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ON CERTAIN CLASS OF ANALYTIC FUNCTIONS BASED ON POISSON DISTRIBUTION

(Mengenai Kelas Tertentu Fungsi Analisis Berdasarkan Taburan Poisson)

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ABSTRACT

In this paper, a power series whose coefficients are the probabilities of the Poisson distribution is considered. A new subclass of univalent holomorphic functions related to Poisson distribution and convolution is introduced. Some properties such as coefficient estimates, distortion bounds, radii of starlikeness, convexity and close-to-convexity are obtained. The neighbourhood property for functions in this new subclass is also given.

Keywords: Univalent function; Poisson distribution series; convolution; radii of starlikeness; convexity and close-to-convexity; neighbourhood

ABSTRAK

Dalam makalah ini, siri kuasa yang pekalinya adalah kebarangkalian taburan Poisson dipertimbangkan. Subkelas fungsi holomorfi univalen berkaitan dengan taburan Poisson dan konvolusi diperkenalkan. Beberapa sifat seperti anggaran pekali, batas erotan, jejari bakbintang, kecembungan dan hampir cembung diperoleh. Sifat kejiranan fungsi bagi subkelas fungsi baharu ini juga ditunjukkan.

Kata kunci: Fungsi Univalen; siri taburan Poisson; konvolusi; jejak bakbintang; kecembungan dan hampir kecembungan; kejiranan.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $\Delta = \{z : |z| < 1\}$.

Also, $\mathcal{N}$ denote the subclass of $\mathcal{A}$ consisting of functions of the type

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad (a_n \geq 0). \tag{1}$$

A variable X is said to be Poisson distributed if it takes the values $0,1,2,\ldots$ with probabilities $\frac{me^{-m}}{1!}, \frac{m^2e^{-m}}{2!}, \ldots$ respectively, where m is called the parameter and it is average number of events per interval. Thus, the probability mass function for a Poisson distribution is of the type

$$P(x = k) = \frac{m^k e^{-m}}{k!}, \qquad (k = 0,1,2,\ldots).$$

Now we consider a power series whose coefficients are probabilities of the Poisson distribution as follows:

$$P(m,z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{n-1!} e^{-m} z^n,$$

where, by ratio test, the radius of convergence is infinity, see Porwal (2014), Porwal and Kumar (2016).and Altunkaya and Yalcin (2018). Also,

$$K(m,z) = 2z - P(m,z) = z - \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n. \tag{2}$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are also univalent in Δ.

The Hadamard product (convolution) of two functions

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n$$

and

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n$$

is defined by

$$(f * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n. \tag{3}$$

Using (2) and (3) we consider the integral operator $I: \mathcal{N} \to \mathcal{N}$ by

$$I(f) = f(z) * \int_0^z \frac{k(m,t)}{t} dt, \tag{4}$$

or equivalently, see Rabha and Kota (2018) and Gangadharan *et al*. (2016),

$$I(f) = z - \sum_{n=2}^{\infty} \frac{e^{-m} m^{n-1}}{n!} a_n z_n. \tag{5}$$

For f and F be analytic in Δ, we say that f is subordinate to F, written $f \prec F$, if there exists a function $w(z)$ analytic in Δ, with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in \Delta$, such that $f(z) = F(w(z))$, for all $z \in \Delta$. If f is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(\Delta) \subset F(\Delta)$.

We consider $\mathcal{A}(\alpha,\beta,t,m)$ as a subclass of $\mathcal{N}$ consisting of all functions in $\mathcal{N}$ for which:

$$\frac{z\big(I(f)\big)'}{f_t(z)} \prec \frac{1+\alpha z}{1+\beta z}, \tag{6}$$

or equivalently,

$$\left|\frac{\dfrac{z\big(I(f)\big)'}{f_t(z)}-1}{\alpha-\beta\dfrac{z\big(I(f)\big)'}{f_t(z)}}\right| < 1, \tag{7}$$

where $-1 \le \beta \le \alpha \le 1$, $0 \le t \le 1$,

$$f_t(z) = (1-t)z + tf(z), \qquad (f(z) \in \mathcal{N}), \tag{8}$$

and $I(f)$ is defined by (4). For more details about Poisson distribution, see Divesh and Saurabh (2015); Saurabh and Manish (2016); Saurabh and Divesh (2017).

