*Journal of Quality Measurement and Analysis*
*Jurnal Pengukuran Kualiti dan Analisis*

*JQMA* 11(1) 2015, 11-17

# CLOSE-TO-CONVEX FUNCTIONS WITH STARLIKE POWERS

(Fungsi Hampir Cembung dengan Kuasa Bak Bintang)

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*ABSTRACT*

Let $K^\alpha$ be the set of functions *f*, analytic in $z \in D = \{z : |z| < 1\}$ satisfying $\operatorname{Re}\left(\frac{f'(z)}{(g(z)/z)^\alpha}\right) > 0$, for $g$ starlike in $D$ and $0 \le \alpha \le 1$. It is shown that such functions form a subset of the close-to-convex functions. Sharp bounds for the coefficients are given and the Fekete-Szegö problem is solved.

*Keywords:* univalent functions; starlike functions; close-to-convex functions; coefficients; Fekete-Szegö

*ABSTRAK*

Andaikan $K^\alpha$ set fungsi *f*, analisis dalam $z \in D = \{z : |z| < 1\}$ memenuhi $Ny\left(\frac{f'(z)}{(g(z)/z)^\alpha}\right) > 0$, untuk $g$ bak bintang dalam $D$ dan $0 \le \alpha \le 1$. Dapat ditunjukkan bahawa fungsi tersebut merupakan subset bagi suatu set fungsi hampir cembung. Batas-batas terbaik bagi pekali diberikan dan masalah Fekete-Szegö diselesaikan.

*Kata kunci*: fungsi univalen; fungsi bak bintang; fungsi hampir cembung; pekali; Fekete-Szegö

## 1. Introduction

Let $S$ be the class of analytic normalised univalent functions $f$ defined in $z \in D = \{z : |z| < 1\}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Denote by $S^*$ the subset of functions, starlike with respect to the origin and by $K$ the subset of close-to-convex functions. Then $f \in S^*$ , if and only if, for $z \in D = \{z : |z| < 1\}$,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$$

and $f \in K$ if and only if, there exists $g \in S^*$ such that

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > 0.$$

Hence $S^* \subset K \subset S$.

For $0 \le \alpha \le 1$, the subset $B(\alpha)$ of Bazilevič functions (1955) has been widely studied and is defined for $z \in D = \{z : |z| < 1\}$ by

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)^{1-\alpha} g(z)^{\alpha}}\right\} > 0$$

for some $g \in S^*$.

In the original paper of Bazilevič (1995), it was shown that $B(\alpha) \subset S$. Taking $g(z) \equiv z$ gives the class $B_1(\alpha)$, which has been extensively studied e.g. Singh (1973), Thomas (1985) and Thomas (1991). We note that $B_1(0)$ is the well-known class $R$ of functions whose derivative has positive real part in $D$.

Thus $f \in B_1(\alpha)$, if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)^{1-\alpha} z^{\alpha}}\right\} > 0.$$

We choose a different route as follows.

## 2. Main Result

**Definition 1**. For $0 \le \alpha \le 1$, denote by $K^{\alpha}$, the set of functions $f$ analytic in $z \in D = \{z : |z| < 1\}$ and given by (1) such that for some $g \in S^*$,

$$\mathrm{Re}\left\{\frac{zf'(z)}{z^{1-\alpha} g(z)^{\alpha}}\right\} > 0,$$

which is equivalent to

$$\mathrm{Re}\left\{\frac{f'(z)}{(g(z)/z)^{\alpha}}\right\} > 0.$$

**Theorem 1.** *Let $f \in K^{\alpha}$. Then $f \in K$ and so is univalent for $z \in D = \{z : |z| < 1\}$.*

**Proof**: Let $G(z) = z(g(z)/z)^{\alpha}$. Then it is easily seen that $G \in S^*(1-\alpha)$ for $0 < \alpha \le 1$, where $S^*(1-\alpha)$ is the class of functions starlike of order $1-\alpha$. Thus $K^{\alpha} \subset K$ and so functions in $K^{a}$ are univalent. □

**Theorem 2.** *Let* $f \in K^{\alpha}$ *and be given by* (1) *and* $F$ *be defined for* $z \in D = \{z : |z| < 1\}$ *by*

$$F'(z) = 1 + \sum_{n=1}^{\infty} n\gamma_n(\alpha)z^n = \frac{(1+z)}{(1-z)^{2\alpha+1}} = \left(1 + \sum_{k=1}^{\infty} \binom{2\alpha+k-1}{k} z^k\right)\left(1 + 2\sum_{k=1}^{\infty} z^k\right) \qquad (2)$$

*with* $\gamma_1(\alpha) = 1$. *Then* $|a_n| \le \gamma_n(\alpha)$ *for* $n \ge 2$, *where* $\gamma_n(\alpha) \sim \dfrac{2n^{2\alpha-1}}{\Gamma(2\alpha+1)}$ *as* $n \to \infty$.

