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FIRST ORDER DIFFERENTIAL SUBORDINATION ASSOCIATED WITH CASSINI CURVE

(Subordinasi Pembeza Peringkat Pertama yang Bersekutu dengan Lengkung Cassini)

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ABSTRACT

Let p be an analytic function defined on the open unit disc with $p(0)=1$. In this paper, we determine the conditions for β such that certain subordination properties hold for $p(z)$ is subjected to certain geometric conditions involving the expressions $1+\beta zp'(z)$, $1+\beta zp'(z)/p(z)$ and $1+\beta zp'(z)/p^2(z)$ of which each is subordinated to $\sqrt{1+cz}$ and the condition for β is determined.

Keywords: analytic functions; univalent functions; differential subordination; Cassini curve

ABSTRAK

Andaikan p fungsi analisis yang tertakrif pada cakera unit terbuka dengan $p(0)=1$. Di dalam makalah ini, syarat β ditentukan sedemikian sehingga sifat-sifat subordinasi adalah benar bagi $p(z)$ tertakluk kepada syarat geometri mengandungi $1+\beta zp'(z)$, $1+\beta zp'(z)/p(z)$ dan $1+\beta zp'(z)/p^2(z)$ yang bersubordinasi kepada $\sqrt{1+cz}$.

Kata kunci: fungsi analisis; fungsi univalen; subordinasi pembezaan; lengkung Cassini

1. Introduction

Let A be the class of analytic functions in $D=\{z\in\mathbb{C}:|z|<1\}$ and normalized by the condition $f(0)=0=f'(0)-1$. Let S be the subset of A denoting the class of univalent functions. Also, we denote by C and S^* respectively the class of convex and starlike functions. If an analytic function f subordinates to function g which also analytic, there must exists an analytic function w so $w(0)=0$ and $|w(z)|<1$ for $|z|<1$ and $f(z)=g(w(z))$. When the function g is univalent in D, then we say that $f(z)\prec g(z)$ and it is commensurate to $f(0)=g(0)$ and $f(D)\subset g(D)$.

In 1935, Goluzin (1935) proved that if the first order differential subordination $zp'(z)\prec zq'(z)$ holds and $zq'(z)$ is convex then the subordination $p(z)\prec q(z)$ holds and the function q is the best dominant. After that, many researchers continued the study and established some generalizations of this first order differential subordination and the general theory was discussed in detail by Miller and Mocanu (1985). If $1+zp'(z)\prec 1+z$ hold, Nunokawa *et al.* (1989) proved the subordination $p(z)\prec 1+z$ also holds. Furthermore, there are lot of results obtained by other researchers, for example see Ali *et al.* (2007), Omar and Halim (2013), Omar *et al.* (2013), Cho *et al.* (2016), Ravichandran and Sharma (2015) and Sharma and Ravichandran (2016).

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A function of $\sqrt{1+z}$ is associated with the class called S_L^* introduced by Sokól and Stankiewics (1996). This class consists $f \in A$ so $w(z) := zf'(z)/f(z)$. It lies in the region that bounded by the right half of the lemniscate of Bernoulli which is $|w^2-1|<1$. The shape of this lemniscate of Bernoulli resembles the symbol ∞.

Besides, Aouf *et al.* (2011) generalized the idea and defined the class $S^*(q_c)$ where $q_c(z)=\sqrt{1+cz}$ for $c \in (0,1]$ as follow:

$$S^*(q_c) = \left\{ f \in A : \left| \left[\frac{zf'(z)}{f(z)} \right]^2 - 1 \right| < c, z \in D \right\} \tag{1}$$

From this we may establish that

$$f \in S^*(q_c) \Leftrightarrow \frac{zf'(z)}{f(z)} \prec \sqrt{1+cz} \ (z \in D). \tag{2}$$

