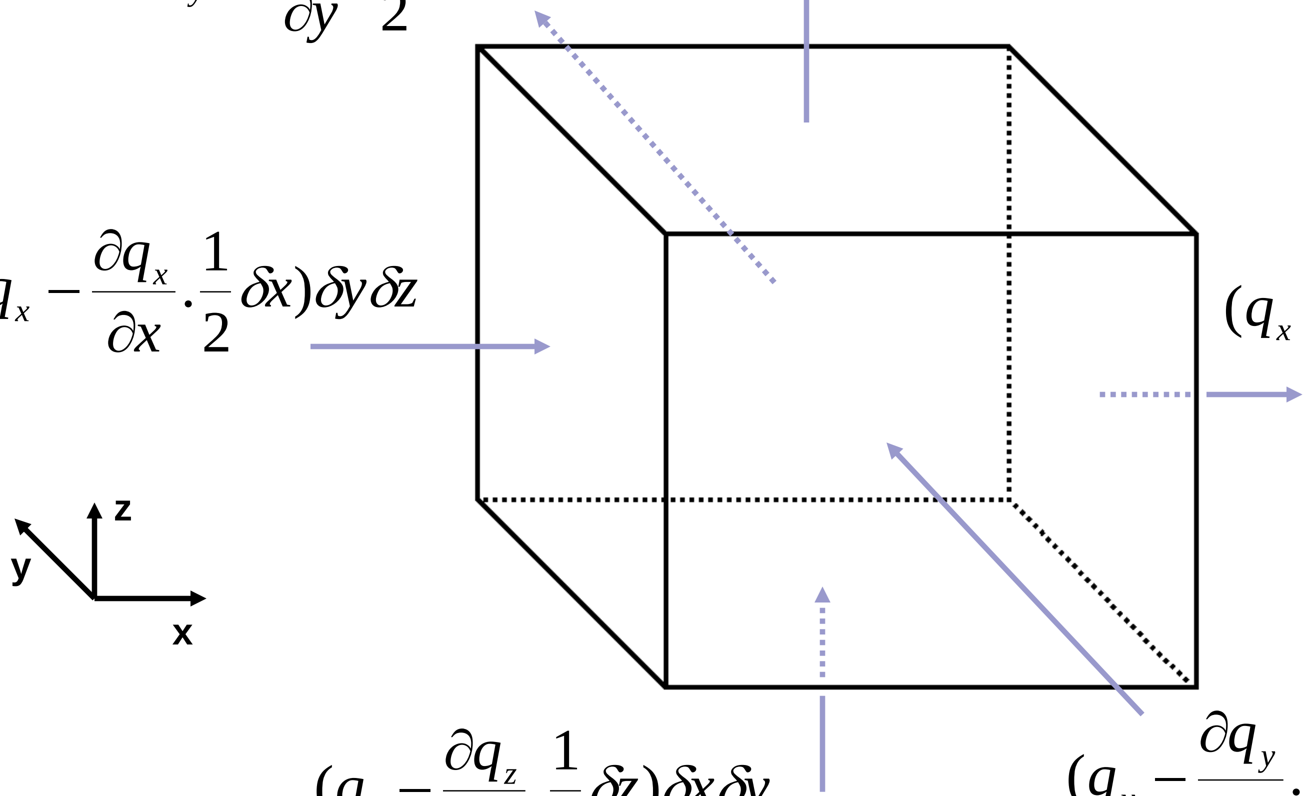
Energy Flux due to Heat Conduction

$$(q_y + \frac{\partial q_y}{\partial y}.\frac{1}{2}\delta y)\delta x \delta z$$

$$(q_z + \frac{\partial q_z}{\partial z}.\frac{1}{2}\delta z)\delta x \delta y$$

$$(q_x - \frac{\partial q_x}{\partial x}.\frac{1}{2}\delta x)\delta y \delta z$$

$$(q_x + \frac{\partial q_x}{\partial x}.\frac{1}{2}\delta x)\delta y \delta z$$



$$(q_z - \frac{\partial q_z}{\partial z}.\frac{1}{2}\delta z)\delta x \delta y$$

$$(q_y - \frac{\partial q_y}{\partial y}.\frac{1}{2}\delta y)\delta x \delta z$$

The heat flux vector q has three components, q_x, q_y, and q_z.

Energy Flux due to Heat Conduction

- Summing all terms and dividing by $\delta x \delta y \delta z$ gives the net rate of heat transfer to the fluid particle per unit volume:

$$-\frac{\partial q_x}{\partial x}-\frac{\partial q_y}{\partial y}-\frac{\partial q_z}{\partial z} = -\nabla\cdot\mathbf{q}$$

- Fourier's law of heat conduction relates the heat flux to the local temperature gradient:

$$q_x=-k\frac{\partial T}{\partial x} \qquad q_y=-k\frac{\partial T}{\partial y} \qquad q_z=-k\frac{\partial T}{\partial z}$$

- In vector form:

$$\mathbf{q}=-k\,\nabla T$$

- Thus, energy flux due to conduction:

$$\boxed{-\nabla\cdot\mathbf{q}=\nabla\cdot(k\,\nabla T)}$$

Energy Equation

- Setting the total derivative for the energy in a fluid particle equal to the previously derived work and energy flux terms, results in the following energy equation:

$$\rho \frac{DE}{Dt} = -\nabla \cdot (p\mathbf{u}) + \left[\frac{\partial (u\tau_{xx})}{\partial x} + \frac{\partial (u\tau_{yx})}{\partial y} + \frac{\partial (u\tau_{zx})}{\partial z} + \frac{\partial (v\tau_{xy})}{\partial x} \right.$$

$$\left. + \frac{\partial (v\tau_{yy})}{\partial y} + \frac{\partial (v\tau_{zy})}{\partial z} + \frac{\partial (w\tau_{xz})}{\partial x} + \frac{\partial (w\tau_{yz})}{\partial y} + \frac{\partial (u\tau_{zz})}{\partial z} \right]$$

$$+ \nabla \cdot (k \nabla T) + S_E$$

- Note that we also added a source term S_E that includes sources (potential energy, sources due to heat production from chemical reactions, etc.).

