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SUFFICIENT CONDITIONS OF STARLIKENESS AND CONVEXITY FOR FUNCTIONS OF COMPLEX ORDER

(Syarat Cukup bagi Kebakbintangan dan Kecembungan untuk Fungsi Peringkat Kompleks)

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ABSTRACT

Let A be the class of normalised and analytic functions defined in the unit disc $U=\{z : |z|<1\}$. In this paper we study the expression $\frac{zf'(z)}{b\left[f(z)\right]^2}-(1+\gamma)\frac{z}{bf(z)}+\gamma,\ z\in U,$ for some $\gamma(\gamma>0)$ and $b(b\in\mathbb{C}\setminus\{0\}$ as a criteria for starlikeness and convexity at analytic functions of complex order.

Keywords: starlikeness; convexity; analytic functions; sufficient conditions

ABSTRAK

Andaikan A kelas fungsi ternormalkan dan analisis yang ditakrifkan dalam cakera unit $U=\{z : |z|<1\}$. Dalam makalah ini dikaji tentang ungkapan $\frac{zf'(z)}{b\left[f(z)\right]^2}-(1+\gamma)\frac{z}{bf(z)}+\gamma,$ $z\in U,$ untuk setiap $\gamma(\gamma>0)$ dan $b(b\in\mathbb{C}\setminus\{0\}$ sebagai suatu kriterium bagi kebakbintangan dan kecembungan untuk fungsi analisis peringkat kompleks.

Kata kunci: kebakbintangan; kecembungan; fungsi analisis; syarat cukup

1. Introduction

Let A denote the class of normalised analytic functions f of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k, \tag{1}$$

which are analytic in the open unit disc $U=\{z : |z|<1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U. A function $f\in A$ is said to be the starlike of complex order b, $\left(b\in\mathbb{C}\setminus\{0\}\right)$ in U if and only if

$$\frac{f(z)}{z}\neq 0 \text{ and } \operatorname{Re}\left[1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right]>0, \quad \left(b\in\mathbb{C}\setminus\{0\},\ z\in U\right). \tag{2}$$

We denote by $S^*(b)$ the subclass of A consisting of functions which are starlike of complex order b in U. Further, let $S_1^*(b)$ denote the class of functions $f\in A$ satisfying the following inequality:

$$\left|\frac{zf'(z)}{f(z)}-1\right|<|b|, \left(b\in\mathbb{C}\setminus\{0\},\ z\in U\right). \tag{3}$$

We note that $S_1^*(b)$ is a subclass of $S^*(b)$.

Also, a function $f \in A$ is said to be the convex of complex order *b*, $\left(b \in \mathbb{C}\setminus\{0\}\right)$ in U if and only if

$$\frac{f(z)}{z} \neq 0 \text{ and } \operatorname{Re}\left[1+\frac{1}{b}\frac{zf''(z)}{f'(z)}\right] > 0, \quad \left(b \in \mathbb{C}\setminus\{0\},\ z \in U\right). \tag{4}$$

We denote by $K(b)$ the subclass of A consisting of functions which are convex of complex order b in U. Further, let $K_1(b)$ denote the class of functions $f \in A$ satisfying the following inequality:

$$\left|\frac{zf''(z)}{f'(z)}\right| < |b|, \quad \left(b \in \mathbb{C}\setminus\{0\},\ z \in U\right), \tag{5}$$

so that, obviously, $K_1(b)$ is a subclass of $K(b)$. The classes $S^*(b)$ and $K(b)$ of starlike and convex functions of a complex order b in U were introduced and investigated earlier by Nasr & Aouf (1982;1985) and Wiatrowski (1970).

Several authors have investigated sufficient conditions for starlikeness and convexity of the class $S^*(b)$ and $K(b)$. For example, Ramesha *et al.* (1995), Silverman (1999), Obradovic (1997), Duren (1983), Mocanu (1988;1992) and Siregar (2011), Siregar and Darus (2011), Siregar and Akbarally (2014), Mohammed Pauzi and Darus (2017), and Bulboaca and Tuneski (2001). In Nunokawa and Sokol (2015) investigated the starlikeness condition of Libera Transform.