Note that $S^*(q_1) = S^*{}_L$. We also denote by θ_c the set of all points in the right half-plane such that the product of the distances from each point to the focuses -1 and 1 is less than c:

$$\theta_c := \left\{ \operatorname{Re} w > 0, |w^2-1| < c \right\} \ (w \in \mathbb{C}), \tag{3}$$

thus the boundary $\partial\theta_c$ is the right loop of the Cassinian ovals $(x^2+y^2)^2 - 2(x^2-y^2) = c^2-1$ and for $c=1$, $S^*(q_1) \equiv S_L^*$. In Cho *et al.* (2019), the authors defined $\varphi_{\sin}(z) := 1+\sin z$ as

$$\varphi_{\sin}(z) := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec 1+\sin z \right\}. \tag{4}$$

A function $f \in \varphi_{\sin}(z)$ if and only if there exists an analytic function q, satisfies $q(z) \prec q_0 = 1+\sin z$, $z \in D$. From this, we can get

$$f(z) = z \exp\left(\int_0^z \frac{q(t)-1}{t} dt \right). \tag{5}$$

In the paper by Ali *et al.* (2007), it showed the conditions of *A*, *B*, *D* and so that $1+\beta zp'(z)/p^n(z)$ ($n=0,1,2$) are subordinated to $\frac{1+Dz}{1+Ez}$ implies $p(z) \prec \frac{1+Az}{1+Bz}$. Likewise, researches in Omar and Halim (2013) found the values of β so that $1+\beta zp'(z)/p^n(z)$ ($n=0,1,2$) subordinate to $\frac{1+Dz}{1+Ez}$ then $p(z) \prec \sqrt{1+z}$. Inspired by these researches, this paper determines the conditions for β by considering the class $S^*(q_c)$ and the function $\varphi_{\sin}(z)$.

2. Preliminary Result

We investigate the subordination properties for the class $S^*(q_c)$ associated with $\varphi_{\sin}(z)$. We will need the following lemma to prove our results.

Lemma 2.1. (Miller & Mocanu 2000) Let q be univalent in D and let θ and ϕ be analytic in a domain U containing $q(D)$ with $\theta(w)\neq 0$ when $w\in q(D)$. Set $Q(z):=zq'(z)\theta(q(z))$ and $h(z):=\phi(q(z))+Q(z)$. Suppose that

i. either h is convex, or Q is starlike univalent in D and

ii. $\operatorname{Re}(zh'(z)/Q(z))>0$ for $z\in D$.

If p is analytic in D, with $p(0)=q(0)$, $p(D)\subset U$ and $\phi(p(z))+zp'(z)\theta(p(z))\prec\phi(q(z))+zq'(z)\theta(q(z))$, then $p(z)\prec q(z)$, and q is best dominant.

The first result gives the bound for β so that $1+\beta zp'(z)\prec\sqrt{1+cz}$ implies that the function p is subordinate to the function $\varphi_{\sin}(z)$.

3. Main Results

Theorem 3.1. *Consider the analytic function* $p\in D$, $p(0)=1$ *such that* $1+\beta zp'(z)\prec\sqrt{1+cz}$. *Then the following subordination result holds:*

if $\beta\geq\dfrac{2\left[\log\left(1+\sqrt{1-c}\right)+1-\sqrt{1-c}-\log 2\right]}{\sin(1)}$, *then* $p(z)\prec\varphi_{\sin}(z)$.

Proof. Define the function $q_B:\bar{D}=\{z\in\mathbb{C}:|z|\leq 1\}\to\mathbb{C}$ as follows:

$$q_B(z)=1+\frac{2}{\beta}\left[\sqrt{1+cz}-\log\left(1+\sqrt{1+cz}\right)+\log 2-1\right] \tag{6}$$

is analytic and it is the solution of $1+\beta zq_B'(z)=\sqrt{1+cz}$. First, denote $\phi(w)=1$, $\theta(w)=\beta$ and we define $Q(z)=zq_B'(z)\theta(q_B(z))=\beta zq_B'(z)$ where $Q:\bar{D}\to\mathbb{C}$.

