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SECOND HANKEL DETERMINANT FOR A SUBCLASS OF ANALYTIC FUNCTIONS INVOLVING *Q*-ANALOGUE OF RUSCHEWEYH OPERATOR

(Penentu Hankel Kedua untuk Subkelas Fungsi Analisis Melibatkan Pengoperasi Ruscheweyh Analog-*q*)

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ABSTRACT

Let S be the class of analytic functions which are univalent and normalised in the open unit disc $U=\{z\in\mathbb{C}:|z|<1\}$. Second Hankel determinant of $|a_2a_4-a_3^2|$ for a class of analytic functions involving *q*-analoque of Ruscheweyh operator is given.

Keywords: *q*-analogue of Ruscheweyh Operator; Fekete-Szego functional; Hankel determinant

ABSTRAK

Andaikan S sebagai kelas fungsi analisis yang univalen dan ternormal dalam cakera unit terbuka $U=\{z\in\mathbb{C}:|z|<1\}$. Diberi penentu Hankel kedua $|a_2a_4-a_3^2|$ untuk kelas fungsi analisis yang melibatkan analog-*q* bagi pengoperasi Ruscheweyh.

Kata kunci: Pengoperasi Ruscheweyh analog-*q*; fungsian Fekete-Szego; penentu Hankel

1. Introduction

The function class is denoted by $\mathcal{A}$ which represented by the following form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k, \qquad (z\in U) \tag{1}$$

that are analytic in $U=\{z\in\mathbb{C}:|z|<1\}$ and satisfy the following normalization conditions $f(0)=0$ and $f'(0)=1$. In addition, let $S\subset\mathcal{A}$ be the class of functions which are univalent in U.

Geometric Function Theory includes the study of a numeral of subclasses within normalised analytic function, using varied approaches. Both *q*-calculus and fractional *q*-calculus are significant methods for examining a range of $\mathcal{A}$ subclasses. Srivastava and Owa (1989) was the first to provide a clear basis for using *q*-calculus within Geometric Function Theory, and to apply the fundamental *q*-hypergeometric functions in this theory. Further, univalent function theory is possible to describe by applying *q*-calculus theory, and more recently the application of a fractional *q*-derivative operator and fractional *q*-integral operator has been seen in creating a number of analytic function subclasses (e.g. in Aldweby and Darus (2013; 2014); Elhaddad *et al.* (2018); Elhaddad and Darus (2019); Mahmood *et al.* (2019); Purohit and Raina (2011; 2013)). Purohit and Raina (2013), for example, examined the use of fractional *q*-calculus operators in defining a number of analytic function classes for U as an open unit disk. Meanwhile, Mohammed and Darus (2013) evaluated *q*-operator characteristics in terms of geometry and approximation with reference to particular analytic function subclasses within

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compact disks. A more complete treatment of applied *q*-analysis within the theory of operators may be found in Aral *et al.* (2013) and Exton (1983).

This work starts by defining key terms and detailed concepts within the *q*-calculus applied here. For the purposes of the report, the following assumption is made: $0<q<1$. Firstly, fractional *q*-calculus operators for a function with complex values $f(z)$ are defined below:

Definition 1.1. The *q*-number $[k]_q$ is defined by

$$[k]_q = \begin{cases} \dfrac{1-q^k}{1-q}, & k \in \mathbb{C}, \\ \sum\limits_{n=0}^{m-1} q^n = 1+q+q^2+\ldots+q^{m-1} & k=m\in\mathbb{N}. \end{cases}$$

Definition 1.2. The *q*-factorial $[k]_q!$ is defined by

$$[k]_q! = \begin{cases} (1+q)\ldots(1+q+\ldots+q^{k-1}), & k=1,2,\ldots, \\ 1, & k=0. \end{cases} \tag{2}$$

Definition 1.3. (Jackson 1908; 1910) The *q*-derivative operator D_q of a function *f* is determined by

$$D_q f(z) = \begin{cases} \dfrac{f(qz)-f(z)}{(q-1)z}, & z\neq 0 \\ f'(z). & z=0. \end{cases} \tag{3}$$

