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ON CERTAIN CLASS OF MEROMORPHIC HARMONIC CONCAVE FUNCTIONS DEFINED BY SALAGEAN OPERATOR

(Mengenai Kelas Fungsi Cekung Harmonik Meromorfi Tertentu yang Ditakrif oleh Pengoperasi Salagean)

KHALID CHALLAB & MASLINA DARUS

ABSTRACT

A class of meromorphic harmonic concave functions defined by Salagean operator in the unit disc is introduced. Coefficient bounds, distortion inequalities, extreme points, geometric convolution, integral convolution and convex combination for the functions belonging to this class have been achieved.

Keywords: Meromorphic function; harmonic function; concave function; Salagean operator

ABSTRAK

Dalam makalah ini diperkenalkan kelas fungsi cekung harmonik meromorfi tertentu yang ditakrif oleh pengoperasi Salagean dalam cakera unit. Batas pekali, ketaksamaan erotan, titik ekstrem, konvolusi geometri, konvolusi kameran dan gabungan cembung bagi fungsi yang terkandung dalam kelas ini diperoleh.

Kata kunci: fungsi meromorfi; fungsi harmonik; fungsi cekung; pengoperasi Salagean

1. Introduction

Conformal maps of the unit disc onto convex domain are a classical topic and several results are found related to this field. Avkhadiev and Wirths (2005) found the conformal mapping of a unit disc onto concave domains (the complements of convex closed sets). This is interesting due to that not many problems have been discussed in this approach.

Let $\mathbb{U}$ denote the open unit disc, where f has the form given by

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1}$$

that maps $\mathbb{U}$ conformally in a domain which complement with respect to C is convex and that be content with the normalisation $f(1)=\infty$. Moreover, they imposed on above functions the condition in order the opening angle of $f(\mathbb{U})$ at infinity is less than or equal to $\alpha\pi, \alpha\in(1,2]$. These families of functions are denoted by $C_0(\alpha)$. The class $C_0(\alpha)$ is referred to as the class of concave univalent functions (see Avkhadiev *et al.* (2006), Avkhadiev and Wirths (2005), and Bhowmik *et al.* (2010)).

Chuaqui *et al.* (2012) defined the concept of meromorphic concave mappings. A conformal mapping of meromorphic function on the unit disc $\mathbb{U}$ is said to be a concave mapping if its image is the complement of a compact, convex set.

If f has the form

$$f(z)=\frac{1}{z}+a_0+a_1z+a_2z^2+\dots,$$

then a necessary and sufficient condition for f to be a concave mapping is

$$1+Re\left\{z\frac{f''(z)}{f'(z)}\right\}<0,\quad |z|<1.$$

A continuous function $f=u+iv$ is a complex valued harmonic function in a domain $\mathbb{U}\subset C$ if both u and v are real harmonic in $\mathbb{U}$. In any simply connected domain, should write $f=h+\overline{g}$ where h and g are analytic in $\mathbb{U}$ (see Clunie and Sheil-Small (1984)). A necessary and sufficient condition for f to be locally univalent and orientation preserving in $\mathbb{U}$ is that $|h'|>|g'|$ in $\mathbb{U}$ (see Clunie and Sheil-Small (1984)). Hengartner and Schober (1987) investigated functions harmonic in the exterior of the unit disc $\tilde{\mathbb{U}}=\{z:|z|>1\}$. They found that complex valued, harmonic, sense preserving, univalent mapping *f* necessity confess the representation

$$f(z)=h(z)+\overline{g(z)}+A\,log\,|z|,$$

where $h(z)$ and $g(z)$ are defined by

$$h(z)=\alpha z+\sum_{n=1}^{\infty}a_n z^{-n},\quad g(z)=\beta\overline{z}+\sum_{n=1}^{\infty}b_n z^{-n}$$

for $0\le\beta<|\alpha|$, $A\in C$ and $z\in\tilde{\mathbb{U}}$.

We call h the analytic part and g the co-analytic part of *f*.