Kinetic Energy Equation

- Separately, a conservation equation for the kinetic energy of the fluid can also be derived.
- In order to do this, multiply the *u*-momentum equation by *u*, the *v*-momentum equation by *v*, and the *w*-momentum equation by *w*.
- We then add the results together. This results in the following equation for the kinetic energy:

$$\rho \frac{D[\frac{1}{2}(u^2+v^2+w^2)]}{Dt} = -\mathbf{u}\cdot\nabla p + u\left(\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\right)$$

$$+ v\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\right) + w\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \tau_{zz}}{\partial z}\right) + \mathbf{u}\cdot\mathbf{S}_M$$

Internal Energy Equation

- Subtract the kinetic energy equation from the energy equation.
- Define a new source term for the internal energy as $S_i = S_E - \mathbf{u}\cdot\mathbf{S_M}$. This results in:

$$\rho \frac{Di}{Dt} = -p\,\nabla\cdot\mathbf{u} + \left[\tau_{xx}\frac{\partial u}{\partial x} + \tau_{yx}\frac{\partial u}{\partial y} + \tau_{zx}\frac{\partial u}{\partial z} + \tau_{xy}\frac{\partial v}{\partial x} + \tau_{yy}\frac{\partial v}{\partial y} + \tau_{zy}\frac{\partial v}{\partial z} + \tau_{xz}\frac{\partial w}{\partial x} + \tau_{yz}\frac{\partial w}{\partial y} + \tau_{zz}\frac{\partial u}{\partial z}\right] + \nabla\cdot(k\,\nabla T) + S_i$$

Enthalpy Equation

- An often used alternative form of the energy equation is the total enthalpy equation.
  - Specific enthalpy $h = i + p/\rho$.
  - Total enthalpy $h_0 = h + ½ (u^2+v^2+w^2) = E + p/\rho$.

$$\frac{\partial(\rho h_0)}{\partial t} + \nabla\cdot(\rho h_0 \mathbf{u}) = \nabla\cdot(k\,\nabla T)$$

$$+\left[\frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{yx})}{\partial y} + \frac{\partial(u\tau_{zx})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x}\right.$$

$$\left. + \frac{\partial(v\tau_{yy})}{\partial y} + \frac{\partial(v\tau_{zy})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(u\tau_{zz})}{\partial z}\right] + S_h$$

Viscous Stresses

- A model for the viscous stresses τ_{ij} is required.
- The viscous stresses can be expressed as functions of the local deformation rate (strain rate) tensor.
- There are two types of deformation:
  - Linear deformation rates due to velocity gradients.
    - Elongating stress components (stretching).
    - Shearing stress components.
  - Volumetric deformation rates due to expansion or compression.
- All gases and most fluids are isotropic: viscosity is a scalar.
- Some fluids have anisotropic viscous stress properties, such as certain polymers and dough – Non-Newtonian fluids.

Viscous Stress Tensor

- Using an isotropic (first) dynamic viscosity μ for the linear deformations and a second viscosity $\lambda = -2/3\mu$ for the volumetric deformations results in:

$$\boldsymbol{\tau} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix}$$

$$= \begin{pmatrix} 2\mu\frac{\partial u}{\partial x} - \frac{2}{3}\mu\left(\nabla\cdot\mathbf{u}\right) & \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & 2\mu\frac{\partial v}{\partial y} - \frac{2}{3}\mu\left(\nabla\cdot\mathbf{u}\right) & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & 2\mu\frac{\partial w}{\partial z} - \frac{2}{3}\mu\left(\nabla\cdot\mathbf{u}\right) \end{pmatrix}$$

Note: div u = 0 for incompressible fluids.

Navier-Stokes Equations

- Including the viscous stress terms in the momentum balance and rearranging, results in the Navier-Stokes equations:

$$x-\text{momentum}: \quad \frac{\partial(\rho u)}{\partial t}+\nabla\cdot(\rho u\mathbf{u})=-\frac{\partial p}{\partial x}+\nabla\cdot(\mu\,\nabla u)+S_{Mx}$$

$$y-\text{momentum}: \quad \frac{\partial(\rho v)}{\partial t}+\nabla\cdot(\rho v\mathbf{u})=-\frac{\partial p}{\partial y}+\nabla\cdot(\mu\,\nabla v)+S_{My}$$

$$z-\text{momentum}: \quad \frac{\partial(\rho w)}{\partial t}+\nabla\cdot(\rho w\mathbf{u})=-\frac{\partial p}{\partial z}+\nabla\cdot(\mu\,\nabla w)+S_{Mz}$$

Viscous Dissipation

- Similarly, substituting the stresses in the internal energy equation and rearranging results in:

$$\text{Internal energy}: \quad \frac{\partial(\rho i)}{\partial t} + \nabla\cdot(\rho i\mathbf{u}) = -p\,\nabla\cdot\mathbf{u} + \nabla\cdot(k\,\nabla T) + \Phi + S_i$$

- Here Φ is the viscous dissipation term. This term is always positive and describes the conversion of mechanical energy to heat.

$$\Phi = \mu\left\{2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2\right\} - \frac{2}{3}\mu(\nabla\cdot\mathbf{u})^2$$

Equations of State

- Fluid motion is described by six partial differential equations for mass, momentum (in x,y,z), and energy.
- Amongst the unknowns are four thermodynamic variables: $\mathbf{u}$ (in form of u,v,w), ρ, p, i, and T.
- The thermodynamic equilibrium can be assumed where the time taken for a fluid particle to adjust to new conditions is short relative to the timescale of the flow.
- We add two equations of state using the two state variables ρ and T: $p=p(\rho,T)$ and $i=i(\rho,T)$.
  - For a perfect gas, these become: $p=\rho RT$ and $i=C_v T$.
- At low speeds (Ma < 0.3), the fluids can be considered incompressible.
  - There is no linkage between the energy equation, and the mass and momentum equation.
  - We then only need to solve for energy if the problem involves heat transfer.

