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IMPLICATION DIAGRAM OF FIVE CHAOS CHARACTERIZATIONS: A SURVEY ON COMPACT METRIC SPACE AND SHIFT OF FINITE TYPE

(Diagram Implikasi bagi Lima Pencirian Kalut: Suatu Tinjauan pada Ruang Matrik Padat dan Ruang Anjakan Terhingga)

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ABSTRACT

We give a summary on the development of the relationships between some chaos characterizations, focusing on Devaney chaos, locally eventually onto, strong dense periodic points, topologically mixing and totally transitive. We represent the relationships in four implication diagrams. We look at compact metric spaces in general and on shifts of finite type in particular. At the end, it is shown that different spaces yield different results.

Keywords: locally eventually onto; strong dense periodic points; totally transitive; topologically mixing; Devaney chaos

ABSTRAK

Kami menyimpulkan hasil kajian yang memberikan hubungan di antara sebilangan pencirian kalut, iaitu kalut Devaney, akhirnya keseluruh secara setempat, titik-titik berkala tumpat kuat, percampuran secara topologi dan transitif sepenuhnya. Hubungan ini diberikan dalam bentuk empat gambar rajah implikasi. Kami menumpukan pada ruang metrik padat secara amnya dan anjakan jenis terhingga secara khususnya. Didapati, ruang metrik yang berbeza memberikan hasil kajian yang berbeza.

Kata kunci: akhirnya keseluruh secara setempat; titik-titik berkala tumpat kuat; percampuran secara topologi; transitif sepenuhnya; kalut Devaney

1. Introduction

It is very important to identify the chaotic behavior of a dynamical system (X, f), for a continuous function f that acting on a compact metric space X into itself. There are many versions of chaos definitions in mathematical perceptions and there is no universal agreement on which definitions are the best to describe chaos. One of the most frequently used is Devaney's, which isolates three essential characteristics of a chaotic function. Due to Devaney (2003), a dynamical system (X, f), is said to be chaotic whenever it satisfies the following;

(i) transitivity (i.e. for any two open nonempty sets U and V, there is some $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$)

(ii) has dense set of periodic points

(iii) sensitive dependence on initial conditions (i.e. there is a constant $\delta > 0$ such that for every $x \in X$ and every $\epsilon > 0$, there is some point y and some $n \in \mathbb{N}$ such that $d(x, y) < \epsilon$ but $d(f^n(x), f^n(y)) \geq \delta$.

It has been proven that transitivity and dense periodic points are independence while sensitive dependence on initial conditions is redundant (Banks *et al*. 1992). Therefore, the three main ingredients for Devaney chaos have been reduced to two of it, which are transitivity and dense periodic points.

Some dynamicists investigate whether strengthening these two ingredients of Devaney chaos will yield more precise chaos characterizations. There are many strong chaos characterizations have been introduced which includes locally eventually onto ($l.e.o$), topologically mixing and totally transitive. These three characterizations are stronger than transitivity. Recently in 2015, another chaos characterizations which is stronger than dense periodic points has been introduced and its properties were not yet discovered in detail. Since all of these notions have been investigated separately, our aim in this work is to study their relationship in general view and compare it to a particular space, shift of finite type.

A dynamical system (X, f) is said to be locally eventually onto (or sometimes locally everywhere onto or topologically exact) if for every open set U, there exists a positive integer n such that $f^n(U) = X$. This notion is stronger than transitivity. A dynamical system (X, f) is said to have a strong dense periodicity whenever for any integer n, periodic points of prime period at least n are dense in the space X. Let us denote P_n as the set of all periodic points of prime period at least n. This notion is stronger than the property of dense set of periodic points. Topologically mixing is whenever for every pair of open sets U and V, there exists a positive integer M such that $f^m(U) \cap V \neq \emptyset$, for all $m \geq M$. It is totally transitive whenever for every integer m, f^m is transitive. These two notions are also stronger than transitivity.

The space shift of finite type, Σ is a compact space which consists of sequences over k symbols where some finite blocks are not allowed to appear in any sequence of the set. Finite block is a finite sequence $a_0 a_1 \cdots a_{l-1}$ of length l and for every $i = 0,1,\cdots,l-1$, the entry $a_i \in \{0,1,\cdots,k-1\}$. The map σ on Σ is a continuous map which shifts every sequence in Σ one step to the right i.e. deleting the first entry of the sequence (i.e. $\sigma(x_0 x_1 \cdots) = x_0 x_1 \cdots$ for every $x_0 x_1 \cdots \in \Sigma$).

