

**SOME RESULTS FOR THE CLASS OF ANALYTIC FUNCTIONS
INVOLVING SALAGEAN DIFFERENTIAL OPERATOR**
(Beberapa Sifat untuk Kelas Fungsi Analisis Melibatkan Pengoperasi Pembeza Salagean)

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ABSTRACT

Let $T_n^\alpha(\beta)$ denote the class of function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, analytic and univalent in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, which are defined involving the Salagean differential operator D^n , $n \in N \cup \{0\}$, such that $\text{Re}(D^n f(z)^\alpha / z^\alpha) \geq \beta$, $z \in U$, $\alpha > 0$, $0 \leq \beta < 1$. In this paper, some properties such as a representation theorem and coefficient estimates for the class $T_n^\alpha(\beta)$ are obtained.

Keywords: Salagean differential operator; coefficient estimate; representation theorem

ABSTRAK

Andaikan $T_n^\alpha(\beta)$ kelas fungsi $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, analisis dan univalen pada cakera unit $U = \{z \in \mathbb{C} : |z| < 1\}$, yang tertakrif melibatkan pengoperasi pembeza Salagean D^n , $n \in N \cup \{0\}$ sedemikian hingga $\text{Re}(D^n f(z)^\alpha / z^\alpha) \geq \beta$, $z \in U$, $\alpha > 0$, $0 \leq \beta < 1$. Dalam makalah ini, beberapa sifat fungsi f di dalam kelas $T_n^\alpha(\beta)$ seperti teorem perwakilan dan anggaran pekali diperoleh.

Kata kunci: Pengoperasi pembeza Salagean, anggaran pekali, teorem perwakilan

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (1)$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, \mathbb{C} is a complex number. We denote the subclass of A consisting of analytic and univalent functions in U by S . Let P be the class of analytic function $p(z)$ of the form

$$p(z) = 1 + \sum_{k=2}^{\infty} p_k z^k, \quad z \in U \quad (2)$$

such that $p(0) = 1$ and $\text{Re}(p(z)) > 0$.

The following classes of functions are well known and have been studied repeatedly by many authors such as Salagean (1983), Opoola (1994), Abdulhalim (2003) and others.

$$\begin{aligned}
 S_0 &= \left\{ f \in A : \operatorname{Re} \left(\frac{f(z)}{z} \right) > 0, z \in U \right\}, \\
 B(\beta) &= \left\{ f \in A : \operatorname{Re} \left(\frac{f(z)}{z} \right) > \beta, 0 \leq \beta < 1, z \in U \right\}, \\
 \delta(\beta) &= \left\{ f \in A : \operatorname{Re}(f'(z)) > \beta, 0 \leq \beta < 1, z \in U \right\}.
 \end{aligned} \tag{3}$$

In 1993, Salagean introduced the following operator which is known as the Salagean differential operator.

Definition 1.1. For a function $f \in A$, the Salagean differential operator $D^n : A \rightarrow A$, $n \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ is defined by

$$D^n f(z) = D[D^{n-1} f(z)] = z[D^{n-1} f(z)]', \tag{4}$$

where

$$D^0 f(z) = f(z) \quad \text{and} \quad D^1 f(z) = Df(z) = zf'(z).$$

The operator D^n has been employed by various authors to define several subclasses of A , see Kanas (1989, Obradovic (1992), Opoola (1994), Babalola and Opoola (2006). For instance, Opoola (1994) introduced the subclass $T_n^\alpha(\beta)$, $\alpha > 0, 0 \leq \beta < 1, n \in \mathbb{N}_0$ of analytic functions which are defined involving Salagean differential operator as follows:

$$T_n^\alpha(\beta) = \left\{ f \in A : \operatorname{Re} \left(\frac{D^n f(z)^\alpha}{z^\alpha} \right) > \beta \right\}, \tag{5}$$

for $z \in U, \alpha > 0, 0 \leq \beta < 1$. The class $T_n^\alpha(\beta)$ is the generalization of the classes of functions as mentioned in (3), where $T_n^\alpha(0) := B_n(\alpha)$ is the class of Bazilevic functions (Abdulhalim 2003). Some properties of this class of functions were also established by Opoola, namely

- (i) T_n^α is a subclass of univalent functions,
- (ii) $T_{n+1}^\alpha \subset T_n^\alpha(\beta)$,
- (iii) If $f \in T_n^\alpha(\beta)$, then the integral operator $F_c = \frac{\alpha + c}{z^\alpha} \int_0^c t^{\alpha-1} f(z)^\alpha dt, c \geq 0$, is also in $T_n^\alpha(\beta)$.

In this paper, we obtain further properties of functions in the class $T_n^\alpha(\beta)$.