Summary of Equations in Conservation Form

$$\text{Mass:} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$x - \text{momentum:} \quad \frac{\partial(\rho u)}{\partial t} + \nabla \cdot (\rho u \mathbf{u}) = -\frac{\partial p}{\partial x} + \nabla \cdot (\mu \nabla u) + S_{Mx}$$

$$y - \text{momentum:} \quad \frac{\partial(\rho v)}{\partial t} + \nabla \cdot (\rho v \mathbf{u}) = -\frac{\partial p}{\partial y} + \nabla \cdot (\mu \nabla v) + S_{My}$$

$$z - \text{momentum:} \quad \frac{\partial(\rho w)}{\partial t} + \nabla \cdot (\rho w \mathbf{u}) = -\frac{\partial p}{\partial z} + \nabla \cdot (\mu \nabla w) + S_{Mz}$$

$$\text{Internal energy:} \quad \frac{\partial(\rho i)}{\partial t} + \nabla \cdot (\rho i \mathbf{u}) = -p \nabla \cdot \mathbf{u} + \nabla \cdot (k \nabla T) + \Phi + S_i$$

$$\text{Equations of state:} \quad p = p(\rho, T) \quad \text{and} \quad i = i(\rho, T)$$

$$\text{e.g. for perfect gas:} \ p = \rho R T \quad \text{and} \quad i = C_v T$$

General Transport Equations

- The system of equations is now closed, with seven equations for seven variables: pressure, three velocity components, enthalpy, temperature, and density.
- There are significant commonalities between the various equations. Using a general variable ϕ, the conservative form of all fluid flow equations can usefully be written in the following form:

$$\frac{\partial(\rho\phi)}{\partial t} + \nabla\cdot(\rho\phi\mathbf{u}) = \nabla\cdot(\Gamma\nabla\phi) + S_\phi$$

- Or, in words:

Rate of increase of ϕ of fluid element	+	Net rate of flow of ϕ out of fluid element (convection)	=	Rate of increase of ϕ due to diffusion	+	Rate of increase of ϕ due to sources

Integral Form

- The key step of the finite volume method is to integrate the differential equation shown in the previous slide, and then to apply Gauss' divergence theorem, which for a vector $\mathbf{a}$ states:

$$\int_{CV} \nabla \cdot \mathbf{a}\, dV = \int_{A} \mathbf{n} \cdot \mathbf{a}\, dA$$

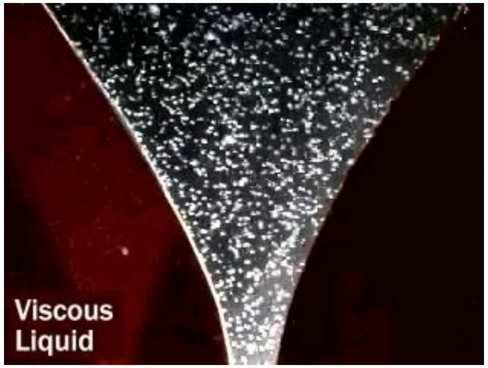
- This then leads to the following general conservation equation in integral form:

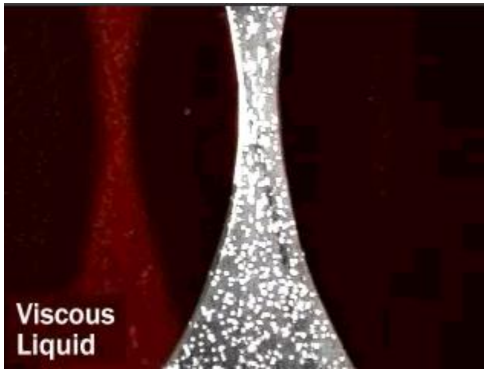
$$\frac{\partial}{\partial t}\left(\int_{CV} \rho\phi\, dV \right) + \int_{A} \mathbf{n} \cdot (\rho\phi\mathbf{u})\, dA = \int_{A} \mathbf{n} \cdot (\Gamma \nabla \phi)\, dA + \int_{CV} S_{\phi}\, dV$$

Rate of increase of ϕ + **Net rate of decrease of ϕ due to convection across boundaries** = **Net rate of increase of ϕ due to diffusion across boundaries** + **Net rate of creation of ϕ**

Classification: Fluid Flow vs. Granular Flow

- Fluid and solid particles: fluid flow vs. granular flow.
- A fluid consists of a large number of individual molecules. These could in principle be modeled as interacting solid particles.
- The interaction between adjacent salt grains and adjacent fluid parcels is quite different, however.





Reynolds Number

- The Reynolds number Re is defined as:

$$\mathrm{Re} = \rho V L / \mu.$$

  - Here L is a characteristic length, and V is the velocity.
- It is a measure of the ratio between inertial forces and viscous forces.
  - If Re >> 1 the flow is dominated by inertia.
  - If Re << 1 the flow is dominated by viscous effects.