By considering $\sqrt{1+cz}-1$, let $f(z)=\sqrt{1+cz}-1$, so we can get $\dfrac{zf'(z)}{f(z)}=\dfrac{1}{2}+\dfrac{(1+cz)^{1/2}}{2(1+cz)}$.

Then let $h(\varepsilon)=\dfrac{(1+\varepsilon)^{1/2}}{2(1+\varepsilon)}$ where $|\varepsilon|=|cz|=|x+yi|=\sqrt{x^2+y^2=1},\varepsilon\neq 1$, so

$\left[h(\varepsilon)\right]^2 = \dfrac{1}{4(1+x+yi)} = \dfrac{1+x-yi}{8(1+x)}$. Thus, $\operatorname{Re}\left[h(\varepsilon)\right]^2 = \dfrac{1}{8}$ and $\operatorname{Re}\left[h(x)\right] = \pm\sqrt{\dfrac{1}{8}}$. Hence, $\operatorname{Re}\left[\dfrac{zf'(z)}{f(z)}\right] = \dfrac{1}{2} \pm \sqrt{\dfrac{1}{8}} > 0$, then $\sqrt{1+cz}-1$ is a starlike function, so Q is also starlike. Function $h(z) = \phi(q_B(z)) + Q(z)$ satisfies $\operatorname{Re}(zh'(z)/Q(z)) > 0$ for $z \in D$.

Hence, from Lemma 2.1 we know that $1+\beta zp'(z) \prec 1+\beta zq_B'(z)$ implies $p(z) \prec q_B(z)$. This concludes that $p(z) \prec P(z)$ for appropriate P. The result holds if $q_B(z) \prec P(z)$ holds. If $q_B(z) \prec P(z)$, then $P(-1) < q_B(-1) < q_B(1) < P(1)$. This gives a necessary condition for $p \prec P$ to hold. So, by taking $P(z) = \varphi_{\sin}(z)$, the inequalities $q_B(-1) \geq \varphi_{\sin}(-1)$ and $q_B(1) \leq \varphi_{\sin}(1)$ reduce to $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{2\left[\log\left(1+\sqrt{1-c}\right)+1-\sqrt{1-c}-\log 2\right]}{\sin(1)} \quad \text{and} \quad \beta_2 = \frac{2\left[\sqrt{1+c}-\log\left(1+\sqrt{1+c}\right)+\log 2-1\right]}{\sin(1)}$$

respectively. The subordination $q_B(z) \prec \varphi_{\sin}(z)$ holds if $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$. The proof is complete. □

When $c=1$, we get the following corollary.

Corollary 3.2. *If c=1 and* $1+\beta zp'(z) \prec \sqrt{1+z}$ *where p is analytic function p in D with* $p(0)=1$. *Then the following subordination result holds:*

$$\text{If } \beta \geq \frac{2(1-\log 2)}{\sin(1)} \approx 0.729325, \text{ then } p(z) \prec \varphi_{\sin}(z)$$

which is stated as Theorem 2.1 in Ahuja et al. (2018).

Next result gives bound on β so that $1+\beta zp'(z)/p(z) \prec \sqrt{1+cz}$ implies p is subordinate to $\varphi_{\sin}(z)$ function.

Theorem 3.3. *Suppose that* p *be analytic function in* D, $p(0)=1$ *and* $1+\beta zp'(z)/p(z) \prec \sqrt{1+cz}$. *Then the following subordination result holds:*

$$\text{if } \beta \geq \frac{2\left[\sqrt{1+c}-\log\left(\sqrt{1+c}+1\right)+\log 2-1\right]}{\log\left(1+\sin(1)\right)}, \text{ then } p(z) \prec \varphi_{\sin}(z).$$