Lemma 1.1 (Cranell 1998):
For a shift of finite type, the following are equivalence:
(i) locally eventually onto
(ii) totally transitive
(iii) totally transitive and dense periodic points
(iv) topologically mixing

Lemma 1.2 (Malouh *et al*. 2016):
For shift of finte type, strong dense periodic points implies Devaney chaotic.

In the next sections, we will see that these results are not necessarily true on general space.

2. Implication of locally eventually onto to other chaos characterizations

In this section we will summarize the implication of $l.e.o$ to the chaos characterizations i.e. Devaney chaos, strong dense periodic points, totally mixing and totally transitive.

Theorem 2.1(Malouh & Syahida 2017):
Locally eventually onto implies totally transitive and topologically mixing.
Proof:
Directly from the definitions. □

We need the following example to show $l.e.o$ relation with Devaney chaos and strong dense periodicity property. Guirao *et al*. (2009) used this example to illustrate some other properties.

Example 2.2:
Let $n \in N$ and Z_{n+1} be a cyclic group with $n+1$ elements. We endow $\mathbb{Z}_{n+1}$ with the discrete topology. Note that "+" and "-" denote addition and subtraction $\mathrm{mod}(n+1)$. Let $X_n = (\mathbb{Z}_{n+1})^{\infty} = \{(x_m)_{m=1}^{\infty}: x_m \in \mathbb{Z}_{n+1}, m \in \mathbb{N}\}$ be the product topological space of countable infinite copies of $\mathbb{Z}_{n+1}$. It is well known that X_n is an compact, perfect and has countable base containing clopen sets (i.e. homeomorphic to the Cantor set). This bases can be chosen to consist of cylinder sets, i.e., sets of the form

$$[z_1, \cdots, z_k] = \{(x_m)_{m=1}^{\infty} \in X_n: x_1 = z_1, \cdots, x_k = z_k\}$$

where $k \in \mathbb{N}$ and $z_1, \cdots, z_k$ is an arbitrary sequence of elements of $\mathbb{Z}_{n+1}$ of length k. Define the map $f_n: X_n \to X_n$, by $f_n((x_m)_{m=1}^{\infty}) = (y_m)_{m=1}^{\infty}$, where

$$y_m = \begin{cases} x_{m+1} & \text{if } x_1 \neq x_{n+1} \\ 1 + x_{m+1} & \text{if } x_1 = x_{n+1} \end{cases}$$

for all $m \in \mathbb{N}$.

Lemma 2.3 (Guirao *et al*. 2009):
Let $n \in N$. Then
(i) the map f_n is continuous
(ii) f_n does not admit any periodic points with prime period n
(iii) the map f_n is $l.e.o.$

Theorem 2.4:
$l.e.o$ does not imply Devaney chaos nor strong dense periodic points.
Proof:
Let us consider the map f_n^n where f_n is as in Example 2.2. By Lemma 2.3, since f_n is l.e.o, so does f_n^n. Since f_n does not admit any periodic points with prime period n, so f_n^n does not have any fix points. Therefore, f_n^n does not have any periodic points. Hence f_n^n is not Devaney chaotic nor has P_n dense. □

We summarize the relation between $l.e.o$ and other chaos characterizations in the diagram as shown in Figure 1. The arrow A ⟶ B means "A implies B" while the arrow A ⟶̸ B means "A does not imply B".

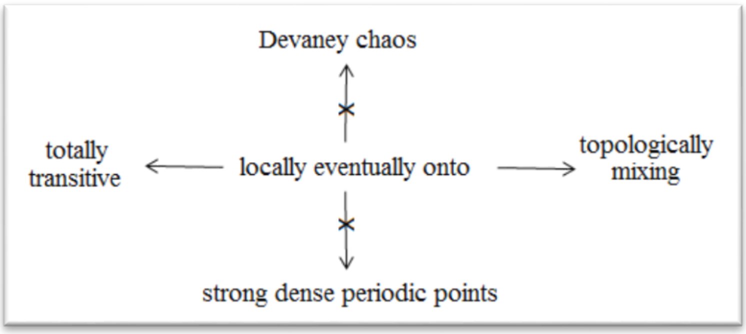

Figure 1: Implication of locally eventually onto

Interestingly, some of the implication that is not true on general compact space is true on shift of finite type.

Theorem 2.5:
On shift of finite type, l.e.o implies Devaney chaos.
Proof:
From Lemma 1.1, l.e.o implies totally transitive and dense periodic points. Therefore, it is Devaney chaos. □

3. Implication of topologically mixing onto to other chaos characterizations

In this section we will summarize the implication of topologically mixing to the other chaos characterizations i.e. Devaney chaos, strong dense periodic points, $l.e.o$ and totally transitive.

