Journal of Quality Measurement and Analysis *JQMA* 11(1) 2015, 61-67
Jurnal Pengukuran Kualiti dan Analisis

OSCILLATION BEHAVIOUR OF FIRST ORDER NEUTRAL DELAY DIFFERENTIAL EQUATIONS

(Gelagat Ayunan bagi Persamaan Pembezaan Tunda Neutral Peringkat Pertama)

ROKIAH@ROZITA AHMAD, FATIMA N. AHMED, UMMUL KHAIR SALMA DIN
& MOHD SALMI MD. NOORANI

ABSTRACT

In this paper, we obtain some sufficient conditions for the oscillation of all solutions of first order neutral delay differential equations. The results attained improve and generalise some of existing results in the literature. Some examples are presented to illustrate our results.

Keywords: oscillation; nonoscillation; neutral delay differential equations; first order

ABSTRAK

Dalam makalah ini, diperoleh beberapa syarat cukup untuk ayunan bagi penyelesaian persamaan pembezaan tunda neutral peringkat pertama. Hasil yang diperoleh dapat menambah baik dan mengitlak beberapa keputusan yang sedia ada dalam susastera. Beberapa contoh dikemukakan bagi menjelaskan hasil tersebut.

Kata kunci: ayunan; bukan ayunan; persamaan pembezaan tunda neutral; peringkat pertama

1. Introduction

A neutral delay differential equation (NDDE) is a differential equation in which the highest order derivative of the unknown function appears in the equation both with and without delays. Recently, increasing numbers of investigations have been carried out in studying the oscillation of NDDEs Agarwal *et al.* (2004), Candan and Dahiya (2009), Gyori and Ladas (1991), Karpuz and Ocalan (2008), Zhou (1999) studied oscillation criteria, while Parhi and Rath (2001), Gopalsamy and Zhang (1990), Tanaka (1999; 2002), Yu *et al.* (1992) focused on the nonoscillatory solutions. The study of oscillatory behaviour of solutions of NDDEs has some important applications, such as networks containing lossless transmission lines, modelling of the transformation of information, and the theory of automatic control (Driver 1984; Hale 1977; Sficas & Stavroulakis 1987; Ocalan 2009; Erbe *et al*. 1995).

Consider the first order NDDE of the form

$$\left[r(t)\big(x(t)+px(t-\tau)\big)\right]'+q(t)x(t-\sigma)\ =0, \qquad t\ge t_0\,, \tag{1}$$

where

$$[r,q]\in C\left[\left[t_0,\infty\right),\left(0,\infty\right)\right],\ \ p\in\mathbb{R},\ \tau,\ \sigma\in\mathbb{R}^{+}. \tag{2}$$

The oscillatory solutions of (1) have been investigated by a number of researchers and some sufficient conditions for the oscillatory and nonoscillatory solutions have been obtained (Grammatikopoulos *et al*. 1986; Kubiaczyk & Saker 2002; Saker & Elabbasy 2001; Graef *et al*. 1986).

Let $m=\max\{\tau,\sigma\}$. By the solution of (1), a function $x\in C\left[\left[t_1-m,\infty\right),\mathbb{R}\right]$ for some $t_1\ge t_0$ such that $x(t)+px(t-\tau)$ is continuously differentiable and (1) is identically satisfied for $t_1 \geq t_0$. Such a solution of (1) is said to be oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative.

The main objective of this article is to give some new sufficient conditions for the oscillatory solutions of (1). Without loss of generality, we will deal only with the positive solutions of (1).

We present some of the well known lemmas, which will be needed in the proof of our main results. They may also have further applications in the analysis. For the proofs see Chuanxi and Ladas (1989), and Gyori and Ladas (1991).

Lemma 1.1. *Let*

$$f, g : [t_0, \infty) \to \mathbb{R},$$

be such that

$$f(t) = g(t) + p\, g(t-c), \quad t \geq t_0 + \max\{0, c\},$$

where $p, c \in \mathbb{R}$, *and* $p \neq 1$. *Assume that* $\lim_{t\to\infty} f(t) \equiv l \in \mathbb{R}$ *exists. Then the following statements hold:*

(1) *If* $\liminf_{t\to\infty} g(t) \equiv a \in \mathbb{R}$, *then* $l = (1+p)a$.

(2) *If* $\limsup_{t\to\infty} g(t) \equiv b \in \mathbb{R}$, *then* $l = (1+p)b$.