*The inequality is sharp.*

**Proof**: Write

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \qquad \left(\frac{g(z)}{z}\right)^{\alpha} = 1 + \sum_{k=1}^{\infty} B_k(\alpha) z^k \qquad \text{and} \qquad h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k,$$

where $h \in P$, the class of functions whose derivative has positive real part in *D*. Then

$$f'(z) = \left(\frac{g(z)}{z}\right)^{\alpha} h(z) = \left(1 + \sum_{k=1}^{\infty} B_k(\alpha) z^k\right)\left(1 + \sum_{k=1}^{\infty} c_k z^k\right). \qquad (3)$$

Equating the coefficients, we obtain for $n \ge 2$, $na_n = \sum_{i=0}^{n-1} B_i(\alpha) c_{n-1-i}$, where $B_0(\alpha) = c_0 = 1$.

A result of Klein (1968) shows that for $g \in S^*$ and $i \ge 1$,

$$|B_i(\alpha)| \le \left|\binom{-2\alpha}{i}\right| = \binom{2\alpha+i-1}{i}$$

and so using the well-known inequality $|c_n| \le 2$ for $h \in P$ and comparing the coefficients in (2) and (3), the result follows. □

We note that elementary analysis shows that $\gamma_n(\alpha) \sim \dfrac{2n^{2\alpha-1}}{\Gamma(2\alpha+1)}$ as $n \to \infty$.

Theorem 1 gives the inequalities $|a_2| \le 1 + \alpha$ and $|a_3| \le \frac{1}{3}(2\alpha^2 + 5\alpha + 2)$, which we will use in the following theorem.

We now solve the Fekete-Szegö problem for functions in $K^{\alpha}$, noting that when $\alpha = 1$ we obtain the classical result of Keogh and Merkes (1969).

**Theorem 3.** *Let* $f \in K^{\alpha}$. *Then*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{1}{3}(2+\alpha)(1+2\alpha) - \mu(1+\alpha)^2 & \text{if } \mu \le \frac{2\alpha}{3(1+\alpha)}, \\ \frac{\alpha}{3} + \frac{2}{3}(1-\alpha^2) + \frac{4\alpha^2}{9\mu} & \text{if } \frac{2\alpha}{3(1+\alpha)} \le \mu \le \frac{2}{3}, \\ \frac{2+\alpha}{3} & \text{if } \frac{2}{3} \le \mu \le \frac{2(2+\alpha)}{3(1+\alpha)}, \\ \mu(1+\alpha^2) - \frac{1}{3}(2+\alpha)(1+2\alpha) & \text{if } \mu \ge \frac{2(2+\alpha)}{3(1+\alpha)}. \end{cases}$$

*All inequalities are sharp.*

**Proof**: We first recall the Fekete-Szegö inequality for starlike functions (see Keogh and Merkes (1969), for example), which states that for $g$ starlike in $D$,

$$\left|b_3 - \mu b_2^2\right| \le \text{ Max } \left(1, \left|3 - 4\mu\right|\right),$$

where $\mu$ is any real number and where, $b_2$ and $b_3$ are coefficients of the starlike function *g*. Equating coefficients in (3), gives

$$\left|a_3 - \mu a_2^2\right| = \left|\frac{\alpha}{3} b_3 - \frac{\alpha}{12}\left(2\left(1-\alpha\right) + 3\alpha\mu\right) b_2^2 + \frac{\alpha}{6} b_2 c_1 \left(2 - 3\mu\right) + \frac{1}{3}\left(c_2 - \frac{3\mu}{4} c_1^2\right)\right|$$

$$\le \frac{\alpha}{3}\left|b_3 - \frac{1}{4}(2(1-\alpha) + 3\boldsymbol{\mu}\ \ )\right| + \frac{\alpha}{3}\left|(2 - 3\mu)\right|\left|c_1\right|$$

$$+ \frac{1}{3}\left|c_2 - \frac{1}{2} c_1^2\right| + \frac{1}{12}\left|2 - 3\mu\right|\left|c_1\right|^2. \tag{4}$$

(i) The case $\dfrac{2\alpha}{3\left(1+\alpha\right)} \le \mu \le \dfrac{2}{3}$.