Example 3.1 (Ruette 2017):
Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of points in $(0,1)$ such that $a_n < a_{n+1}$ for all $n \in \mathbb{Z}$, and $\lim_{n\to-\infty} a_n = 0$ and $\lim_{n\to+\infty} a_n = 1$. For all $n \in \mathbb{Z}$, we set $I_n = [a_n, a_{n+1}]$ and define $f_n: I_n \to I_{n-1} \cup I_n \cup I_{n+1}$ by $f_n(a_n) = a_n$, $f_n(a_{n+1}) = a_{n+1}$, $f_n\left(\frac{2a_n+a_{n+1}}{3}\right) = a_{n+2}$, $f_n\left(\frac{a_n+2a_{n+1}}{3}\right) = a_{n-1}$ and f_n is linear between the points where it has already been defined.
Then we define the map $f: [0,1] \to [0,1]$ by

$$f(0) = 0, f(1) = 1$$
$$\forall n \in \mathbb{Z}, \forall x \in I_n, f(x) = f_n(x)$$

Lemma 3.2 (Ruette 2017):
The interval function $f: [0,1] \to [0,1]$ defined in Example 3.1 is continuous, topologically mixing but not $l.e.o$.

Theorem 3.3:
Topologically mixing does not imply $l.e.o$.
Proof:
It is proven by the counterexample in Example 2. □

Theorem 3.4:
Topologically mixing does not imply Devaney chaos nor strong dense periodic points.
Proof:
Let us consider the map f_n^n where f_n is as in Example 2.2. By the proof in Theorem 2.4 f_n^n is $l.e.o$. By Theorem 2.1, f_n^n is mixing. Since f_n does not admit any periodic points with prime period n, so f_n^n does not have any periodic points. Hence f_n^n is not Devaney chaotic nor has P_n dense. □

Theorem 3.5 (Thomson 2013):
Topologically mixing implies totally transitive.
Proof:
Suppose that f is topologically mixing and that $N \in \mathbb{N}$ such that for every $n \geq \mathbb{N}$, $f^n(U) \cap V \neq \emptyset$. To see that f^m is transitive for every integer m, let M be the smallest multiple of m which is greater than N. Therefore $f^M(U) \cap V \neq \emptyset$ and hence f^m is transitive. □

We summarize the relation between topologically mixing and other chaos characterization in the following diagram in Figure 2.

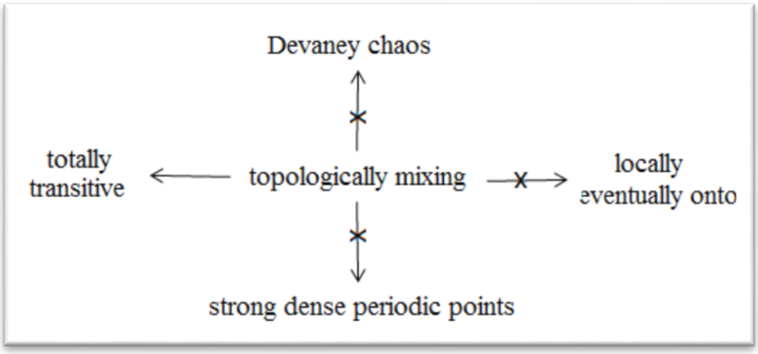

Figure 2: Implication of topologically mixing

Interestingly, some of the implication that is not true on general compact space is true on shift of finite type.

Theorem 3.6:
On shift of finite type, topologically mixing implies Devaney chaos and l.e.o.
Proof:
From Lemma 1.1, topologically mixing implies totally transitive, dense periodic points and l.e.o. So does Devaney chaos. □

4. Implication of totally transitive to other chaos characterizations

In this section we summarize the implication of totally transitive to other chaos notions i.e. Devaney chaos, strong dense periodic points, topologically mixing and $l.e.o$.