In the present paper, the expressions

$$\frac{z^2 f'(z)}{b\left[f(z)\right]^2} - \left(1+\gamma\right)\frac{z}{bf(z)} + \gamma$$

and

$$\frac{zf''(z)}{b\left[f'(z)\right]^2} - \gamma\frac{1}{bf'(z)} + \gamma$$

are studied and their sufficient conditions that will place $f(z)$ in the classes $S^*(b,\gamma)$ and $K(b,\gamma)$, respectively defined above are given.

The work of Nishiwaki and Owa (2014), Singh and Tuneski (2004), and Tuneski (2009) have motivated us to come to determine sufficient conditions for the classes above.

2. Preliminaries

To prove our main results, we will need the following definition and lemmas in this section.

Lemma 2.1. (*Theorem* 2*: Miller et al.* (1984)) *Let* $F(z)$ *and* $G(z)$ *be analytic functions in* U, $\gamma \geq 0$ *and* $G'(z) \neq 0$. *Furthermore, in the case of* $\gamma = 0$, $F(0) = G(0)$. *If*

$$\operatorname{Re}\left(1+\frac{zG''(z)}{G'(z)}\right) > \kappa(\gamma) = \begin{cases} -\dfrac{\gamma}{2} & \left(\gamma \leq 1\right); \\[2mm] -\dfrac{1}{2\gamma} & \left(\gamma \geq 1\right), \end{cases}$$

and $F(z) \prec G(z)$, *then* $z^{-\gamma}\int_0^z t^{\gamma-1}F(t)\,dt \prec z^{-\gamma}\int_0^z t^{\gamma-1}G(t)\,dt.$

The following lemma was studied by Singh and Tuneski (2004).

Lemma 2.2. (Singh & Tuneski 2004). *Let* $\gamma \geq 0$, $p(z), G(z)$ *be analytic functions in* U *and* $G'(z) \neq 0.$ *Furthermore, in the case of* $\gamma = 0$, $F(0) = G(0)$, *if*

$$\operatorname{Re}\left(1+\frac{zG''(z)}{G'(z)}\right) > \kappa(\gamma)$$

and $1-\gamma p(z) - zp'(z) \prec G(z)$, *then* $p(z) - Cz^{-\gamma} \prec z^{-\gamma}\int_0^z t^{\gamma-1}\left(1-G(t)\right)dt,$ *where* $C = p(0)$ *for* $\gamma = 0$ *and* $C = 0$ *for* $\gamma > 0.$

Lemma 2.3. (Tuneski 2009). *Let* $f(z) \in A$ *and* $\frac{f(z)}{z} \neq 0$ *satisfies*

$$\left| f'(z) - (1-\gamma)\frac{f(z)}{z} - \gamma \right| < \lambda, \quad z \in U,$$

for some γ $(\gamma > 0)$ *and* λ $(\lambda > 0)$. *Then*

$$\left| \frac{f(z)}{z} - 1 \right| < \frac{\lambda}{1+\gamma}, \quad z \in U,$$

and

$$|f(z)| < 1 + \frac{\lambda}{1+\gamma}, \qquad z \in U.$$

3. Main Results

Our main result is stated as the following:

Theorem 3.1 *Let* $f(z) \in A$, $f(z) \neq 0$, *satisfies*

$$\left| \frac{z^2 f'(z)}{b[f(z)]^2} - (1+\gamma)\frac{z}{bf(z)} + \gamma \right| < \lambda, \qquad z \in U, \tag{6}$$

for some γ $(\gamma > 0)$ *and* λ $(\lambda > 0)$, b $\left(b \in \mathbb{C}\setminus\{0\}\right)$. *Then*

$$\left| \frac{z}{bf(z)} \right| < \frac{\lambda}{1+\gamma}, \qquad z \in U. \tag{7}$$

Proof. Let us define the function $G(z)$ by $G(z)=1-\gamma+\lambda z,\ \lambda>0,$ then $G'(0)=\lambda$ and

$$\mathrm{Re}\left(1+\frac{zG''(z)}{G'(z)}\right)=1, \qquad z\in U.$$

Furthermore, let us suppose that $p(z)=\dfrac{z}{bf(z)}$, then $p(z)\in H(U),\ f(z)\neq 0$ and

$$1-\gamma\, p(z)-zp'(z)=1-(1+\gamma)\frac{z}{bf(z)}+\frac{z^2 f'(z)}{b\left[f(z)\right]^2}.$$