Lemma 1.2. *Assume that* ρ *is a positive constant. Let* $h \in C\left[[t_0, \infty), \mathbb{R}^+\right]$, *and suppose that*

$$\liminf_{t\to\infty} \int_{t-\rho}^{t} h(s)ds > \frac{1}{e}.$$

Then

(1) *the delay differential inequality* $x'(t) + h(t)x(t-\rho) \leq 0,\ t \geq t_0$ *has no eventually positive solution,*

(2) *the advanced differential inequality* $x'(t) - h(t)x(t-\rho) \geq 0,\ t \geq t_0$ *has no eventually positive solution.*

Lemma 1.3. *Assume that*

$$\int_{t_0}^{\infty} q(s)ds = \infty, \tag{3}$$

holds and let $x(t)$ *be an eventually positive solution of NDDE*

$$\left(x(t) + px(t-\tau)\right)' + q(t)x(t-\sigma) = 0; \qquad t \geq t_0, \tag{4}$$

where

$$p \in \mathbb{R}, p \neq 1,\ q \in C\left[[t_0, \infty), \mathbb{R}^+\right] \text{ and } \tau \in (0, \infty),\ \sigma \in \mathbb{R}^+.$$

Let $z(t) = x(t) + px(t-\tau)$ *and* $w(t) = z(t) + pz(t-\tau)$. *Then*

(1) $z(t)$ *is a decreasing function and either*

$$\lim_{t\to\infty} z(t) = -\infty, \tag{5}$$

or $$\lim_{t\to\infty} z(t) = 0. \tag{6}$$

(2) *The following statements are equivalent*

(i) (5) *holds,*

(ii) $p < -1$,

(iii) $\lim_{t\to\infty} x(t) = \infty$,

(iv) $w(t) > 0,\ w'(t) > 0$ *and* $w''(t) > 0$.

(3) *The following statements are equivalent:*

(i) (6) *holds,*

(ii) $p > -1$,

(iii) $\lim_{t\to\infty} x(t) = 0$,

(iv) $w(t) > 0,\ w'(t) < 0$ *and* $w''(t) > 0$.

2. Main Results

In this section, we give some new sufficient conditions for all solutions of (1) to be oscillatory.

Theorem 2.1. *Consider the equation* (1) *where* $r(t) \equiv r$ *positive constant and* $p = -1$. *Assume that* (3) *holds. Then every solution of* (1) *is oscillatory.*

Proof: Assume, for the sake of a contradiction, that (1) has an eventually positive solution $x(t) > 0,\ \forall\ t \ge t_0 > 0$. Let

$$z(t) = x(t) - x(t-\tau).$$

Then

$$z'(t) = -\frac{q(t)x(t-\sigma)}{r} < 0.$$

Hence for all $t \ge t_0$, we have

$z(t) > 0$ or $z(t) < 0$.

Let $z(t) > 0$. This implies that

$$\int_{t_0}^{\infty} q(s)x(s-\sigma)ds < r\,z(t_0) < \infty. \tag{7}$$

On the other hand, $z(t) > 0$ gives $x(t) > x(t-\tau)$ and hence $\liminf_{t\to\infty} x(t) > 0$. Thus, there exists a positive constant, k such that $x(t) > k > 0$. Then

$$\int_{t+\sigma}^{\infty} q(t)x(t-\sigma)dt > k\int_{t+\sigma}^{\infty} q(t)dt,$$

which leads to

$$\int_{t+\sigma}^{\infty} q(t)x(t-\sigma)dt = \infty.$$

This is a contradiction with (7). Therefore $z(t) < 0$, which implies $x(t) < x(t-\tau)$. Then $x(t)$ is bounded and hence $\liminf_{t\to\infty} x(t)$ and $\liminf_{t\to\infty} z(t)$ exist. From Lemma 1.1, we get

$$\lim_{t\to\infty} z(t) = 0.$$

This contradicts the fact that $z(t)$ is a negative and monotonic decreasing function. The proof is completed. □

Example 2.1. Consider the NDDE

$$\left(x(t) - x(t-\pi)\right)' + e^{t+2}x(t-\pi/2) = 0, \quad t > 0. \tag{8}$$

Here we have

$$q(t) = e^{t+2}, \ r = 1, \ \tau = \pi \text{ and } \sigma = \frac{\pi}{2}.$$

Then all the hypothesis of Theorem 2.1 are satisfied, where

$$\int_{t_0}^{\infty} q(t)\,dt = \int_{0}^{\infty} e^{t+2}dt = \infty.$$

Hence every solution of (8) is oscillatory.