For $z\in\tilde{\mathbb{U}}\setminus\{0\}$, let M_H be the class of functions:

$$f(z)=h(z)+\overline{g(z)}=\frac{1}{z}+\sum_{k=1}^{\infty}a_k z^k+\overline{\sum_{k=1}^{\infty}b_k z^k}, \tag{2}$$

which are harmonic in the punctured unit disc $\mathbb{U}\setminus\{0\}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{U}\setminus\{0\}$ and $\mathbb{U}$, respectively, and $h(z)$ has a simple pole at the origin with residue 1 here (see Al-Shaqsi and Darus 2008).

A function $f\in M_H$ is said to be in the subclass MS_H^* of meromorphically harmonic starlike functions in $\mathbb{U}\setminus\{0\}$ if it satisfies the condition

$$Re\left\{-\frac{zh'(z)-\overline{zg'(z)}}{h(z)+g(z)}\right\}>0,\qquad \left(z\in\mathbb{U}\setminus\{0\}\right).$$

Also, a function $f\in M_H$ is said to be in the subclass MC_H of meromorphically harmonic convex functions in $\mathbb{U}\setminus\{0\}$ if it satisfies the condition

$$Re\left\{-\frac{zh''(z)+h'(z)-\overline{zg''(z)+g'(z)}}{h'(z)+g'(z)}\right\}>0,\qquad \left(z\in\mathbb{U}\setminus\{0\}\right).$$

Note that the classes of harmonic meromorphic starlike functions, harmonic meromorphic convex functions and harmonic meromorphic concave functions (MHC_0) have been studied by Jahangiri and Silverman (1999), Jahangiri (2000), Jahangiri (1998) and recently by Aldawish and Darus (2015).

Salagean (1983) introduced the operator S^n for $f \in MHC_\circ$ which is the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disc $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0$.

Now, we define S^n for $f = h + \overline{g}$ given by (2) as

$$S^n f(z) = S^n h(z) + S^n \overline{g}(z), \; n = 0,1,2,\ldots, \; z \in U \setminus \{0\},$$

where

$$S^n h(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k z^k,$$

$$S^n \overline{g}(z) = \sum_{k=1}^{\infty} k^n \overline{b_k z^k}.$$

This work is an attempt to give a connection between harmonic function and meromorphic concave functions defined by Salagean operator by introducing a class $S^n MHC_0$ of meromorphic harmonic concave functions defined by Salagean operator.

Definition 1.1. Let $S^n MHC_0$ denote the class of meromorpic harmonic concave functions $S^n f(z)$ defined by Salagean differential operator of the form

$$S^n f(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k z^k + \sum_{k=1}^{\infty} k^n \overline{b_k z^k} \tag{3}$$

such that

$$1 + Re\left\{ \frac{z\left(S^n f(z)\right)''}{(S^n f(z))'} \right\} < 0, \; n \geq 0.$$

2. Coefficient Conditions

In this section, sufficient coefficient condition for a function $S^n f(z) \in S^n MHC_0$ is derived.

Theorem 2.1. *Let $S^n f(z) = h + \overline{g}$ be of the form*

$$S^n f(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k z^k + \sum_{k=1}^{\infty} k^n \overline{b_k z^k}.$$

If $\sum_{k=1}^{\infty} k^{n+2}(|a_k|+|b_k|) \le 1$*, then* $S^n f(z)$ *is harmonic univalent, sense preserving in* $\mathbb{U}\setminus\{0\}$.