Reynolds Number

Re = 0.05 Re = 10 Re = 200 Re = 3000

Newton's Second Law

- For a solid mass: **F** = m.**a**
- For a continuum:

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}.\nabla\mathbf{u}\right]=\nabla.\sigma+f$$

Force per volume (body force)

Mass per volume (density)

acceleration

Force per area (stress tensor)

- For an incompressible Newtonian fluid, this becomes:

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}.\nabla\mathbf{u}\right]=-\nabla p+\mu\nabla^2\mathbf{u}+\rho\mathbf{g}$$

- Here p is the pressure and μ is the dynamic viscosity. In this form, the momentum balance is also called the Navier-Stokes equation.

Newton's Second Law

- The flow is then inviscid, $\mu = 0$, and the Navier-Stokes equations becomes the Euler equations:

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u}\right] = -\nabla p + \rho\mathbf{g}$$

- When:
  - The flow is steady: $\partial u / \partial t = 0$
  - The flow is irrotational: the vorticity $\omega = \nabla \times \mathbf{u} = 0$
  - The flow is inviscid: $\mu = 0$

And using: $\mathbf{u}.\nabla\mathbf{u} = \frac{1}{2}\nabla(\mathbf{u}.\mathbf{u}) - \mathbf{u} \times \nabla \times \mathbf{u}$

The Navier-Stokes equation becomes the Bernoulli equation :

$$\nabla\left(\frac{p}{\rho} + \frac{\mathbf{u}.\mathbf{u}}{2} + gz\right) = 0$$

Basic Quantities

- The Navier-Stokes equations for incompressible flow involve four basic quantities:
  - Local (unsteady) acceleration.
  - Convective acceleration.
  - Pressure gradients.
  - Viscous forces.
- The ease with which solutions can be obtained and the complexity of the resulting flows often depend on which quantities are important for a given flow.

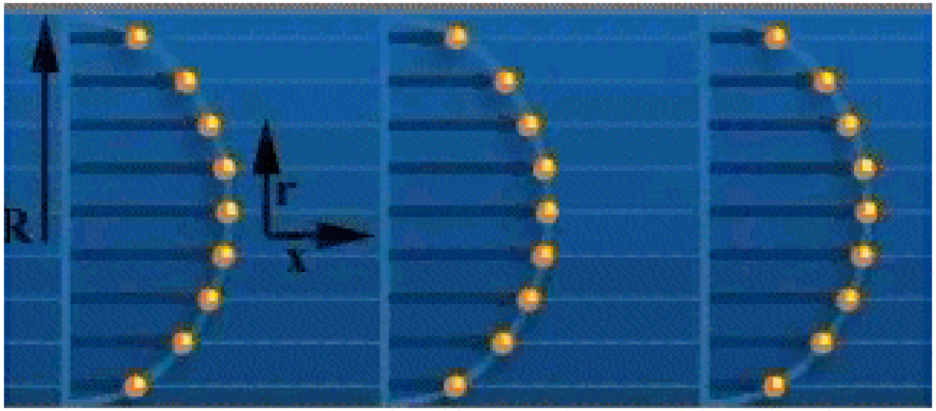
$$\rho\frac{\partial \boldsymbol{u}}{\partial t} + \rho\boldsymbol{u}\cdot\nabla\boldsymbol{u} = -\nabla p + \mu\nabla^2\boldsymbol{u}$$

Flow	Local Acceleration	Convective Acceleration	Pressure Gradients	Viscous Forces
Pipe (steady laminar flow)			✓	✓
Plate (impulsively started)	✓			✓
Plate (boundary layer)		✓		✓
Airfoil (inviscid)		✓	✓	
Pipe (inviscid, impulsively started)	✓		✓	
Cylinder (steady viscous flow)		✓	✓	✓
Airfoil (unsteady flow)	✓	✓	✓	✓

✓ *Means term is important for this flow.*

Steady Laminar Flow

- Steady viscous laminar flow in a horizontal pipe involves a balance between the pressure forces along the pipe and viscous forces.
- The local acceleration is zero because the flow is steady.
- The convective acceleration is zero because the velocity profiles are identical at any section along the pipe.



Pressure Gradients and Viscous Forces

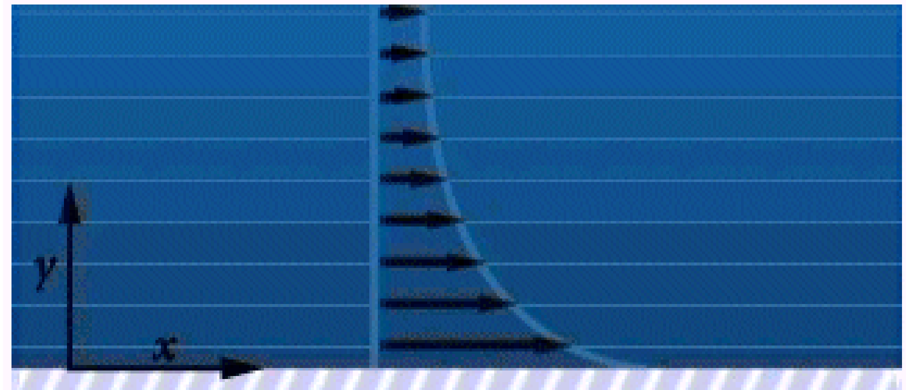
$$0 = -\nabla p + \mu \nabla^2 u$$

$$u = \left(-\frac{dp}{dx}\right)\frac{R^2}{4\mu}\left[1-\left(\frac{r}{R}\right)^2\right]$$

velocity profile is independent of x

Flow Past an Impulsively Started Flat Plate

- Flow past an impulsively started flat plate of infinite length involves a balance between the local (unsteady) acceleration effects and viscous forces. Here, the development of the velocity profile is shown.
- The pressure is constant throughout the flow.
- The convective acceleration is zero because the velocity does not change in the direction of the flow, although it does change with time.