We note from Definition 1.3 that

$$\lim_{q\to 1^-}(D_q f)(z) = \lim_{q\to 1^-}\frac{f(zq)-f(z)}{(q-1)z} = f'(z).$$

From (1) and (3), we get

$$D_q f(z) = 1+\sum_{k=2}^{\infty}[k]_q a_k z^{k-1}.$$

Aldweby and Darus (2014) defined the *q*-analogue of Ruscheweyh Operator $\mathcal{R}_q^\delta$ by

$$\mathcal{R}_q^\delta f(z) = z+\sum_{k=2}^{\infty}\frac{[k+\delta-1]_q!}{[\delta]_q![k-1]_q!}a_k z^k,$$

where $\delta \geq 0$ and $[k]_q!$ is defined by (2).

Also, as $q\to 1^-$ we have

$$\lim_{q\to 1^-}\mathcal{R}_q^{\delta} f(z)=z+\lim_{q\to 1^-}\left[\sum_{k=2}^{\infty}\frac{[k+\delta-1]_q\,!}{[\delta]_q\,![k-1]_q\,!}a_k z^k\right]=z+\sum_{k=2}^{\infty}\frac{[k+\delta-1]!}{[\delta]![k-1]!}a_k z^k=\mathcal{R}^{\delta} f(z),$$

where $\mathcal{R}^{\delta} f(z)$ is Ruscheweyh differential operator described by Ruscheweyh (1975) and studied by several authors, for example Mogra (1999), and Shukla and Kumar (1983).

Noonan and Thomas (1976) examined the following q^{th} Hankel determinant

$$H_q(r)=\begin{vmatrix} a_r & a_{r+1} & \cdots & a_{r+q-1} \\ a_{r+1} & a_{r+2} & \cdots & a_{r+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{r+q-1} & a_{r+q} & \cdots & a_{r+2q-2} \end{vmatrix} \tag{4}$$

in which $r \geq 1$ and $q \geq 1$. This determinant has been the subject of study by a range of researchers. Specifically, a number of works provided sharp upper limits for $H_2(2)$ (e.g. Abubaker and Darus (2011); Bansal (2013); Janteng *et al.* (2006; 2007); Mohammed and Darus (2012); Pommerenke (1966; 1967); Raducanu and Zaprawab (2017) and Srivastava *et al.* (2018)) in a range of normalised analytic function classes. Fekete-Szego functional is well-established as $\left|a_3-a_2^2\right|=H_2(1)$, which is generalisable to $\left|a_3-\mu a_2^2\right|$ to certain real and complex μ. Further, sharp estimation were provided by Fekete and Szego for $\left|a_3-\mu a_2^2\right|$ in real μ as well as $f \in S$, representing U 's normalised univalent function class. This effectively combines two coefficients describing area problems as Gronwall previously put forward in 1914/15. Further, $\left|a_2a_4-a_3^2\right|$ as the functional has equivalence with $H_2(2)$. For the current analysis, $H_2(2)$ Hankel determinant upper bounds are determined for an analytic function subclass through the following:

Definition 1.4. Let $f \in \mathcal{A}$. Then f is said to be within the class $R_q(\delta)$ if it is satisfied the condition

$$Re\left\{D_q\left(\mathcal{R}_q^{\delta} f(z)\right)\right\}>0 \qquad z\in U. \tag{5}$$

Note that, when $\delta=0$ and $q\to 1^-$ the class $R_q(\delta)$ is reduced to the class studied by MacGregor (1962) and Janteng *et al.* (2006).

Following preliminary results are required to prove and validate the above results.

2. Preliminaries

Let $\mathcal{P}$ be the family of all functions p analytic in U for which $Re\{p(z)\}>0$ and

$$p(z)=1+c_1z+c_2z^2+\dots\dots. \tag{6}$$

Lemma 2.1. (Duren 1983) *Let* p *within the class* $\mathcal{P}$ *, then* $|c_k|\le 2$ *for each* $k\in\mathbb{N}$.

Lemma 2.2. (Libera & Zlotkiewicz 1982; 1983*) Let* $p\in\mathcal{P}$ *be given by* (6). *Then*

$$2c_2=c_1^2+(4-c_1^2)x, \tag{7}$$

for some $x, |x|<1$*, and*

$$4c_3=c_1^3+(4-c_1^2)2c_1x-(4-c_1^2)c_1x^2+2(4-c_1^2)(1-|x|^2)z, \tag{8}$$

for some $z, |z|<1$.