If $0 \le \mu \le \frac{2}{3}$, then $\left|3\alpha\mu - 2\alpha - 1\right| \ge 1$ and so using the well-known bound $\left|c_2 - \frac{1}{2} c_1^2\right| \le 2 - \frac{1}{2}\left|c_1\right|^2$ for $h \in P$, (4) gives

$$\left|a_3 - \mu a_2^2\right| \le \frac{\alpha}{3}\left|3\alpha\mu - 2\alpha - 1\right| + \frac{\alpha}{3}\left(2 - 3\mu\right)\left|c_1\right| + \frac{1}{3}\left(2 - \frac{1}{2}\left|c_1\right|^2\right) + \frac{1}{12}\left(2 - 3\mu\right)\left|c_1\right|^2.$$

Since $3\alpha\mu - 2\alpha - 1 \le 0$, when $0 \le \mu \le \frac{2}{3}$, we have

$$\left|a_3-\mu a_2^2\right| \le \frac{\alpha}{3}(1+2\alpha-3\alpha\mu)+\frac{\alpha}{3}(2-3\mu)|c_1|+\frac{1}{3}\left(2-\frac{1}{2}|c_1|^2\right)+\frac{1}{12}(2-3\mu)|c_1|^2$$

$$=\frac{\alpha}{3}(1+2\alpha-3\alpha\mu)+\frac{\alpha}{3}(2-3\mu)|c_1|+\frac{2}{3}-\frac{\mu}{4}|c_1|^2$$

$$=\Phi(x) \text{ say,}$$

with $|c_1|=x$. It is easy to see that $\Phi(x)$ is maximum at $x_0=\frac{2\alpha}{3\mu}(2-3\mu)$ and $\Phi(x_0)=\frac{\alpha}{3}+\frac{2}{3}(1-\alpha^2)+\frac{4\alpha^2}{9\mu}$, but since $x_0\le 2$, we have $\mu\ge\frac{2\alpha}{3(1+\alpha)}$. Equality is attained when $c_1=\frac{2\alpha}{3\mu}(2-3\mu)$, $c_2=b_2=2$ and $b_3=3$.

(ii) The case $\mu\le\frac{2\alpha}{3(1+\alpha)}$.

First note that the above shows that when $\mu=\frac{2\alpha}{3(1+\alpha)}$,

$$\left|a_3-\frac{2\alpha}{3(1+\alpha)}a_2^2\right|\le\frac{2+3\alpha}{3},$$

and so writing

$$a_3-\mu a_2^2=\frac{3\mu(1+\alpha)}{2\alpha}\left(a_3-\frac{2\alpha}{3(1+\alpha)}a_2^2\right)+\left(1-\frac{3\mu(1+\alpha)}{2\alpha}\right)a_3,$$

and using the bound $|a_3|\le\frac{1}{3}(2+\alpha)(1+2\alpha)$ obtained from Theorem 1, we have

$$\left|a_3-\mu a_2^2\right|\le\frac{1}{3}(2+\alpha)(1+2\alpha)-\mu(1+\alpha)^2.$$

Equality is attained when $b_3=3$, $b_2=c_1=c_2=2$.

(iii) The case $\frac{2}{3}\le\mu\le\frac{2(2+\alpha)}{3(1+\alpha)}$.

Since $\text{Max}\{1,|3\alpha\mu-2\alpha-1|\}=1$ on this interval,

$$\left|a_3-\mu a_2^2\right| \le \alpha+\frac{2\alpha}{3}(3\mu-2)|c_1|+\frac{1}{3}\left(2-\frac{1}{2}|c_1|^2\right)+\frac{1}{12}(3\mu-2)\left|c_1^2\right|$$

$$=\alpha+\frac{2\alpha}{3}(3\mu-2)x+\frac{1}{3}\left(2-\frac{1}{2}x^2\right)+\frac{1}{12}(3\mu-2)x^2$$

$$=\Psi(x) \text{ say,}$$

where again $x=|c_1|$. Since $\Psi'(x)=0$ at $x_0=\dfrac{2\alpha(3\mu-2)}{4-3\mu}$, and $x_0\le 2$, it follows that $\mu\le\dfrac{2(2+\alpha)}{3(1+\alpha)}$. We finally note that $\Psi(x_0)=\dfrac{2+\alpha}{3}$ and so the proof for $\dfrac{2}{3}\le\mu\le\dfrac{2(2+\alpha)}{3(1+\alpha)}$ is complete. Equality is attained when $c_1=b_2=0$, $c_2=2$ and $b_3=1$.