Local Acceleration and Viscous Forces

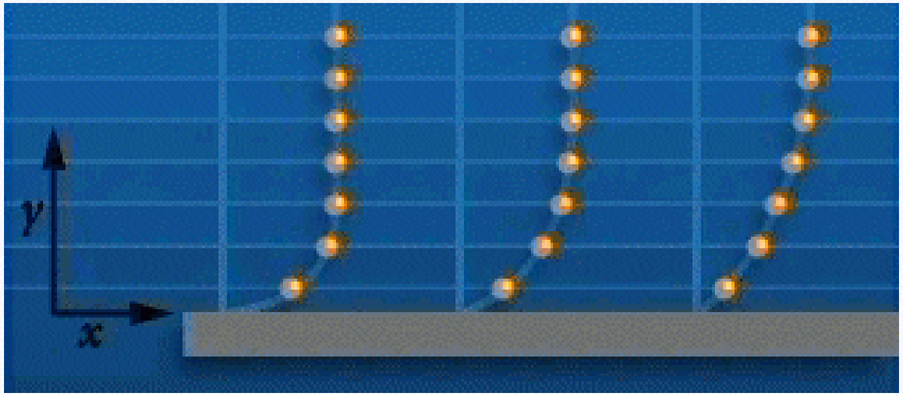
$$\frac{\partial u}{\partial t} = \nu \nabla^2 u$$

$$\nu = \mu / \rho$$

$$u = V_0 \left[1 - erf \left(\frac{y}{\sqrt{4 \nu t}} \right) \right]$$

Boundary Layer Flow along a Flat Plate

- Boundary layer flow along a finite flat plate involves a balance between viscous forces in the region near the plate and convective acceleration effects.
- The boundary layer thickness grows in the downstream direction.
- The local acceleration is zero because the flow is steady.



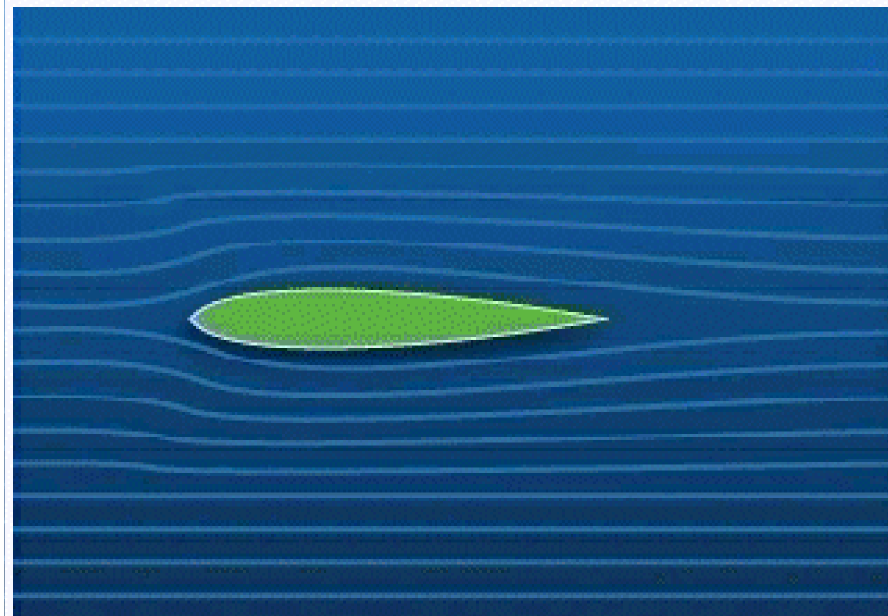
Convective Acceleration and Viscous Forces

$$u \cdot \nabla u = \nu \nabla^2 u$$

$$u = u(x, y)$$

Inviscid Flow past an Airfoil

- Inviscid flow past an airfoil involves a balance between pressure gradients and convective acceleration.
- Since the flow is steady, the local (unsteady) acceleration is zero.
- Since the fluid is inviscid (μ=0) there are no viscous forces.

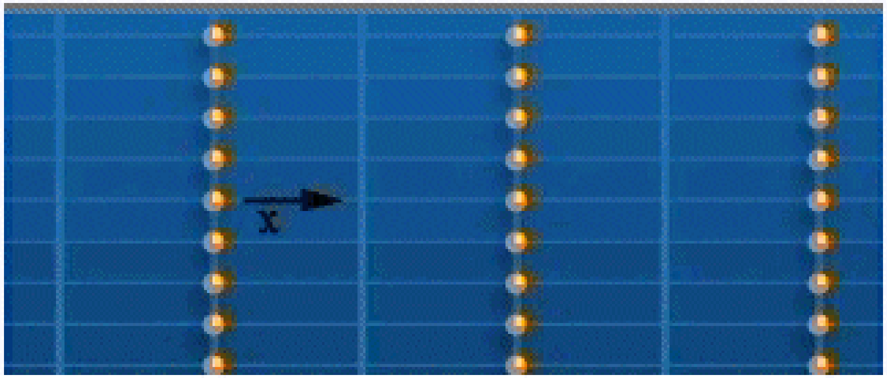


Convective Acceleration and Pressure Gradients $\rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p$

$$\boldsymbol{u} = \boldsymbol{u}(x, y)$$

Impulsively Started Flow of an Inviscid Fluid

- Impulsively started flow of an inviscid fluid in a pipe involves a balance between local (unsteady) acceleration effects and pressure differences.
- The absence of viscous forces allows the fluid to slip along the pipe wall, producing a uniform velocity profile.
- The convective acceleration is zero because the velocity does not vary in the direction of the flow.
- The local (unsteady) acceleration is not zero since the fluid velocity at any point is a function of time.