Theorem 4.1:
Totally transitive does not imply $l.e.o$.
Proof:
Let us consider the interval function $f:[0,1] \to [0,1]$ defined in Example 3.1 which is continuous, topologically mixing but not $l.e.o$. By Theorem 3.5, f is totally transitive. Hence f is the counterexample to prove the theorem. □

Example 4.2 (Cranell 1998):
Let us define the following set

$$\Sigma_* = \{\{x_i\}_{i\in\mathbb{N}}: x_i \in \{0,1,2\} \text{ and } x_i = 0, x_{i+1} = 2 \text{ implies } |i-j| \neq 2^p \text{ for any } p \in \mathbb{N}\}$$

and endow with a continuous map σ such that $\sigma(x_0x_1\cdots) = x_1x_2\cdots$.
The metric on Σ_* is $d(\mathbf{s},\mathbf{t}) = \frac{1}{2^j}$ where j is the first entry of **s** and **t** in Σ_* such that $s_j \neq t_j$. Therefore the open ball in Σ_* is a cylinder $C_l = x_0x_l\cdots x_{l-1}$ of length l, where C_l is a set of sequences in Σ_* which started with admissible word $x_0x_l\cdots x_{l-1}$. Cranell [4] show that Σ_* is compact as it is homeomorphic to the Cantor set.

Lemma 4.3 (Cranell 1998):
(Σ_*, σ) is totally transitive but is not topologically mixing.
Proof:
To show that (Σ_*, σ) is totally transitive, fix $r > 0$ and pick two admissible cylinders $C_1 = x_0x_l\cdots x_{l-1}$ and $C_2 = y_0y_1\cdots y_{m-1}$. Pick p so that $2^p > rm + l$. We construct a new cylinder, $C = C_1 1^{2p}$. This cylinder of length $2^p + l$ is admissible. The next $2^p - l > rm$ slots

are unrestricted, and that there are more than rm slots into which we can place cylinder C_2. Hence $(\sigma^r)^j(C_1) \cap C_2 \neq \emptyset$ for some j.
The dynamical system is not mixing because $\sigma^{2p}[0] \cap [2] = \emptyset$, for every $p > 0$. □

Theorem 4.4:
Totally transitive does not imply topologically mixing.
Proof:
The counterexample is (Σ_*, σ) as given in Example 4.2. □

Example 4.5:
The irrational rotation $R_\alpha: S_1 \to S_1$ is an automorphism of the unit circle defined by $\theta \in S_1$, $R_\alpha: \theta \to \theta + \alpha \pmod{2\pi}$ where $\frac{\alpha}{2\pi} \in \mathbb{R}\backslash\mathbb{Q}$.

Lemma 4.6 (Thomson 2013):
(S_1, R_α) *is totally transitive but has no periodic points*.
Proof:
To show that R_α is totally transtive, let us consider the forward orbit of $\theta \in S_1$. The orbit of θ is a sequence bounded by 0 and 2π, so contains a convergent subsequence. Say for any $\epsilon > 0$, $|R_\alpha^s(\alpha) - R_\alpha^r(\alpha)| < \epsilon$. If we let $m = r - s > 0$, then $|R_\alpha^m(\theta) - \theta| = \left|R_\alpha^s\left(R_\alpha^m(\theta)\right) - R_\alpha^s(\theta)\right| = |R_\alpha^s(\alpha) - R_\alpha^r(\alpha)| < \epsilon$ since R_α preserves arc lengths. So if we look at the orbit of the arc $\left(\theta, R_\alpha^m(\theta)\right)$, we see that it partitions the circle into arcs of length less than ϵ. Since we let ϵ be arbitrarily small, we can choose it for any open arc V, ϵ is less than the length of V. Then the orbit of the arc less than the size ϵ evetually intersects V, and any open arc U msut also eventually intersects V. So R_α is transitive. Since $R_\alpha^m = R_{m\alpha}$ for any integer $m > 0$, and $m\alpha$ is also irrational relative to 2π, the irrational rotation is totally tgransitive.
To see that R_α does not have any periodic points, let us assume the opposite and $\theta \in S_1$ such that $R_\alpha^n(\theta) = \theta$ for some integer $n > 0$. Then $\theta + n\alpha = \theta \pmod{2\pi}$ or $n\alpha = 0 \pmod{2\pi}$. If this were the case, however, then $\alpha = \frac{2m\pi}{n}$ for some integer m. This is contradicting to our assumption that R_α is an irrational rotation. So R_α does not have any periodic points. □

Theorem 4.7:
Totally transitive does not implies Devaney chaos nor strong dense periodic points.
Proof:
(S_1, R_α) in Example 4.5 is the counterexample as it is totally transitive but does not have any periodic points. □

We summarize these relations in the following diagram in Figure 3.