Theorem 2.2. *Assume that* (1) *holds with* $p \neq -1$, $r(t) \equiv r$ *positive constant,* $q(t) \equiv q > 0$ *and*

$$\frac{q}{r(1+p)}(\sigma-\tau) > \frac{1}{e}, \tag{9}$$

then every solution of (1) *is oscillatory.*

Proof: Assume, for the sake of contradiction that (1) has an eventually positive solution $x(t) > 0, \ \forall\ t \geq t_0 > 0$. Let $z(t) = x(t) + px(t-\tau)$ and $w(t) = z(t) + pz(t-\tau)$. By direct substitution where $z(t)$ and $w(t)$ are solutions of (1), p and r are constants, we have

$$rz'(t) + prz'(t-\tau) + qz(t-\sigma) = 0.$$

From Lemma 1.3 we have $w(t) > 0$, w' is eventually increasing and

$$rw'(t) + prw'(t-\tau) + qw(t-\sigma) = 0 \tag{10}$$

Since $z(t)$ is decreasing and either $\lim_{t\to\infty} z(t) = -\infty$ or $\lim_{t\to\infty} z(t) = 0$, it is claimed that

$$w'(t-\tau) \le w'(t).$$

Hence,

$$r(1+p)w'(t-\tau) + qw(t-\sigma) \le 0$$

and so

$$w'\left(t-\tau\right) + \frac{q}{r(1+p)} w\left(t-\sigma\right) \le 0.$$

Letting $t = t+\tau$, we obtain

$$w'(t) + \frac{q}{r(1+p)} w\left(t-(\sigma-\tau)\right) \le 0 \ \text{ if } 1+p>0 \tag{11}$$

or

$$w'(t) - \left[\frac{q}{-r(1+p)}\right] w\left(t+(\tau-\sigma)\right) \ge 0 \ \text{ if } 1+p<0 \tag{12}$$

In view of Lemma 1.2 (i) and (ii), and also the condition (9), it is impossible for (11) and (12) to have eventually positive solutions. This contradicts the fact that $w(t) > 0$. Thus, the proof is completed. □

Example 2.1. Consider the NDDE

$$\left[\frac{e}{4}\left(x(t) + \left(\frac{1}{3\pi e} - 1\right)x(t-\pi)\right)\right]' + \frac{1}{3\pi e}x(t - \frac{3\pi}{2}) = 0, \quad t > 0. \tag{13}$$

Here we have

$$p = \frac{1}{3\pi e} - 1,\ q = \frac{1}{3\pi e},\ r = \frac{e}{4},\ \tau = \pi \text{ and } \sigma = {}^{3\pi}\!/_{2}$$

Then all the hypothesis of Theorem 2.2 are satisfied, where

$$\frac{q}{r(1+p)}\left(\sigma-\tau\right) = \frac{{}^{1}\!/_{3\pi e}}{\frac{e}{4}\left(1 + \frac{1}{3\pi e} - 1\right)}\left(\frac{3\pi}{2} - \pi\right) = \frac{2\pi}{e} > \frac{1}{e}.$$

Hence every solution of (13) is oscillatory.

Remark 2.1. Theorem 2.2 is an extent for Theorems 3 and 4 in Ladas and Sficas (1986) and Theorem 6.1.3 in Gyori and Ladas (1991).

Theorem 2.3. *Assume that* (2) *and* (3) *hold,* $-1 < p < 0$ *and*

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} \frac{q(s)}{r(s-\sigma)} ds > \frac{1}{e}, \tag{14}$$

then every solution of (1) *oscillates.*

Proof: Assume, for the sake of contradiction that (1) has an eventually positive solution $x(t) > 0, \forall\ t \geq t_0 > 0$. Let $z(t) = x(t) + px(t-\tau)$. Then by Lemma 1.3, we obtain $z(t) > 0$. As $x(t) > z(t)$, it follows from (1) that

$$\left(r(t)\,z(t)\right)' + q(t)\,z(t-\sigma) \leq 0$$

Dividing the last inequality by $r(t) > 0$, we obtain

$$z'(t) + \frac{r'(t)}{r(t)} z(t) + \frac{q(t)}{r(t)} z(t-\sigma) \leq 0. \tag{15}$$

Let $z(t) = e^{-\int_{t_0}^{t} \frac{r'(s)}{r(s)} ds} y(t)$. This implies that $y(t) > 0$. Substituting in (15) yields, for all $t \geq t_0$

$$y'(t) + \frac{q(t)}{r(t-\sigma)} y(t-\sigma) \leq 0, \qquad t \geq t_0. \tag{16}$$

In view of Lemma 1.2 (i) and (14), it is impossible for (16) to have an eventually positive solution. This contradicts the fact that $y(t) > 0$ and the proof of Theorem 2.3 is completed. □

Example 2.2. Consider the NDDE as follows,

$$\left[\frac{1}{t}\left(x(t) - \frac{e}{5} x\left(t - \frac{\pi}{2}\right)\right)\right]' + \frac{1}{t-\pi} x\left(t-\pi\right) = 0, \qquad t \geq t_0 > \pi. \tag{17}$$

Here we have

$$r(t) = \frac{1}{t},\ q(t) = \frac{1}{t-\pi}, \quad -1 < p = -\frac{e}{5} < 0, \quad \tau = \frac{\pi}{2} \text{ and } \sigma = \pi.$$

Then all the hypothesis of Theorem 2.3 are satisfied where

$$\liminf_{t\to\infty} \int_{t-\sigma}^{t} \frac{q(s)}{r(s-\sigma)} ds = \liminf_{t\to\infty} \int_{t-\frac{\pi}{2}}^{t} ds = \frac{5\pi}{2} > \frac{1}{e}.$$

Hence every solution of (17) is oscillatory.