Proof:

First for $0<|z_1|\le|z_2|<1$, we have

$$\left|S^n f(z_1)-S^n f(z_2)\right| = \left|\frac{(-1)^n}{z_1}-\frac{(-1)^n}{z_2}+\sum_{k=1}^{\infty}k^n a_k(z_1^k-z_2^k)+\sum_{k=1}^{\infty}k^n b_k(z_1^k-z_2^k)\right|$$

$$\ge \frac{\left|(-1)^n\right|}{|z_1|}-\frac{\left|(-1)^n\right|}{|z_2|}-\sum_{k=1}^{\infty}k^n\left(|a_k|+|b_k|\right)\left|z_1^k-z_2^k\right|$$

$$> \frac{|z_1-z_2|}{|z_1||z_2|}\left[1-|z_1||z_2|\sum_{k=1}^{\infty}k\cdot k^n\left(|a_k|+|b_k|\right)\right]$$

$$> \frac{|z_1-z_2|}{|z_1||z_2|}\left[1-|z_2|^2\sum_{k=1}^{\infty}k^{n+1}\left(|a_k|+|b_k|\right)\right]$$

$$> \frac{|z_1-z_2|}{|z_1||z_2|}\left[1-\sum_{k=1}^{\infty}k^{n+2}\left(|a_k|+|b_k|\right)\right].$$

The last expression is non negative by $\sum_{k=1}^{\infty}k^{n+2}(|a_k|+|b_k|)\le 1$ and $S^n f(z)$ is univalent in $\mathbb{U}\setminus\{0\}$.

Now we want to show that $S^n f(z)$ is sense preserving in $\mathbb{U}\setminus\{0\}$. So we need to show that $\left|h'(z)\right|\ge\left|g'(z)\right|$ in $\mathbb{U}\setminus\{0\}$.

$$\left|h'(z)\right| \ge \frac{1}{|z|^2}-\sum_{k=1}^{\infty}k^{n+1}|a_k||z|^{k-1}$$

$$=\frac{1}{r^2}-\sum_{k=1}^{\infty}k^{n+1}|a_k|r^{k-1} > 1-\sum_{n=1}^{\infty}k^{n+1}|a_k|$$

$$\ge 1-\sum_{k=1}^{\infty}k^{n+2}|a_k|$$

$$\ge \sum_{k=1}^{\infty}k^{n+2}|b_k| > \sum_{k=1}^{\infty}k^{n+1}|b_k|r^{k-1}$$

$$=\sum_{k=1}^{\infty}k^{n+1}|b_k||z|^{k-1}\ge\left|g'(z)\right|.$$

Thus this completes the proof of the theorem. □

Theorem 2.2. *Let* $S^n f(z) = h + \overline{g}$ *be of the form* (3), *then* $S^n f(z) \in S^n MHC_{\circ}$ *if the inequality*

$$\sum_{k=1}^{\infty} k^{n+2}\left(|a_k| + |b_k|\right) \le 1 \qquad (4)$$

holds for coefficient $S^n f(z) = h + \overline{g}$.

Proof:

Suppose that the inequality (4) holds by using the fact that $Re(w) < 0 \leftrightarrow \left|\frac{w+1}{w-1}\right| < 1$, so it suffices to show that $\left|\frac{w+1}{w-1}\right| < 1$. We have

$$w = 1 + \frac{z\left(S^n f(z)\right)''}{(S^n f(z))'},$$

$$w = \frac{zg'(z)}{g(z)}, \text{ where } g(z) = z\left(S^n f(z)\right)'$$

$$\left|\frac{w+1}{w-1}\right| = \left| \frac{\dfrac{\dfrac{(-1)^{n+2}}{z} + \sum_{k=1}^{\infty} k^{n+2} a_k z^k - \sum_{k=1}^{\infty} k^{n+2}\overline{b_k z^k}}{\dfrac{(-1)^{n+1}}{z} + \sum_{k=1}^{\infty} k^{n+1} a_k z^k - \sum_{k=1}^{\infty} k^{n+1}\overline{b_k z^k}} + 1}{\dfrac{\dfrac{(-1)^{n+2}}{z} + \sum_{k=1}^{\infty} k^{n+2} a_k z^k - \sum_{k=1}^{\infty} k^{n+2}\overline{b_k z^k}}{\dfrac{(-1)^{n+1}}{z} + \sum_{k=1}^{\infty} k^{n+1} a_k z^k - \sum_{k=1}^{\infty} k^{n+1}\overline{b_k z^k}} - 1} \right|$$