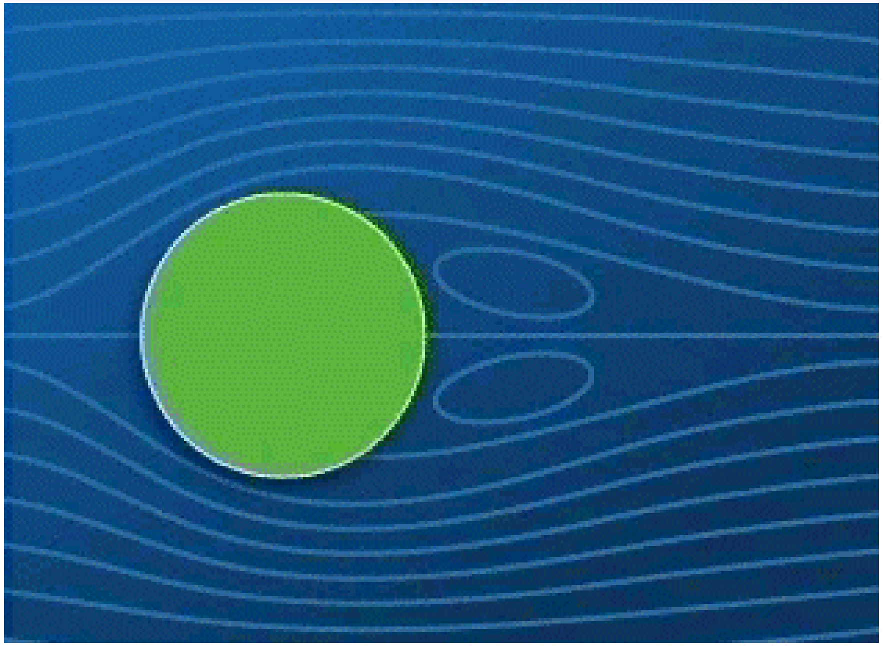
Local Acceleration and Pressure Gradients

$$\rho \frac{\partial u}{\partial t} = -\nabla p$$

$$u = u(t)$$

Steady Viscous Flow past a Cylinder

- Steady viscous flow past a circular cylinder involves a balance among convective acceleration, pressure gradients, and viscous forces.
- For the parameters of this flow (density, viscosity, size, and speed), the steady boundary conditions (i.e. the cylinder is stationary) give steady flow throughout.
- For other values of these parameters the flow may be unsteady.

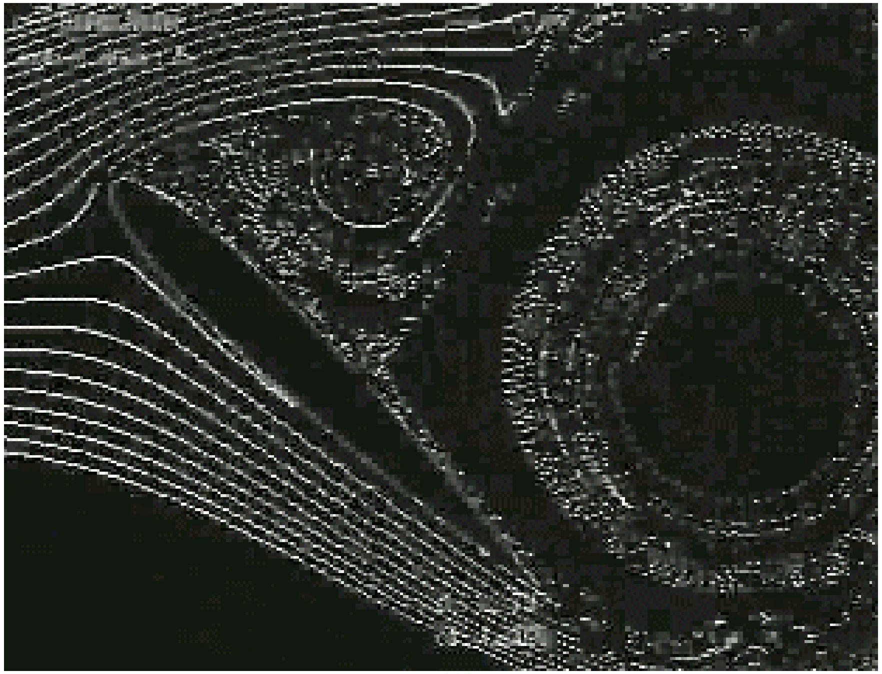


Convective Acceleration, Pressure Gradients and Viscous Forces

$$\rho u \cdot \nabla u = -\nabla p + \mu \nabla^2 u$$

Unsteady Flow past an Airfoil

- Unsteady flow past an airfoil at a large angle of attack (stalled) is governed by a balance among local acceleration, convective acceleration, pressure gradients and viscous forces.
- A wide variety of fluid mechanics phenomena often occurs in situations such as these where all of the factors in the Navier-Stokes equations are relevant.