2. Preliminaries

We observed that for $f \in A$ and of the form (1), and applying the operator (4) then we have

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k \tag{6}$$

for $z \in U, n \in \mathbb{N}_0 = \{0,1,2,\dots\}$. Also, suppose that $\alpha > 0$, then by using binomial expansion, from (1) we can write

$$f(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} A_k(\alpha) z^{\alpha+k-1}, \quad (7)$$

where the coefficients $A_k(\alpha), k \in \{2,3,4,\dots\}$ with $A_k(1) = a_k$, depend on the coefficients a_k of $f(z)$ and the parameter $\alpha > 0$. Now, if we apply the operator (4) for the function (7), then we obtain

$$D^n f(z)^\alpha = \alpha^n z^\alpha + \sum_{k=2}^{\infty} (\alpha+k-1)^n A_k(\alpha) z^{\alpha+k-1} \quad (8)$$

for $z \in U, \alpha > 0, n \in \mathbb{N}_0$.

In order to prove our results, we shall need the following lemmas.

Lemma 2.1. (Hayami *et al.* 2007) *A function $p(z) \in P$ given by (2) satisfies the condition $Ny(p(z)) > 0, z \in U$ if and only if*

$$p(z) \neq \frac{\psi-1}{\psi+1}, \quad z \in U, \psi \in \mathbb{C}, |\psi|=1. \quad (9)$$

Lemma 2.2. (Alfors 1966) *Let $f(z)$ be an analytic function in the unit disk U with $f(0) = 0$ and $|f(z)| < 1$. Then, $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ in U . Strict inequality holds in both estimates unless $f(z)$ is a rotation of the disk $f(z) = e^{i\theta} z$. If $|f(z)| = |z|$ for some $z \neq 0$, then $f(z) = cz$, with a constant c of absolute value 1.*

3. Main Results

In this section, we establish some properties of functions in the class $T_n^\alpha(\beta)$. First, we prove a sufficient condition for functions belong to this class.

Theorem 3.1. *A function $f \in A$ is in the class $T_n^\alpha(\beta), \alpha > 0, 0 \leq \beta < 1, n \in \mathbb{N}_0$ if and only if*

$$1 + \sum_{k=2}^{\infty} Q_k z^{k-1} \neq 0,$$

where

$$Q_k = \frac{(\psi+1)}{2(1-\beta)} \left(\frac{\alpha+k-1}{\alpha} \right)^n A_k(\alpha) \quad (10)$$

for some $\alpha > 0, 0 \leq \beta < 1, n \in \mathbb{N}_0$ with $\psi \in \mathbb{C}, |\psi|=1$.

Proof. From (5), it suggests that there exists a function $p(z) \in P$ such that

$$\frac{D^n f(z)^\alpha}{z^\alpha} = \alpha^n [\beta + (1-\beta)p(z)]$$

for $z \in U, \alpha > 0, 0 \leq \beta < 1, n \in \mathbb{N}_0$. Upon setting

$$p(z) = \left(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - \beta \right) \left(\frac{1}{1-\beta} \right),$$

then from Lemma 2.1, we get

$$\left(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - \beta \right) \left(\frac{1}{1-\beta} \right) \neq \frac{\psi-1}{\psi+1}$$

for $z \in U, \alpha > 0, 0 \leq \beta < 1, n \in \mathbb{N}_0$ with $\psi \in \mathbb{C}, |\psi| = 1$. It is equivalent to

$$(\psi+1)D^n f(z)^\alpha - [\beta(\psi+1) + (1-\beta)(\psi-1)]\alpha^n z^\alpha \neq 0. \quad (11)$$

Substituting (8) into (11) yields that

$$2(1-\beta)\alpha^n z^\alpha + \sum_{k=2}^{\infty} (\psi+1)(\alpha+k-1)^n A_k(\alpha) z^{\alpha+k-1} \neq 0 \quad (12)$$

Now, dividing both sides of (12) by $2(1-\beta)\alpha^n z^\alpha \neq 0$, we obtain

$$1 + \sum_{k=2}^{\infty} \frac{(\psi+1)}{2(1-\beta)} \left(\frac{\alpha+k-1}{\alpha} \right)^n A_k(\alpha) z^{\alpha+k-1} \neq 0$$

for any $\psi \in \mathbb{C}$ such that $|\psi| = 1, z \in U, \alpha > 0, 0 \leq \beta < 1, n \in \mathbb{N}_0$. \square

Remark 3.2. For the case of $\beta = 0$ in (10), the result has been proved by Singh *et al.* (2009).

The following property of functions in the class $T_n^\alpha(\beta)$ is also established.