$$= \left| \frac{\dfrac{\dfrac{(-1)^{n+2}}{z} + \sum_{k=1}^{\infty} k^{n+2} a_k z^k - \sum_{k=1}^{\infty} k^{n+2}\overline{b_k z^k} + \dfrac{(-1)^{n+1}}{z} + \sum_{k=1}^{\infty} k^{n+1} a_k z^k - \sum_{k=1}^{\infty} k^{n+1}\overline{b_k z^k}}{\dfrac{(-1)^{n+1}}{z} + \sum_{k=1}^{\infty} k^{n+1} a_k z^k - \sum_{k=1}^{\infty} k^{n+1}\overline{b_k z^k}}}{\dfrac{\dfrac{(-1)^{n+2}}{z} + \sum_{k=1}^{\infty} k^{n+2} a_k z^k - \sum_{k=1}^{\infty} k^{n+2}\overline{b_k z^k} - \dfrac{(-1)^{n+1}}{z} - \sum_{k=1}^{\infty} k^{n+1} a_k z^k + \sum_{k=1}^{\infty} k^{n+1}\overline{b_k z^k}}{\dfrac{(-1)^{n+1}}{z} + \sum_{k=1}^{\infty} k^{n+1} a_k z^k - \sum_{k=1}^{\infty} k^{n+1}\overline{b_k z^k}}} \right|$$

$$= \left| \frac{\sum_{k=1}^{\infty}\left(k^{n+2} + k^{n+1}\right) a_k z^k - \sum_{k=1}^{\infty}\left(k^{n+2} + k^{n+1}\right)\overline{b_k z^k}}{\dfrac{-2(-1)^{n+1}}{z} + \sum_{k=1}^{\infty}\left(k^{n+2} - k^{n+1}\right) a_k z^k - \sum_{k=1}^{\infty}\left(k^{n+2} - k^{n+1}\right)\overline{b_k z^k}} \right|$$

$$\Rightarrow\left|\frac{w+1}{w-1}\right|<\frac{\sum_{k=1}^{\infty}\left(k^{n+2}+k^{n+1}\right)|a_k|+\sum_{k=1}^{\infty}\left(k^{n+2}+k^{n+1}\right)|b_k|}{2-\sum_{k=1}^{\infty}\left(k^{n+2}-k^{n+1}\right)|a_k|-\sum_{k=1}^{\infty}\left(k^{n+2}-k^{n+1}\right)|b_k|}.$$

The last expression is bounded above by 1 if

$$\sum_{k=1}^{\infty}\left(k^{n+2}+k^{n+1}\right)|a_k|+\sum_{k=1}^{\infty}\left(k^{n+2}+k^{n+1}\right)|b_k|\le$$

$$2-\sum_{k=1}^{\infty}\left(k^{n+2}-k^{n+1}\right)|a_k|-\sum_{k=1}^{\infty}\left(k^{n+2}-k^{n+1}\right)|b_k|,$$

which is equivalent to our condition by

$$\sum_{k=1}^{\infty}k^{n+2}(|a_k|+|b_k|)\le 1. \square$$

Theorem 2.3. *Let* $S^n f(z)=h+\overline{g}$ *be of the form* (3). *A necessary and sufficient condition for* $S^n f(z)$ *to be in* $S^n MHC_0$ *is that*

$$\sum_{k=1}^{\infty}k^{n+2}(|a_k|+|b_k|)\le 1.$$

Proof:

In view of Theorem 2.2, we need only to show that $S^n f(z)\notin S^n MHC_0$, if the coefficient inequality (4) does not hold, to this end we have if $S^n f(z)$ meromorphic harmonic concave function, then $1+Re\left\{\frac{z\left(S^n f(z)\right)''}{(S^n f(z))'}\right\}$, equivalent to

$$Re\frac{zg'(z)}{g(z)}=Re\frac{z\left(\frac{(-1)^{n+2}}{z^2}+\sum_{k=1}^{\infty}k^{n+2}a_k z^{k-1}+\sum_{k=1}^{\infty}k^{n+2}\overline{b_k z^{k-1}}\right)}{\frac{(-1)^{n+1}}{z}+\sum_{k=1}^{\infty}k^{n+1}a_k z^k+\sum_{k=1}^{\infty}k^{n+1}\overline{b_k z^k}}$$

$$=Re\frac{\frac{(-1)^{n+2}}{z}+\sum_{k=1}^{\infty}k^{n+2}a_k z^k+\sum_{k=1}^{\infty}k^{n+2}\overline{b_k z^k}}{\frac{(-1)^{n+1}}{z}+\sum_{k=1}^{\infty}k^{n+1}a_k z^k+\sum_{k=1}^{\infty}k^{n+1}\overline{b_k z^k}}.$$

For $|z|=r>1$, the above expression reduce to

$$Re\frac{-(-1)^{n+1}+\sum_{k=1}^{\infty}k^{n+2}(|a_k|+|b_k|)r^k}{(-1)^{n+1}+\sum_{k=1}^{\infty}k^{n+1}(|a_k|+|b_k|)r^k}=\frac{A(r)}{B(r)}\le 0.$$

If the condition (4) does not hold, then $A(r)$ and $B(r)$ are positive for r sufficiently close to 1. Thus there exists a $z_0 = r_0 > 1$ for which the quotient $\frac{A(r)}{B(r)}$ is positive. This contradicts the required condition that $\frac{A(r)}{B(r)} \le 0$, and so the proof is completed .□

3. Distortion bounds and extreme points

Bounds and extreme points for functions belonging to the class $S^n MHC_0$ are estimated in this section.

Theorem 3.1. *If* $S^n f_k = S^n h_k + S^n \overline{g}_k \in S^n MHC_0$ *and* $0 < |z| = r < 1$ *then*

$$\left|S^n f_k(z)\right| \le \frac{1+r^2}{r}, \text{ and } \left|S^n f_k(z)\right| \ge \frac{1-r^2}{r}.$$

Proof:

Let $S^n f_k = S^n h_k + S^n \overline{g}_k \in S^n MHC_0$. Taking the absolute value of $S^n f_k$, we obtain

$$\left|S^n f_k(z)\right| = \left|\frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k z^k + \sum_{k=1}^{\infty} k^n \overline{b_k z^k}\right|$$

$$\ge \frac{1}{r} - \sum_{k=1}^{\infty} k^n \left(|a_k| + |b_k|\right) r^k \ge \frac{1}{r} - \sum_{k=1}^{\infty} k^n \left(|a_k| + |b_k|\right) r$$

$$\ge \frac{1}{r} - \sum_{k=1}^{\infty} k^{n+2} \left(|a_k| + |b_k|\right) r,$$

$$\ge \frac{1}{r} - r = \frac{1-r^2}{r};$$

and

$$\left|S^n f_k(z)\right| = \left|\frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k z^k + \sum_{k=1}^{\infty} k^n \overline{b_k z^k}\right|$$

$$\le \frac{1}{r} + \sum_{k=1}^{\infty} k^n \left(|a_k| + |b_k|\right) r^k$$

$$\le \frac{1}{r} + \sum_{k=1}^{\infty} k^{n+2} \left(|a_k| + |b_k|\right) r,$$

$$\le \frac{1}{r} + r = \frac{1+r^2}{r}. \square$$

Theorem 3.2. *Let* $S^n f_n = S^n h_n + S^n \overline{g}_n$ *where* $S^n h_n$ *and* $S^n \overline{g}_n$ *are given by*

$$S^n f_n(z) = S^n h_n(z) + S^n \overline{g}_n(z)$$

$$= \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k z^k + \sum_{k=1}^{\infty} k^n \overline{b_k z^k}.$$