Local Acceleration, Convective Acceleration, Pressure Gradients and Viscous Forces

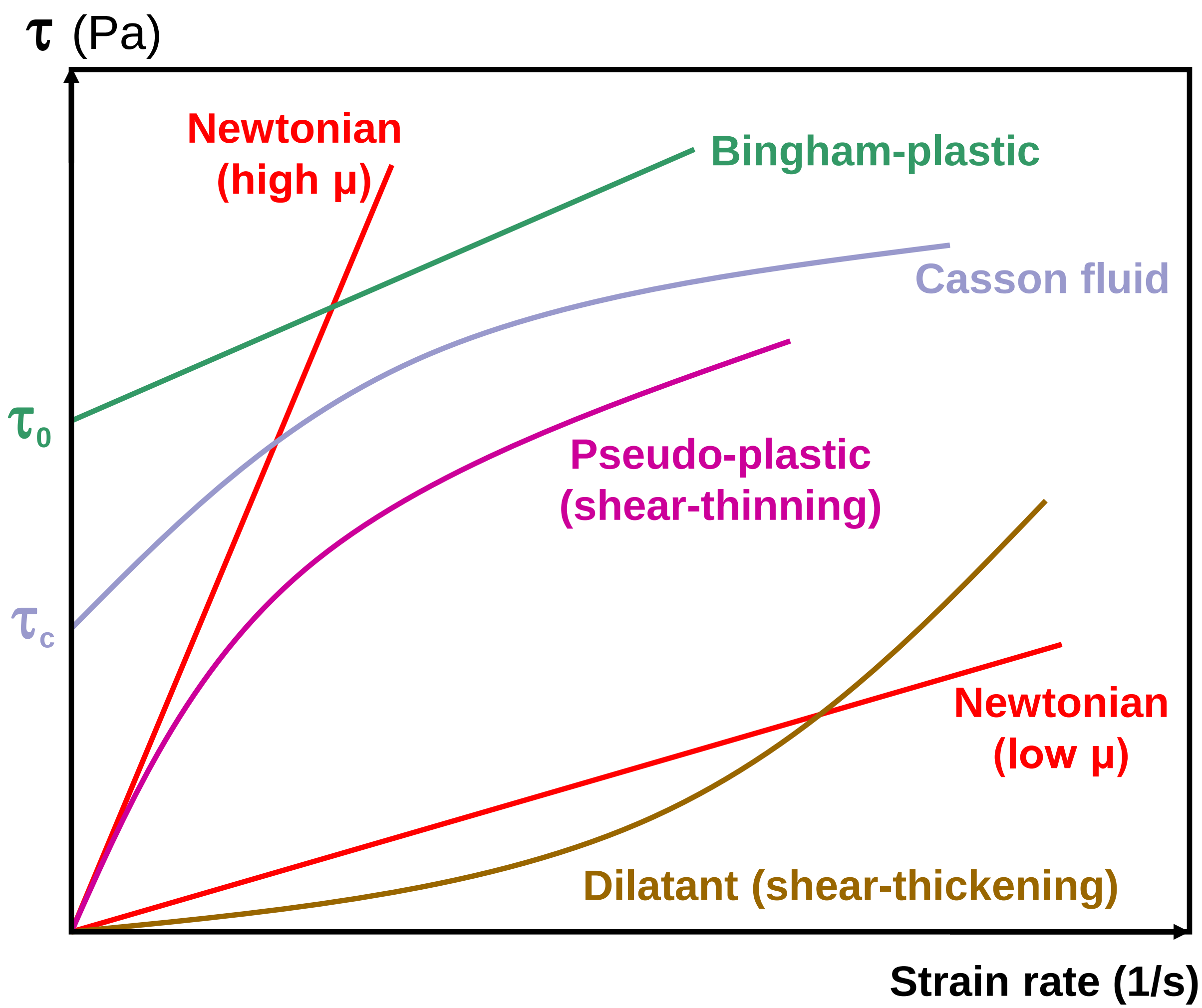
$$\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \mu \nabla^2 \boldsymbol{u}$$

Flow Classifications

- Laminar vs. turbulent flow.
  - Laminar flow: fluid particles move in smooth, layered fashion (no substantial mixing of fluid occurs).
  - Turbulent flow: fluid particles move in a chaotic, “tangled” fashion (significant mixing of fluid occurs).
- Steady vs. unsteady flow.
  - Steady flow: flow properties at any given point in space are constant in time, e.g. $p = p(x,y,z)$.
  - Unsteady flow: flow properties at any given point in space change with time, e.g. $p = p(x,y,z,t)$.

Flow Classifications

- Newtonian vs. Non-Newtonian
  - Newtonian fluids: water, air.
  - Pseudoplastic fluids: paint, printing ink.
  - Dilatant fluids: dense slurries, wet cement.
  - Bingham fluids: toothpaste, clay.
  - Casson fluids: blood, yogurt.
  - Visco-elastic fluids: polymers (not shown in graph because viscosity is not isotropic).



Flow Classifications

- Incompressible vs. compressible flow.
  - Incompressible flow: volume of a given fluid particle does not change.
    - Implies that density is constant everywhere.
    - Essentially valid for all liquid flows.
  - Compressible flow: volume of a given fluid particle can change with position.
    - Implies that density will vary throughout the flow field.
    - Compressible flows are further classified according to the value of the Mach number (M), where.

$$M = \frac{V}{c}$$

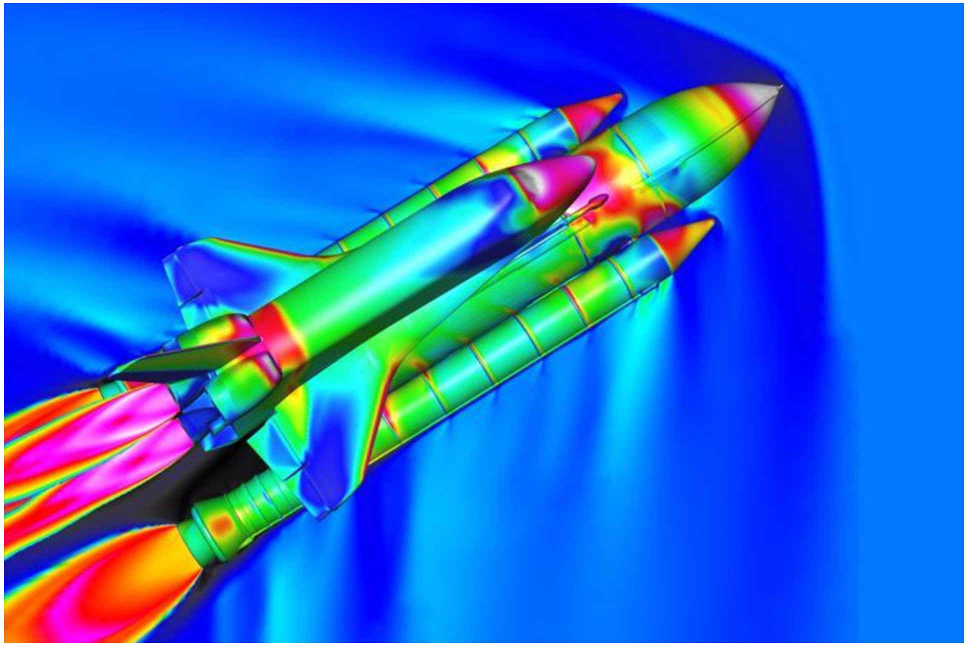
    - $M < 1$ - Subsonic.
    - $M > 1$ - Supersonic.