On the other hand, we have

$$1-(1+\gamma)\frac{z}{bf(z)}+\frac{z^2 f'(z)}{b\left[f(z)\right]^2}\prec 1-\gamma+\lambda z$$

from the inequality (6). Applying Lemma 2.1, we obtain

$$\frac{z}{bf(z)}-Cz^{-\gamma}\prec z^{-\gamma}\int_0^z t^{\gamma-1}(1-G(t))\,dt,$$

$$=1-\frac{\lambda}{1+\gamma}z-C_1 z^{-\gamma}$$

where $C=p(0)$ for $\gamma=0$ and $C=0$ for $\gamma>0$.
Thus, we get

$$\left|\frac{z}{bf(z)}\right|<\frac{\lambda}{1+\gamma}.$$

The left hand side of the inequality (6) holds true for λ if we take the function

$$f(z)=\frac{z}{1+\dfrac{\lambda}{1+\gamma}e^{i\theta}z},$$

from the inequality (7), implying that, this result is sharp. □

Theorem 3.2 *Let* $f\in A,\ \ \gamma\geq 0,\ \ b\in\mathbb{C}\setminus\{0\},\ G\in H(U)$ *and* $G'(0)\neq 0,\ \ f(z)\neq 0,\ \ f'(z)\neq 0.$ *Also let*

$$\mathrm{Re}\left(1+\frac{zG''(z)}{G'(z)}\right)>\kappa(\gamma), \qquad z\in U.$$

If

$$\left(1-\frac{1}{b}\right)+\frac{\left(\dfrac{1}{b}-\gamma\right)+\dfrac{zf''(z)}{f'(z)}}{bf'(z)/f(z)}\prec G(z) \tag{8}$$

then

$$\frac{z}{bf(z)}-Cz^{-\gamma}\prec z^{-\gamma}\int_0^z t^{\gamma-1}(1-G(t))\,dt.$$

Proof. Let us put $p(z)=\dfrac{z}{bf(z)}$, then $p(z)\in H(U)$, $f(z)\neq 0$, $f'(z)\neq 0$, and

$$1-\gamma p(z)-zp'(z)=\left(1-\frac{1}{b}\right)+\frac{\left(\frac{1}{b}-\gamma\right)+\frac{zf''(z)}{f'(z)}}{bf'(z)/f(z)}.$$

The rest follows from Lemma 2.2. □

Choosing $G(z)=1-\gamma+\lambda\left(z+\dfrac{mz^2}{2}\right)$ in (8), we obtain the following result.

Corollary 3.3. *Let* $f\in A$, $f(z)\neq 0$, $f'(z)\neq 0$, $\gamma\geq 0$, $b\in\mathbb{C}\setminus\{0\}$, $\lambda>0$, $m\in\mathbb{R}$ *and* $|m|\leq\dfrac{1-\kappa(\gamma)}{2-\kappa(\gamma)}$. *Then*

$$\left(1-\frac{1}{b}\right)+\frac{\left(\frac{1}{b}-\gamma\right)+\frac{zf''(z)}{f'(z)}}{bf'(z)/f(z)}\prec 1-\gamma+\lambda\left(z+\frac{mz^2}{2}\right) \tag{9}$$

implies

$$\frac{z}{bf(z)}\prec 1-\frac{\lambda}{1+\gamma}z-\frac{\lambda m}{2(2+\gamma)}z^2\equiv h(z).$$

Additionally, if

$$m\leq\left(1-\frac{\lambda}{1+\gamma}\right)\frac{2(2+\gamma)}{\lambda},$$

then $f\in S_b^*$, *i.e.,* f *is a starlike functions of complex order.*

Theorem 3.4 *Let* $f(z)\in A$ *and* $f(z)\neq 0$ *satisfies*

$$\left|\frac{z^2 f'(z)}{b\left[f(z)\right]^2}-(1+\gamma)\frac{z}{bf(z)}+\gamma\right|<\lambda,\qquad z\in U, \tag{10}$$

for some γ $(\gamma\geq 0)$ *and* λ $(0<\lambda<\frac{1}{2})$, b $(b\in\mathbb{C}\setminus\{0\})$ *then* $f\in S^*\left(\dfrac{b(1+\gamma-\lambda)-\lambda(1+2\gamma)}{b(1+\gamma-\lambda)}\right)$.