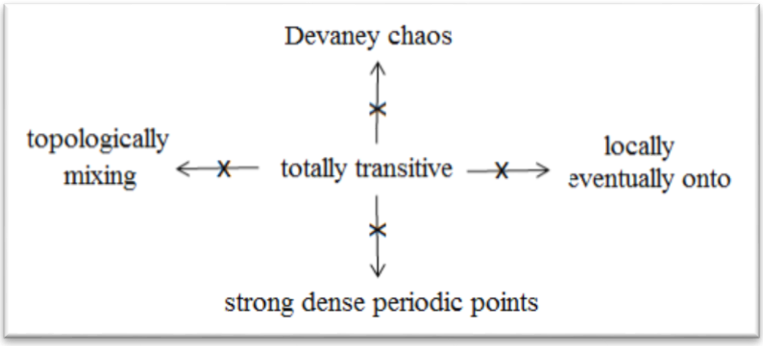

Figure 3: Implication of totally transitive

Interestingly, some of the implication that is not true on general compact space is true on shift of finite type.

Theorem 4.8:
On shift of finite type, totally transitive implies Devaney chaos, locally eventually onto and topologically mixing
Proof:
From Lemma 1.1, totally transitive implies l.e.o, topologically mixing = and dense periodic points. So dees Devaney chaos. □

5. Implication of strong dense periodic points to other chaos characterizations

In this section we will summarize the implication of strong dense peridic points to the other chaos characterizations i.e. Devaney chaos, totally transitive, topologically mixing and $l.e.o$.

Example 5.1 (Syahida & Good 2015):
Let $D = \{re^{i\theta} \in \mathbb{C}: r \in [0,1], \theta \in [0,2\pi)\}$ be the closed unit disk in the complex plane. Define $f: D \to D$ by $f(re^{i\theta}) = re^{i(\theta+2r\pi)}$. Then f is a homeomorphism of D and the restriction f_r of f to $C_r = \{z: |z| = r\}$ is a rotation of order r. If r is irrational, then each point of C_r has infinite orbit and f_r is transitive. If $r = \frac{p}{q}$ is rational, with p and q in lowest terms, then every point in C_r has period q.

Lemma 5.2 (Syahida & Good 2015):
Let f and D are as defined in Example 5.1. Then (f, D) has strong dense periodicity property but not transitive.
Proof:
Since the set of rationals with denominator at least n is dense in [0, 1], the the set $P_n(f)$ is dense for all n.

It is also clear that f is not transitive; for example, for no n does $f^n\left(B_{\frac{1}{8}}\left(\frac{1}{4}\right)\right)$ meet $B_{\frac{1}{8}}\left(\frac{3}{4}\right)$. □

Theorem 5.3:
Strong dense periodicity property does not imply $l.e.o$, Devaney chaos, topologically transitive nor topologically mixing.
Proof:

Since $l.e.o$, Devaney chaos, topologically transitive and topologically mixing are stronger than transitivity, then (D,f) as defined in Example 5.1 is the counterexample. □

We summarize the relation between strong dense periodic points and other chaos characterization in the following diagram in Figure 4.

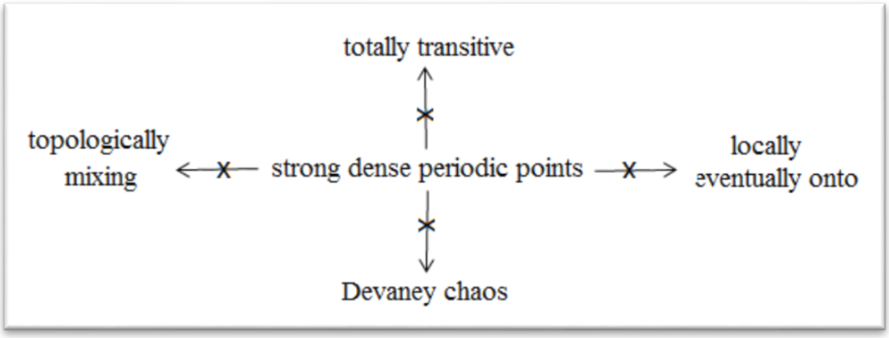

Figure 4: Implication of totally transitive

Interestingly, some of the implication that is not true on general compact space is true on shift of finite type, i.e. strong dense periodic points implies Devany chaos as stated in Lemma 1.2.

6. Conclusion

All four diagrams in Figures 1,2,3 and 4 are completely summarize the relation between five chaos notions; $l.e.o$, Devaney chaos, totally transitive, topologically mixing and strong dense periodicity property on general compact metric spaces. Counterexamples we used to illustrate some properties are ranges from variety of spaces; unit circle, shift space, interval, unit disk in complex plane and cyclic group. Most of the implications are not true in general. It is turn out however that the implications are true in some other spaces, for example on shift of finite type.

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