Set

$$S^n h_{n,0} = S^n g_{n,0} = \frac{(-1)^n}{z},\ S^n h_{n,k}(z) = \frac{(-1)^n}{z} + \frac{1}{k^{n+2}} z^k,$$

for $k = 1,2,3,\ldots$*, and*

$$S^n g_{n,k}(z) = \frac{(-1)^n}{z} + \frac{1}{k^{n+2}} \overline{z^k},$$

for $k = 1,2,3,\ldots$*, then* $S^n f_n \in S^n MHC_0$ *if and only if* $S^n f_n$ *can be expressed as*

$$S^n f_{n,k} = \sum_{k=0}^{\infty} (\lambda_k S^n h_{n,k}(z) + \gamma_k S^n g_{n,k}(z))$$

where $\lambda_k \geq 0,\ \gamma_k \geq 0$ *and* $\sum_{k=0}^{\infty} (\lambda_k + \gamma_k) = 1$*. In particular the extreme points of* $S^n MHC_0$ *are* $\{S^n h_{n,k}\}$ *and* $\{S^n g_{n,k}\}$.

Proof:

For functions $S^n f_n = S^n h_n + S^n \overline{g}_n$ where $S^n h_n$ and $S^n \overline{g}_n$ are given by (3), we have

$$S^n f_{n,k}(z) = \sum_{k=0}^{\infty} (\lambda_k S^n h_{n,k}(z) + \gamma_k S^n g_{n,k}(z))$$

$$= \lambda_0 S^n h_{n,0} + \gamma_0 S^n g_{n,0} + \sum_{k=1}^{\infty} \lambda_k \left(\frac{(-1)^n}{z} + \frac{1}{k^{n+2}} z^k \right) + \sum_{k=1}^{\infty} \gamma_k \left(\frac{(-1)^n}{z} + \frac{1}{k^{n+2}} \overline{z^k} \right)$$

$$= \sum_{k=0}^{\infty} (\lambda_k + \gamma_k) \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} \frac{1}{k^{n+2}} \left(\lambda_k z^k + \gamma_k \overline{z^k} \right).$$

Now by Theorem 2.2,

$$\sum_{k=1}^{\infty} \lambda_k \frac{1}{k^{n+2}} k^{n+2} + \gamma_k \frac{1}{k^{n+2}} k^{n+2} = \sum_{k=1}^{\infty} \lambda_k + \gamma_k = 1 - \lambda_0 - \gamma_0 \leq 1.$$

So $S^n f_{n,k}(z) \in S^n MHC_0$.

Conversely, suppose that $S^n f_{n,k}(z) \in S^n MHC_0$, setting

$$\lambda_k = k^{n+2}\left|k^n a_k\right|,\ k \geq 1 \ \text{ and } \ \gamma_k = k^{n+2}\ |\left|k^n b_k\right|,\ k \geq 1,$$

we define

$$\lambda_0 + \gamma_0 = 1 - \sum_{k=1}^{\infty} \lambda_k - \sum_{k=1}^{\infty} \gamma_k .$$

Therefore, $S^n f_n$ can be written as

$$S^n f_n(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} \left|k^n a_k\right| z^k + \sum_{k=1}^{\infty} \left|k^n b_k\right| \overline{z^k}$$

$$= \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} \lambda_k \frac{1}{k^{n+2}} z^k + \gamma_k \frac{1}{k^{n+2}} z^k$$

$$= \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} \left(S^n h_{n,k}(z) - \frac{(-1)^n}{z} \right) \lambda_k + \sum_{k=1}^{\infty} \left(S^n \mathrm{g}_{n,k}(z) - \frac{(-1)^n}{z} \right) \gamma_k$$

$$= \frac{(-1)^n}{z} \left(1 - \sum_{k=1}^{\infty} \lambda_k - \sum_{k=1}^{\infty} \gamma_k \right) + \sum_{k=1}^{\infty} S^n h_{n,k}(z) \lambda_k + \sum_{k=1}^{\infty} S^n \mathrm{g}_{n,k}(z) \gamma_k$$

$$= (\lambda_0 + \gamma_0) \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} S^n h_{n,k}(z) \lambda_k + \sum_{k=1}^{\infty} S^n \mathrm{g}_{n,k}(z) \gamma_k$$

$$= \sum_{k=0}^{\infty} \left(\lambda_k S^n h_{n,k}(z) + \gamma_k S^n \mathrm{g}_{n,k}(z) \right).$$