Theorem 3.3. Let $f \in A$ belong to the class $T_n^\alpha(\beta), \alpha > 0, 0 \leq \beta < 1, n \in \mathbb{N}_0$. Then, there exists an analytic function $\phi(z)$ with $|\phi(z)| \leq 1, z \in U$ such that

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = 2\beta - 1 + \frac{2(1-\beta)}{1-z\phi(z)}, \quad (12)$$

for $z \in U, \alpha > 0, 0 \leq \beta < 1, n \in \mathbb{N}_0$.

Proof. Let us define the functions $A(z)$ and $B(z)$ as follows:

$$A(z) = \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - \beta \quad (13)$$

and

$$B(z) = z \left(\frac{A(z) - (-\beta)}{A(z) + (1-\beta)} \right) \quad (14)$$

for $z \in U, \alpha > 0, 0 \leq \beta < 1, n \in \mathbb{N}_0$. Substituting the equation (13) into (14), we get

$$B(z) = z \left(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - 1 \right) \left(\frac{1}{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - (2\beta - 1)} \right)$$

for any $z \in U$. $B(z)$ is an analytic function for $z \in U$. Also, since $f(0) = 0$ and $f'(0) = 1$, we have that $B(0) = 0$ and $|B(z)| < 1$ for $z \in U$. Hence, by Schwarz's Lemma (Lemma 2.2), $|B(z)| < |z|$, $z \in U$ which gives that

$$\left(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - 1 \right) \left(\frac{1}{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - (2\beta - 1)} \right) < |z|$$

or equivalently

$$\left(\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - 1 \right) \left(\frac{1}{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - (2\beta - 1)} \right) = z\phi(z),$$

where $\phi(z)$ is analytic and $|\phi(z)| \leq 1$ for $z \in U$. Therefore,

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - 1 = z\phi(z) \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - (2\beta - 1)z\phi(z). \quad (15)$$

Solving for $D^n f(z)^\alpha / \alpha^n z^\alpha$ from (15), then we get

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \frac{1 - (2\beta - 1)z\phi(z)}{1 - z\phi(z)} = (2\beta - 1) + \frac{2(1 - \beta)}{1 - z\phi(z)}.$$

Thus, we obtain the desired result (12). \square

The following result is called the integral representation theorem for functions in the class $T_n^\alpha(\beta)$, which provides further property of functions in this class.

Theorem 3.4. *Let $f \in A$ belong to the class $T_n^\alpha(\beta)$, $\alpha > 0, 0 \leq \beta < 1, n \in \mathbb{N}_0$. Then, there exists an analytic function $\phi(z)$ with $|\phi(z)| \leq 1, z \in U$ such that*

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \frac{1}{\alpha^n z^\alpha} \exp \int_0^z \left(\frac{\alpha}{t} - C(t) \right) dt, \quad (16)$$

for $\alpha > 0, 0 \leq \beta < 1, n \in \mathbb{N}_0$, where

$$C(t) = \frac{2(\beta - 1)[z\phi'(t) + \phi(t)]}{1 - 2\beta t\phi(t) + (2\beta - 1)t^2\phi^2(t)}. \quad (17)$$

Proof: Let $f(z) \in T_n^\alpha(\beta)$. Then, from Theorem 3.3 we have

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = 2\beta - 1 + \frac{2(1-\beta)}{1-z\phi(z)} = \frac{1-(2\beta-1)z\phi(z)}{1-z\phi(z)}. \quad (18)$$

Taking the logarithmic differentiation, we get from (18)

$$\frac{[D^n f(z)^\alpha]'}{D^n f(z)^\alpha} = \frac{\alpha}{z} - \frac{2(1-\beta)[z\phi'(z) + \phi(z)]}{1-2\beta z\phi(z) + (2\beta-1)z^2\phi^2(z)}. \quad (19)$$

Now, integrating both sides of (19) along the line segment from 0 to z , we obtain

$$\ln[D^n f(z)^\alpha] = \int_0^z \left(\frac{\alpha}{t} - \frac{2(1-\beta)[t\phi'(t) + \phi(t)]}{1-2\beta t\phi(t) + (2\beta-1)t^2\phi^2(t)} \right) dt.$$

This gives us that

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \frac{1}{\alpha^n z^\alpha} \exp \int_0^z \left(\frac{\alpha}{t} - C(t) \right) dt$$

for $z \in U$, $\alpha > 0$, $0 \leq \beta < 1$, $n \in \mathbb{N}_0$, where

$$C(t) = \frac{2(1-\beta)[t\phi'(t) + \phi(t)]}{1-2\beta t\phi(t) + (2\beta-1)t^2\phi^2(t)}$$

for an analytic function $\phi(z)$ with $|\phi(z)| \leq 1$, $z \in U$. Thus, the proof of Theorem 3.4 is completed. \square

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