Proof. The function $q_B : \overline{D} = \{z \in \mathbb{C} : |z| \leq 1\} \to \mathbb{C}$ defined by

$$q_B(z) = \exp\left[\frac{2}{\beta}\left[\sqrt{1+cz}-\log\left(\sqrt{1+cz}+1\right)+\log 2-1\right]\right]$$

is analytic and it is the solution of $1+\beta zq_B'(z)/q_B(z) = \sqrt{1+cz}$. The function $\phi(w)=1$ and $\theta(w) = \beta/w$ is defined and note that

$Q(z) := zq_B'(z)\theta(q_B(z)) = \beta zq_B'(z)/q_B(z) = \sqrt{1+cz} - 1$ is starlike in D. The function $h(z) := \phi(q_B(z)) + Q(z) = 1 + Q(z)$ satisfies $\mathrm{Re}(zh'(z)/Q(z)) > 0$ for $z \in D$. So, Lemma 2.1 is satisfied and the following subordination

$$1+\beta\frac{zp'(z)}{p(z)} \prec 1+\beta\frac{zq_B'(z)}{q_B(z)} \tag{7}$$

implies $p(z) \prec q_B(z)$. The, similar as the proof of Theorem 3.1, we obtain the desired results.
The proof is complete. □

Corollary 3.4. *For* c = 1, *the result reduces to Theorem 2.3 in Ahuja et al. (2018) and gives the following subordination:*

$$\textit{if } \beta \ge \frac{2\left(\sqrt{2}-1+\log 2-\log\left(\sqrt{2}+1\right)\right)}{\log\left(1+\sin(1)\right)} \approx 0.740256, \textit{ then } p(z) \prec \varphi_{\sin}(z).$$

Next, a bound on β is determined such that $1+\beta zp'(z)/p^2(z) \prec \sqrt{1+cz}$ implies p is subordinate to $\varphi_{\sin}(z)$ function.

Theorem 3.5. *Let* $1+\beta zp'(z)/p^2(z) \prec \sqrt{1+cz}$, *then the following subordination result holds:*

$$\textit{if } \beta \ge \frac{2\left(1+\sin(1)\left(\sqrt{1+c}-\log\sqrt{1+c}-1\right)+\log 2-1\right)}{\sin(1)}, \textit{ then } p(z) \prec \varphi_{\sin}(z)$$

where p *is an analytic function in* D *and* $p(0)=1$.

Proof. The function $q_B : \overline{D} \to \mathbb{C}$ defined by

$$q_B(z) = \left(1-\frac{2}{\beta}\left(\sqrt{1+cz}-\log\left(\sqrt{1+cz}+1\right)+\log 2-1\right)\right)^{-1} \tag{8}$$

is analytic and is the solution of $1+\beta zq_B'(z)/q^2{}_B(z) = \sqrt{1+cz}$. Similar as previous method, we define the function $\phi(w)=1$ and $\theta(w)=\beta/w^2$, also $Q(z) := zq_B'(z)\theta(q_B(z)) = \beta zq_B'(z)/q^2{}_B(z) = \sqrt{1+cz}-1$ is starlike in D, Q is starlike function. The condition $\mathrm{Re}(zh'(z)/Q(z)) > 0$ for $z \in D$ is satisfied. Therefore, applying Lemma 2.1, we obtain

$$1+\beta\frac{zp'(z)}{p^2(z)} \prec 1+\beta\frac{zq_B'(z)}{q_B{}^2(z)} \tag{9}$$

implies $p(z) \prec q_B(z)$. Thence, similar as the proof in Theorem 3.1, we obtain the desired result.
The proof is complete. □

Corollary 3.6. *Let c =1 and making use Theorem 3, the following subordination result holds: if* $\beta \geq \dfrac{2\left(1+\sin(1)\right)\left(\sqrt{2}-\log\left(1+\sqrt{2}\right)+\log 2-1\right)}{\sin(1)} \approx 0.989098$*, then* $p(z) \prec \varphi_{\sin}(z)$ *which is reduced to the result in Ahuja et al. (2018).*

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