The proof is complete, therefore $\left\{S^n h_{n,k}\right\}$ and $\left\{S^n \mathrm{g}_{n,k}\right\}$ are extreme points .□

4. Convolution Properties

In this section, we define and study convolution, geometric convolution and integral convolution of the class $S^n MHC_0$. The harmonic functions $S^n f_n, S^n F_n$ are defined as follows:

$$S^n f_n(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n \left|a_k\right| z^k + \sum_{k=1}^{\infty} k^n \left|b_k\right| \overline{z^k}\ , \tag{5}$$

$$S^n F_n(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n \left|A_k\right| z^k + \sum_{k=1}^{\infty} k^n \left|B_k\right| \overline{z^k}\ . \tag{6}$$

The convolution of $S^n f_n$ and $S^n F_n$ is given by

$$\left(S^nf_n * S^nF_n\right)_{(z)} = S^nf_n\left(z\right) * S^nF_n\left(z\right)$$

$$= \frac{\left(-1\right)^n}{z} + \sum_{k=1}^{\infty} k^n \left|a_k\right|\left|A_k\right| z^k + \sum_{k=1}^{\infty} k^n \left|b_k\right|\left|B_k\right| \overline{z^k} . \quad (7)$$

The geometric convolution of S^nf_n and S^nF_n is given by

$$\left(S^nf_n\, S^nF_n\right)_{(z)} = S^nf_n\left(z\right) S^nF_n\left(z\right)$$

$$= \frac{\left(-1\right)^n}{z} + \sum_{k=1}^{\infty} k^n \sqrt{\left|a_k A_k\right|} z^k + \sum_{k=1}^{\infty} k^n \sqrt{\left|b_k B_k\right|} \overline{z^k}. \quad (8)$$

The integral convolution of S^nf_n and S^nF_n is given by

$$\left(S^nf_n \Diamond S^nF_n\right)_{(z)} = S^nf_n\left(z\right) \Diamond S^nF_n\left(z\right)$$

$$= \frac{\left(-1\right)^n}{z} + \sum_{k=1}^{\infty} k^n \frac{\left|a_k A_k\right|}{k} z^k + \sum_{k=1}^{\infty} k^n \frac{\left|b_k B_k\right|}{k} \overline{z^k} . \quad (9)$$

Theorem 4.1. *Let* $S^nf_n \in S^nMHC_0$ *and* $S^nF_n \in S^nMHC_0$, *then the convolution* $S^nf_n * S^nF_n \in S^nMHC_\circ$.

Proof:

For S^nf_n and S^nF_n are given by (5) and (6), then the convolution $S^nf_n * S^nF_n$ is given by (7), we show that the coefficients of $S^nf_n * S^nF_n$ satisfy the required condition given in Theorem 2.2. For $S^nF_n \in S^nMHC_0$, we note that $\left|A_k\right| \le 1$ and $\left|B_k\right| \le 1$. Now for convolution function $S^nf_n * S^nF_n$, we obtain

$$\sum_{k=1}^{\infty} k^{n+2} \left|a_k\right|\left|A_k\right| + \sum_{k=1}^{\infty} k^{n+2} \left|b_k\right|\left|B_k\right| \le \sum_{k=1}^{\infty} k^{n+2} \left|a_k\right| + \sum_{k=1}^{\infty} k^{n+2} \left|b_k\right| \le 1.$$

The proof is complete .□

Theorem 4.2. *If* S^nf_n *and* S^nF_n *of the form* (5) *and* (6) *belong to the class* S^nMHC_0, *then the geometric convolution* $S^nf_n\, S^nF_n$ *also belong to the class* S^nMHC_0.