2. Main results

First, we obtain sharp coefficient estimates for $f(z) \in \mathcal{A}(\alpha,\beta,t,m)$.

Theorem 1. *Let $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ be analytic in Δ. Then $f(z) \in \mathcal{A}(\alpha,\beta,t,m)$ if and only if:*

$$\sum_{n=2}^{\infty}\left\{\left[\frac{e^{-m}m^{n-1}}{(n-1)!}-t\right](1-\beta)+t(\alpha-\beta)\right\}a_n \le \alpha-\beta, \tag{9}$$

where $-1 \le \beta \le \alpha \le 1$, $0 \le t \le 1$ and m is the parameter of Poisson distribution.

Proof. Let $f(z) \in \mathcal{A}(\alpha,\beta,t,m)$, thus:

$$\left|\frac{z\big(I(f)\big)'-f_t(z)}{\alpha f_t(z)-\beta z(I(f))'}\right| = \frac{\left|a\left(1-\sum_{n=2}^{\infty}\dfrac{ne^{-m}m^{n-1}}{n!}a_n z^{n-1}\right)-[(1-t)z+tf(z)]\right|}{\left|\alpha[(1-t)z+tf(z)]-\beta z\left(1-\sum_{n=2}^{\infty}\dfrac{ne^{-m}m^{n-1}}{n!}a_n z^{n-1}\right)\right|}$$

$$= \frac{\left|\sum_{n=2}^{\infty}\left(\dfrac{e^{-m}m^{n-1}}{(n-1)!}-t\right)a_n z^n\right|}{\left|(\alpha-\beta)z-\sum_{n=2}^{\infty}\left(\alpha t-\dfrac{e^{-m}m^{n-1}}{(n-1)!}\beta\right)a_n z^n\right|} < 1,$$

for all $z \in \Delta$. Since $\mathrm{Re}\{z\} \le |z|$ for all z, so we have

$$\operatorname{Re}\left\{\frac{\sum_{n=2}^{\infty}\left(\frac{e^{-m}m^{n-1}}{(n-1)!}-t\right)a_n z^n}{(\alpha-\beta)z-\sum_{n=2}^{\infty}\left(\alpha t-\frac{e^{-m}m^{n-1}}{(n-1)!}\right)a_n z^n}\right\}<1.$$

By letting $z\to 1$, through positive values and choose the values of z such that $\frac{z(I(f))'}{f_t(z)}$ is real we obtain

$$\sum_{n=2}^{\infty}\left[\frac{\mathrm{e}^{-\mathrm{m}}m^{n-1}}{(n-1)!}-t\right]a_n\le(\alpha-\beta)-\sum_{n=2}^{\infty}\left[\alpha t-\frac{e^{-m}m^{n-1}}{(n-1)!}\beta\right]a_n.$$

Hence

$$\sum_{n=2}^{\infty}\left\{\left[\frac{\mathrm{e}^{-\mathrm{m}}m^{n-1}}{(n-1)!}-t\right](1-\beta)+t(\alpha-\beta)\right\}a_n\le\alpha-\beta.$$

Conversely, let (9) holds true. Then,

$$\begin{aligned}&\left|z(I(f))'-f_t(z)\right|-\left|\alpha f_t(z)-\beta z(I(f))'\right|\\&=\left|\sum_{n=2}^{\infty}\left[\frac{\mathrm{e}^{-\mathrm{m}}m^{n-1}}{(n-1)!}-t\right]a_n z^n\right|-\left|(\alpha-\beta)-\sum_{n=2}^{\infty}\left[\alpha t-\frac{e^{-m}m^{n-1}}{(n-1)!}\beta\right]a_n z^n\right|.\end{aligned}$$

By letting $|z|=1$, the above expression by (9) reduces to

$$\le\left|\sum_{n=2}^{\infty}\left\{\left[\frac{\mathrm{e}^{-\mathrm{m}}m^{n-1}}{(n-1)!}-t\right](1-\beta)+t(\alpha-\beta)\right\}a_n-(\alpha-\beta)\right|\le 0.$$

So the proof is complete. □

Remark 2. *The function*

$$f(z)=z-\frac{\alpha-\beta}{(me^{-m}-t)(1-\beta)+t(\alpha-\beta)}z^2, \tag{10}$$

shows that the inequality (9) is sharp.