3. Main results

Theorem 3.1. *Let* $f\in R_q(\delta)$. *Then*

$$\left|a_2a_4-a_3^2\right|\le\frac{4[2]_q^2}{[3]_q^2[\delta+1]_q^2[\delta+2]_q^2}.$$

Proof. Since $f\in R_q(\delta)$, then from (5) we have

$$D_q(\mathcal{R}_q^\delta f(z))=p(z). \tag{9}$$

By replacing $\mathcal{R}_q^\delta f(z)$ and $p(z)$ with their series in (9), we get

$$1+\sum_{k=2}^{\infty}\frac{[k+\delta-1]_q!}{[\delta]_q![k-1]_q!}[k]_q a_k z^{k-1}=1+\sum_{k=1}^{\infty}c_kz^k. \tag{10}$$

Equating the coefficients on both side of (10) yields

$$\begin{cases} a_2 = \dfrac{c_1}{[2]_q[\delta+1]_q}, \\ a_3 = \dfrac{[2]_q c_2}{[3]_q[\delta+1]_q[\delta+2]_q}, \\ a_4 = \dfrac{[2]_q[3]_q c_3}{[4]_q[\delta+1]_q[\delta+2]_q[\delta+3]_q}. \end{cases} \tag{11}$$

From (11), we observe the following

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{[\delta+1]_q^2[\delta+2]_q}\left|\frac{[3]_q c_1 c_3}{[4]_q[\delta+3]_q} - \frac{[2]_q^2 c_2^2}{[3]_q^2[\delta+2]_q}\right|. \tag{12}$$

Since $p(z)$ is within $\mathcal{P}$ concurrently, we suppose that c_1 is greater than zero without the loss of generality. For accessibility of notation, take $c_1 = c \quad (c \in [0,2])$. By means of substituting the values of c_1 and c_2 respectively, from (7) and (8), we have

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| &= \frac{1}{4[\delta+1]_q^2[\delta+2]_q}\left|\frac{[3]_q}{[4]_q[\delta+3]_q}\left\{c^4 + 2c^2(4-c^2)x - c^2(4-c^2)x^2 + 2c(4-c^2)(1-|x|^2)z\right\}\right. \\ &\quad \left. - \frac{[2]_q^2}{[3]_q^2[\delta+2]_q}\left\{c^4 + 2c^2(4-c^2)x + (4-c^2)^2x^2\right\}\right| \\ &= \frac{1}{4[\delta+1]_q^2[\delta+2]_q}\left|\left(\frac{[3]_q}{[4]_q[\delta+3]_q} - \frac{[2]_q^2}{[3]_q^2[\delta+2]_q}\right)c^4\right. \\ &\quad + \left(\frac{[3]_q}{[4]_q[\delta+3]_q} - \frac{[2]_q^2}{[3]_q^2[\delta+2]_q}\right)2c^2(4-c^2)x \\ &\quad \left. - \left(\frac{[3]_q c^2}{[4]_q[\delta+3]_q} + \frac{[2]_q^2(4-c^2)}{[3]_q^2[\delta+2]_q}\right)(4-c^2)x^2 + \frac{2[3]_q c(4-c^2)(1-|x|^2)z}{[4]_q[\delta+3]_q}\right|. \end{aligned}$$

By using triangle inequality, $|z| \le 1$ and replacement of $|x|$ by v, we get

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &= \frac{1}{4[\delta+1]_q^2[\delta+2]_q}\left\{\left(\frac{[3]_q}{[4]_q[\delta+3]_q}-\frac{[2]_q^2}{[3]_q^2[\delta+2]_q}\right)c^4\right.\\ &+\left(\frac{[3]_q}{[4]_q[\delta+3]_q}-\frac{[2]_q^2}{[3]_q^2[\delta+2]_q}\right)2c^2(4-c^2)v\\ &\left.+\left(\frac{[3]_qc^2}{[4]_q[\delta+3]_q}+\frac{[2]_q^2(4-c^2)}{[3]_q^2[\delta+2]_q}\right)(4-c^2)v^2+\frac{2[3]_qc(4-c^2)(1-v^2)}{[4]_q[\delta+3]_q}\right\}\end{aligned}$$