Proof:

Since $S^nf_n, S^nF_n \in S^nMHC_0$, it follows that

$$\sum_{k=1}^{\infty} k^{n+2} (\left|a_k\right| + \left|b_k\right|) \le 1, \quad \sum_{k=1}^{\infty} k^{n+2} (\left|A_k\right| + \left|B_k\right|) \le 1.$$

Hence by Cauchy-Schwartz's inequality, it is noted that

$$\sum_{k=1}^{\infty} k^{n+2}(\sqrt{|a_k A_k|}+\sqrt{|b_k B_k|})\leq 1.$$

The proof is complete .□

Theorem 4.3. *If* $S^n f_n$ *and* $S^n F_n$ *of the form* (5) *and* (6) *belong to the class* $S^n MHC_0$*, then the integral convolution* $S^n f_n \Diamond S^n F_n$ *also belong to the class* $S^n MHC_0$.

Proof:

Since $S^n f_n, S^n F_n \in S^n MHC_0$, it follows that $|A_k|\leq 1$ and $|B_k|\leq 1$, then $S^n f_n \Diamond S^n F_n \in S^n MHC_0$ because

$$\sum_{k=1}^{\infty} k^{n+2}\frac{|a_k A_k|}{k}+\sum_{k=1}^{\infty} k^{n+2}\frac{|b_k B_k|}{k} \leq \sum_{k=1}^{\infty} k^{n+2}\frac{|a_k|}{k}+\sum_{k=1}^{\infty} k^{n+2}\frac{|b_k|}{k}$$
$$\leq \sum_{k=1}^{\infty} k^{n+2}|a_k|+\sum_{k=1}^{\infty} k^{n+2}|b_k|\leq 1.$$

This proved the required result .□

5. Convex Combinations

In this section, we show that the class $S^n MHC_0$ is invariant under convex combinations of its members.

Theorem 5.1. *The class* $S^n MHC_0$ *is closed under convex combinations.*

Proof:

For $i=1,2,3,\ldots$ suppose that $S^n f_i(z)\in S^n MHC_0$ where $S^n f_i$ is given by

$$S^n f_i(z)=\frac{(-1)^n}{z}+\sum_{k=1}^{\infty} k^n a_{ik} z^k+\sum_{k=1}^{\infty} k^n \overline{b_{ik} z^k},\ a_{ik}\geq 0,\ b_{ik}\geq 0,$$

then by Theorem 2.2,

$$\sum_{k=1}^{\infty} k^{n+2}\left(|a_{ik}|+|b_{ik}|\right)\leq 1. \tag{10}$$

For $\sum_{i=1}^{\infty} t_i=1,\ 0\leq t_i\leq 1$, the convex combinations of $S^n f_i$ may be written as

$$\sum_{i=1}^{\infty} t_i S^n f_i(z)=\frac{(-1)^n}{z}+\sum_{k=1}^{\infty} k^n(\sum_{i=1}^{\infty} t_i a_{ik})z^k+\sum_{k=1}^{\infty} k^n\overline{(\sum_{i=1}^{\infty} t_i b_{ik})z^k},$$

then by (10)

$$\sum_{k=1}^{\infty} k^{n+2}\left(\left|\sum_{i=1}^{\infty} t_i a_{ik}\right| + \left|\sum_{i=1}^{\infty} t_i b_{ik}\right|\right) = \sum_{i=1}^{\infty} t_i \left[\sum_{k=1}^{\infty} k^{n+2}\left(|a_{ik}| + |b_{ik}|\right)\right]$$

$$\leq \sum_{i=1}^{\infty} t_i = 1.$$

Thus, $\sum_{i=1}^{\infty} t_i S^n f_i \quad z \quad \in S^n MHC_0 .$□

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School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600 UKM Bangi
Selangor DE, MALAYSIA
E-mail: khalid_math1363@yahoo.com, maslina@ukm.edu.my[*]

[*]Corresponding author