Remark 2.2. Theorem 2.3 is an extent for Theorem 7 in Ladas and Sficas (1986).

Acknowledgements

This research has been completed with the support of these grants: FRGS/2/2013/SG04/UKM/02/3 and DLP-2014-012.

References

Agarwal R.P., Bohner M. & Li W.T. 2004. *Nonoscillation and Oscillation: Theory for Functional Differential Equations*. New York: Marcel Dekker.

Candan T. & Dahiya R. S. 2009. Positive solutions of first order neutral differential equations. *Appl. Math. Lett.* **22**: 1266-1270.

Chuanxi Q. & Ladas G. 1989. Oscillations of neutral differential equations with variable coefficients. *Appl. Anal.* **32**: 215-228.

Driver R.D. 1984. A mixed neutral system. *Nonlinear Anal.* **8**: 155-158.

Erbe L.H., Kong Q. & Zhang B.G. 1995. *Oscillation Theory for Functional Differential Equations*. New York: Marcel Dekker.

Gopalsamy K. & Zhang B.G. 1990. Oscillation and nonoscillation in first order neutral differential equations. *J. Math. Anal. Appl.* **151**: 42-57.

Grammatikopoulos M.K., Grove E.A. & Ladas G. 1986. Oscillations of first order neutral delay differential equations. *J. Math. Anal. Appl.* **120**: 510-520.

Graef J.R., Grammatikopoulos M.K. & Spikes P.W. 1986. On the behavior of solutions of a first order nonlinear neutral delay differential equation. *Appl. Anal*. **40**: 111-121.

Gyori I & Ladas G. 1991. *Oscillation Theory of Delay Differential Equations*. Oxford Mathematical Monographs. New York: Clarendon Press.

Hale J.K. 1977. *Theory of Functional Differential Equations*. New York: Springer.

Karpuz B. & Ocalan O. 2008. Oscillation criteria for some classes of linear delay differential equations of first order. *Bull. Inst. Math. Acad. Sin.* **3**(2): 293-314.

Kubiaczyk I. & Saker S.H. 2002. Oscillation of solutions of neutral delay differential equations. *Math. Slovaca* **52**: 343-359.

Ladas G. & Sficas Y.G. 1986. Oscillations of neutral delay differential equations. *Canad. Math. Bull*. **29**: 438-445.

Ocalan O. 2009. Existence of positive solutions for a neutral differential equation with positive and negative coefficients. *Appl. Math. Lett.* **22**: 84-90.

Parhi N. & Rath R.N. 2001. On oscillation and asymptotic behaviour of solutions of forced first order neutral differential equations. *Proc. Indian Acad. Sci.(Math. Sci.)* **111**: 337-350.

Saker S.H. & Elabbasy E.M. 2001. Oscillation of first order neutral delay differential equations. *Kyungpook Math. J.* **41**: 311-321.

Sficas Y.U. & Stavroulakis I.P. 1987. Necessary and sufficient conditions for oscillations of neutral differential equations. *J. Math. Anal. Appl.* **123**: 494-507.

Tanaka S. 2002. Oscillation of solutions of first order neutral differential equations. *Hiroshima Math. J.* **32**: 79-85.

Tanaka S. 1999. Existence of positive solutions for a class of first order neutral functional differential equations. *J. Math. Anal. App.* **229**: 501-518.

Yu J.S., Wang Z.C. & Chuanxi Q. 1992. Oscillation of neutral delay differential equations. *Bull. Austral. Math. Soc.* **45**: 195-200.

Zhang B.G. 1989. Oscillation of first order neutral functional differential equations. *J. Math. Anal. Appl*. **139**: 311-318.

Zhou Y. 1999. The distribution of zeros of neutral differential equations. *Hiroshima Math. J.* **29**: 361-370.

Pusat Pengajian Sains Matematik
Fakulti Sains dan Teknologi
Universiti Kebangsaan Malaysia
43600 UKM Bangi
Selangor DE, MALAYSIA
E-mail: rozy@ukm.edu.my, zahra80zahra@yahoo.com, ummul@ukm.edu.my, msn@ukm.edu.my*

*Corresponding author