$$\begin{aligned}&=\frac{1}{4[\delta+1]_q^2[\delta+2]_q}\left\{\left(\frac{[3]_q}{[4]_q[\delta+3]_q}-\frac{[2]_q^2}{[3]_q^2[\delta+2]_q}\right)c^4\right.\\ &+\left(\frac{[3]_q}{[4]_q[\delta+3]_q}-\frac{[2]_q^2}{[3]_q^2[\delta+2]_q}\right)2c^2(4-c^2)v\\ &\left.+\left(\frac{[3]_qc(c-2)}{[4]_q[\delta+3]_q}+\frac{[2]_q^2(4-c^2)}{[3]_q^2[\delta+2]_q}\right)(4-c^2)v^2+\frac{2[3]_qc(4-c^2)}{[4]_q[\delta+3]_q}\right\}\\ &=H(c,v), \qquad (13)\end{aligned}$$

where $v=|x|\le 1$ and $0\le c\le 2$.

We next maximize the function $H(c,v)$ on $[0,2]\times[0,1]$. Differentiating $H(c,v)$ in (13) partially with respect to v, yields

$$\begin{aligned}\frac{\partial H(c,v)}{\partial v} &= \frac{1}{4[1+\delta]_q^2[2+\delta]_q}\left\{\left(\frac{[3]_q}{[4]_q[\delta+3]_q}-\frac{[2]_q^2}{[3]_q^2[\delta+2]_q}\right)2c^2(4-c^2)\right.\\ &\left.+\left(\frac{[3]_qc(c-2)}{[4]_q[\delta+3]_q}+\frac{[2]_q^2(4-c^2)}{[3]_q^2[\delta+2]_q}\right)2(4-c^2)v\right\}.\end{aligned}$$

It is clear that $\frac{\partial H(c,v)}{\partial v}\ge 0.$ This means that H is an increasing function of $v.$ Then $H(c,v)$ can not have a maximum in the interior of $[0,2]\times[0,1].$ Furthermore, for fixed $c\in[0,2]$.

$$\max_{0\le v\le 1} H(c,v)=H(c,1)=K(c).$$

Then

$$K(c)=\frac{1}{4[1+\delta]_q^2[2+\delta]_q}\left\{\left(\frac{[3]_q}{[4]_q[\delta+3]_q}-\frac{[2]_q^2}{[3]_q^2[\delta+2]_q}\right)c^4\right.$$

$$+\left(\frac{[3]_q}{[4]_q[\delta+3]_q}-\frac{[2]_q^2}{[3]_q^2[\delta+2]_q}\right)2c^2(4-c^2)$$
$$\left.+\left(\frac{[3]_q c^2}{[4]_q[\delta+3]_q}+\frac{[2]_q^2(4-c^2)}{[3]_q^2[\delta+2]_q}\right)(4-c^2)\right\}.$$

Then

$$K'(c)=\frac{2}{[1+\delta]_q^2[2+\delta]_q}\left\{\frac{[3]_q c(3-c^2)}{[4]_q[\delta+3]_q}-\frac{[2]_q^2 c(4-c^2)}{[3]_q^2[\delta+2]_q}\right\},$$

by computing the above equation, the value of $K'(c)$ <0 is obtained when $0<c<2$ and $K(c)$ has real critical point at $c=0$. Also observe that $K(c)>K(2)$. Accordingly, $\max_{0\le c\le 2}K(c)$ occurs at $c=0$. Then the upper bound of (13) corresponds to $v=1$ and $c=0$.

Hence,

$$\left|a_2a_4-a_3^2\right|\le\frac{4[2]_q^2}{[3]_q^2[\delta+1]_q^2[\delta+2]_q^2}. \ \square$$

Setting $\delta=0$ and $q\to 1^-$, we get the following result.

Corollary 3.1. (Janteng *et al.* 2006) *If* $f\in R_q(0)$, *then*

$$\left|a_2a_4-a_3^2\right|\le\frac{4}{9}.$$

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