Corollary 3. *If* $\mathcal{A}(\alpha,\beta,t,m)$, *then:*

$$a_n\le\frac{\alpha-\beta}{\left[\frac{e^{-m}m^{n-1}}{(n-1)!}-t\right](1-\beta)+t(\alpha-\beta)},\qquad(n\ge 2). \tag{11}$$

Next, we find the distortion bounds for $I(f)$ where $\mathcal{A}(\alpha,\beta,t,m)$.

Theorem 4. *Let $f(z) \in \mathcal{A}(\alpha,\beta,t,m)$ then for $|z| = r < 1$,*

$$r - \frac{\frac{m}{2}(\alpha-\beta)}{(me^{-m}-t)(1-\beta)+t(\alpha-\beta)} r^2 \le |I(f)| \le r + \frac{\frac{m}{2}(\alpha-\beta)}{(me^{-m}-t)(1-\beta)+t(\alpha-\beta)} r^2, \tag{12}$$

for $-1 \le \beta \le \alpha \le 1$, $0 \le t \le 1$, and m is the parameter of Poisson distribution.

Proof. Let f be in $\mathcal{A}(\alpha,\beta,t,m)$. By (5) and (11), for $|e^{-m}| = 1$, we have

$$\begin{aligned} |I(f)| &= \left| z - \sum_{n=2}^{\infty} \frac{\mathrm{e}^{-\mathrm{m}} m^{n-1}}{(n-1)!} a_n z^n \right| \\ &\le |z| + \sum_{n=2}^{\infty} \left| \frac{\mathrm{e}^{-\mathrm{m}} m^{n-1}}{(n-1)!} \right| a_n |z|^n \\ &\le r + \left(\sum_{n=2}^{\infty} \frac{m^{n-1}}{n!} a_n \right) r^n \\ &\le r + \frac{\frac{m}{2}(\alpha-\beta)}{(me^{-m}-t)(1-\beta)+t(\alpha-\beta)} r^2 \end{aligned}$$

and

$$|I(f)| \ge r - \frac{\frac{m}{2}(\alpha-\beta)}{(me^{-m}-t)(1-\beta)+t(\alpha-\beta)} r^2. \square$$

Now, we obtain the radii of starlikeness, convexity and close-to-convexity for $\mathcal{A}(\alpha,\beta,t,m)$.

Theorem 5. *If $f(z) \in \mathcal{A}(\alpha,\beta,t,m)$, then $f(z)$ is univalent starlike of order δ ($0 \le \delta < 1$) in $|z| < R_1$, where*

$$R_1 = \inf_{n\ge 2} \left\{ \frac{(1-\delta)\left[\left(\frac{e^{-m} m^{n-1}}{(n-1)!} - t\right)(1-\beta) + t(\alpha-\beta)\right]}{(n-\delta)(\alpha-\beta)} \right\}^{\frac{1}{n-1}}, \tag{13}$$

for $-1 \le \beta \le \alpha \le 1$, $0 \le t \le 1$ and and m is the parameter of Poisson distribution.

Proof. It is sufficient to show that $\left|\frac{zf'}{f} - 1\right| \le 1-\delta$ for $|z| < R_1$. But

$$\left|\frac{zf'}{f} - 1\right| = \left| \frac{\sum_{n=2}^{\infty}(n-1)a_n z^n}{z - \sum_{n=2}^{\infty} a_n z^n} \right| \le \frac{\sum_{n=2}^{\infty}(n-1)a_n |z|^n}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}} \le 1-\delta$$

or

$$\sum_{n=2}^{\infty} \left(\frac{n-\delta}{1-\delta}\right) a_n |z|^{n-1} \le 1$$

or

$$|z|^{n-1} \leq \frac{(1-\delta)\left[\left(\frac{e^{-m}m^{n-1}}{(n-1)!} - t\right)(1-\beta) + t(\alpha-\beta)\right]}{(n-\delta)(\alpha-\beta)}.$$

So the proof is complete. □

Since f is convex if and only if zf' is starlike, we can easily get the following result, so the proof is omitted.