Flow Classifications

- Single phase vs. multiphase flow.
  - Single phase flow: fluid flows without phase change (either liquid or gas).
  - Multiphase flow: multiple phases are present in the flow field (e.g. liquid-gas, liquid-solid, gas-solid).
- Homogeneous vs. heterogeneous flow.
  - Homogeneous flow: only one fluid material exists in the flow field.
  - Heterogeneous flow: multiple fluid/solid materials are present in the flow field (multi-species flows).

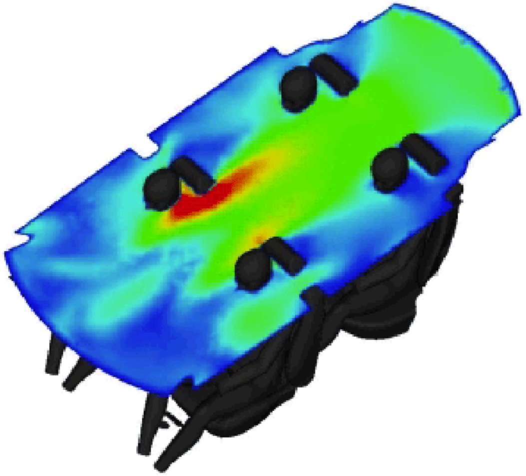
Flow configurations: External Flow

- Fluid flows over an object in an unconfined domain.
- Viscous effects are important only in the vicinity of the object.
- Away from the object, the flow is essentially inviscid.
- Examples: flows over aircraft, projectiles, ground vehicles.



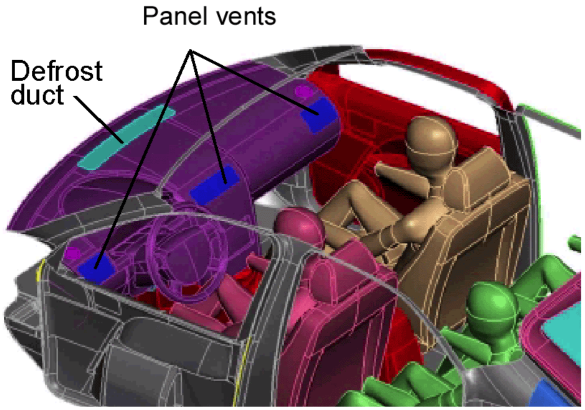
Flow Configurations: Internal Flow

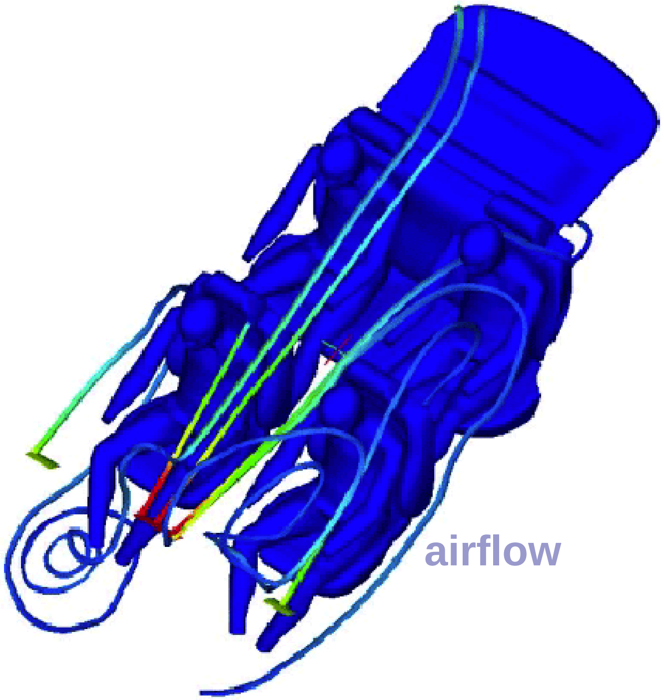
- Fluid flow is confined by walls, partitions, and other boundaries.
- Viscous effects extend across the entire domain.
- Examples: flows in pipes, ducts, diffusers, enclosures, nozzles.



temperature profile

car interior





airflow

Conclusion

- CFD simulations satisfy conservation laws of physics:
  - Mass conservation – the continuity equation
  - Momentum conservation (Newton’s Second Law of Motion) – the Navier-Stokes/Euler equations
  - First Law of Thermodynamics – the energy equation

- Fluid flows can be classified in a variety of ways:
  - Laminar vs. turbulent.
  - Compressible vs. incompressible.
  - Steady vs. unsteady.
  - Supersonic vs. transonic vs. subsonic.
  - Single-phase vs. multiphase.
  - Internal vs. external.

Thank You