Proof. Let defined the function $f(z)$ satisfies the inequality (10), $f(z)\neq 0$. Thus, there is exists an analytic function $w(z)$ in U such that $w(0)=0$ and $|w(0)|<1$, then we have

$$\left(1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)-\left(1+\frac{\gamma}{b}\right)\right)=\frac{1}{b}\left(\lambda w(z)-\gamma\right)\frac{bf(z)}{z}.$$

It follows that

$$\left|1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)-\left(1+\frac{\gamma}{b}\right)\right|=\left|\frac{1}{b}\left(\lambda w(z)-\gamma\right)\right|\left|\frac{bf(z)}{z}\right|.$$

$$<\frac{(1+\gamma)(\gamma+\lambda)\frac{1}{b}}{(1+\gamma-\lambda)}.$$

This shows that

$$\operatorname{Re}\left[1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)\right]>\left(1+\frac{\gamma}{b}\right)-\frac{(1+\gamma)(\gamma+\lambda)}{b(1+\gamma-\lambda)}$$

$$=\left(\frac{b(1+\gamma-\lambda)-\lambda(1+2\gamma)}{b(1+\gamma-\lambda)}\right).$$

We complete the proof of the theorem. □

Putting $zf'(z)$ instead $f(z)$ in Theorem 3.1, we get the following theorem.

Theorem 3.5 *Let* $f(z)\in A$ *and* $f'(z)\neq 0$ *satisfies*

$$\left|\frac{zf''(z)}{b\left[f'(z)\right]^2}-\gamma\frac{z}{bf'(z)}+\gamma\right|<\lambda,\ z\in U, \tag{11}$$

for some $\gamma\ (\gamma>0)$ *and* $\lambda\ (\lambda>0)$, $b\ \left(b\in\mathbb{C}\setminus\{0\}\right)$ *then*

$$\left|\frac{1}{bf'(z)}\right|<\frac{\lambda}{1+\lambda},\quad z\in U. \tag{12}$$

Proof. Let us put $p(z)=\dfrac{1}{bf'(z)}$, $f'(z)\neq 0,$ in the proof of Theorem 3.1, we arrive

$$\left|\frac{1}{bf'(z)}\right|<\frac{\lambda}{1+\lambda},\ z\in U.$$

The left hand side of the inequality (11) holds true for λ if we take the function $f(z)=\dfrac{z}{1+\frac{\lambda}{1+\gamma}e^{i\theta}z}$ and from the inequality (12), implying that, this result is sharp. We complete the proof of the theorem. □

In view of Theorem 3.5, we obtain the sufficient condition of convexity as follows.

Theorem 3.6 *Let* $f(z)\in A$ *and* $f'(z)\neq 0$ *satisfies*

$$\left|\frac{zf''(z)}{b\left[f'(z)\right]^2}-\gamma\frac{1}{bf'(z)}+\gamma\right|<\lambda,\qquad z\in U, \tag{13}$$

for some $\gamma\ (\gamma\geq 0)$ *and* $\lambda\ (0<\lambda<\frac{1}{2})$, $b\ \left(b\in\mathbb{C}\setminus\{0\}\right)$. *Then* $f\in K\left(\dfrac{b(1+\gamma-\lambda)-\lambda(1+2\gamma)}{b(1+\gamma-\lambda)}\right).$

Proof. Apply the same technique as in the proof at Theorem 3.4, we see that

$$\left|1+\frac{1}{b}\frac{zf''(z)}{f'(z)}-\left(1+\frac{\gamma}{b}\right)\right|<\frac{(1+\gamma)(\gamma+\lambda)\frac{1}{b}}{(1+\gamma-\lambda)}.$$

This shows that

$$\operatorname{Re}\left(1+\frac{1}{b}\frac{zf''(z)}{f'(z)}\right) > \left(\frac{b(1+\gamma-\lambda)-\lambda(1+2\gamma)}{b(1+\gamma-\lambda)}\right).$$

which proves the theorem. □

Taking $b=1$ and $\lambda=\frac{1}{2}$ in (12), we get to the following remark.

Remark 3.7 *Let* $f(z)\in A$ *and* $f'(z)\neq 0$ *satisfies*

$$\left|\frac{zf''(z)}{\left[f'(z)\right]^2}-\gamma\frac{1}{f'(z)}+\gamma\right|<\frac{1}{2}, \qquad z\in U, \tag{14}$$

for some γ $(\gamma\geq 0)$. *Then* f is convex of complex order 1 in U, $f\in K(1)$.

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