(iv) The case $\mu\ge\dfrac{2(2+\alpha)}{3(1+\alpha)}$.

Writing $a_3-\mu a_2^2=a_3-\dfrac{2(2+\alpha)}{3(1+\alpha)}a_2^2+\left(\dfrac{2(2+\alpha)}{3(1+\alpha)}-\mu\right)a_2^2$

$$\left|a_3-\mu a_2^2\right|\le\left|a_3-\frac{2(2+\alpha)}{3(1+\alpha)}a_2^2\right|+\left(\mu-\frac{2(2+\alpha)}{3(1+\alpha)}\right)\left|a_2^2\right|$$

$$\le\frac{2+\alpha}{3}+\left(\mu-\frac{2(2+\alpha)}{3(1+\alpha)}\right)(1+\alpha)^2 \;=\mu(1+\alpha)^2-\frac{1}{3}(2+\alpha)(1+2\alpha),$$

where we have used the bound $|a_2|\le 1+\alpha$ obtained from Theorem 1. Choosing $b_2=-2i$, $b_3=-3$, $c_1=2i$ and $c_2=-2$ shows that the inequality is sharp. □

We next define a related class $C^\alpha$.

**Definition 2.** For $0\le\alpha\le1$, denote by $C^\alpha$, the set of function $f$ analytic in $z\in D=\{z:|z|<1\}$ and given by (1) such that for some $g\in S^*$,

$$\operatorname{Re}\left\{\frac{z\left(zf'(z)\right)'}{z^{1-\alpha}g(z)^\alpha}\right\}>0,$$

which is equivalent to

$$\operatorname{Re}\left\{\frac{z\left(zf'(z)\right)'}{\left(g(z)/z\right)^\alpha}\right\}>0$$

and so $f \in C^{\alpha}$ if and only if, $zf' \in K^{\alpha}$.

Using the same methods as in the case of $K^{\alpha}$ it is easily established that $C^{\alpha} \subset C^{*}$, the set of quasi-convex functions first studied by Noor and Thomas (1980), and since $C^{*} \subset K$, functions in $C^{\alpha}$ are also close-to-convex and hence univalent.

We note that since $f \in C^{\alpha}$ if and only if, $zf' \in K^{\alpha}$, the coefficients of functions in $C^{\alpha}$ satisfy $n|a_n| \le \gamma_n(\alpha)$ for $n \ge 2$.

Similar techniques as those employed in Theorem 3 gives the following Fekete-Szegö theorem, the proof of which we omit.

**Theorem 4.** *Let $f \in C^{\alpha}$. Then*

$$|a_3 - \mu a_2^2| \le \begin{cases} \frac{1}{9}(2+\alpha)(1+2\alpha) - \frac{\mu}{4}(1+\alpha)^2 & \text{if } \mu \le \frac{8\alpha}{9(1+\alpha)}, \\ \frac{\alpha}{3} + \frac{2}{3}(1-\alpha^2) + \frac{16\alpha^2}{81\mu} & \text{if } \frac{8\alpha}{9(1+\alpha)} \le \mu \le \frac{8}{9}, \\ \frac{2+\alpha}{3} & \text{if } \frac{8}{9} \le \mu \le \frac{8(2+\alpha)}{9(1+\alpha)}, \\ \mu(1+\alpha^2) - \frac{1}{3}(2+\alpha)(1+2\alpha) & \text{if } \mu \ge \frac{8(2+\alpha)}{9(1+\alpha)}. \end{cases}$$

*All inequalities are sharp.*

## References


Bazilevič I. E. 1955. On a case of integrability in quadratures of the Löwner-Kufarev equation. *Mat. Sb.* **37**(79): 471-476. (Russian) **MR 17**: 356.

Keogh F. R. & Merkes E. P. 1969. A coefficient inequality for certain classes of analytic functions. *Proc. Amer. Math. Soc.* **20**: 8-12.

Klein M. 1968. On starlike functions of order alpha. *Trans. Amer. Math. Soc*. **131**: 99-106.

Noor K. I. & Thomas D. K. 1980. Quasi convex univalent functions. *Int. Journal Math. & Math. Sci.* **3**(2): 255-266.

Singh R. 1973. On Bazilevič functions. *Proc. Amer. Math. Soc.* **38**(2): 261-171.

Thomas D. K. 1985. On a subclass of Bazilevič functions. *Int. Journal. Math. & Math. Sci.* **8**(4): 779-783.

Thomas D. K. 1991. In *New Trends in Geometric Function Theory and Applications*: 146-158. Singapore: World Scientific.



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