Theorem 6. *If $f(z) \in \mathcal{A}(\alpha,\beta,t,m)$, then $f(z)$ is univalent convex of order δ ($0 \leq \delta < 1$) in $|z| < R_2$, where:*

$$R_2 = \inf_{n\geq 2} \left\{\frac{(1-\delta)\left[\left(\frac{e^{-m}m^{n-1}}{(n-1)!} - t\right)(1-\beta) + t(\alpha-\beta)\right]}{n(n-\delta)(\alpha-\beta)}\right\}^{\frac{1}{n-1}}, \tag{14}$$

for $-1 \leq \beta \leq \alpha \leq 1$, $0 \leq t \leq 1$, and m is the parameter of Poisson distribution.

Remark 7. The bounds (13) and (14) for $|z| < 1$ are sharp for each n with the function of the form (10).

Theorem 8. *If $f(z) \in \mathcal{A}(\alpha,\beta,t,m)$, then $f(z)$ is close-to-convex of order δ ($0 \leq \delta < 1$) in $|z| < R_3$, where:*

$$R_3 = \inf_{n\geq 2} \left\{\frac{(1-\delta)\left[\left(\frac{e^{-m}m^{n-1}}{(n-1)!} - t\right)(1-\beta) + t(\alpha-\beta)\right]}{n(\alpha-\beta)}\right\}^{\frac{1}{n-1}}, \tag{15}$$

for $-1 \leq \beta \leq \alpha \leq 1$, $0 \leq t \leq 1$, and m is the parameter of Poisson distribution.

Proof. We must show that $|f'(z) - 1| \leq 1 - \delta$ for $|z| < R_3$, where R_3 is is given by (15). But

$$|f'(z) - 1| = \left|-\sum_{n=2}^{\infty} na_n z^{n-1}\right| \leq \sum_{n=2}^{\infty} na_n |z|^{n-1}.$$

Thus, $|f'(z) - 1| < 1 - \delta$ if $\sum_{n=2}^{\infty} \frac{na_n}{1-\delta}|z|^{n-1} \leq 1$. But, by (11) the above inequality holds true if

$$|z|^{n-1} \leq \frac{(1-\delta)\left[\left(\frac{e^{-m}m^{n-1}}{(n-1)!} - t\right)(1-\beta) + t(\alpha-\beta)\right]}{n(\alpha-\beta)}.$$

This gives the required result. □

3. Neighbourhood property and convexity of $\mathcal{A}(\alpha, \beta, t, m)$

In the last section, we define the neighbourhood of a function $f(z) \in \mathcal{N}$ and prove an interest theorem regarding this concept. Also, we show that the class $\mathcal{A}(\alpha, \beta, t, m)$ is a convex set.

Definition 9.
(1) The (n, λ) –neighbourhood of $f(z) \in \mathcal{N}$ is defined by

$$\mathcal{N}_{n,\lambda}(f) = \left\{ g(z) \in \mathcal{N} : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad and \quad \sum_{n=2}^{\infty} n|a_n - b_n| \leq \lambda \right\}. \tag{16}$$

(2) The function $g(z)$ defined by $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ is said to be a member of the class $\mathcal{A}^{\theta}(\alpha, \beta, t, m)$ if there exists a function $f \in \mathcal{A}(\alpha, \beta, t, m)$ such that:

$$\left|\frac{g(z)}{f(z)} - 1\right| \leq 1 - \theta, \qquad (z \in \Delta, \quad 0 \leq \theta < 1). \tag{17}$$

Theorem 10. *If $f(z) \in \mathcal{A}(\alpha, \beta, t, m)$ and*

$$\theta = 1 - \frac{\lambda[(me^{-m} - t)(1-\beta) + t(\alpha - \beta)]}{2[(me^{-m} - t)(1-\beta) + (\alpha - \beta)(t-1)]}, \tag{18}$$

then $\mathcal{A}^{\theta}(\alpha, \beta, t, m) \supset \mathcal{N}_{n,\lambda}(f)$.

Proof. Let $g \in \mathcal{N}_{n,\lambda}(f)$, then we have from (16) that $\sum_{n=2}^{\infty} n|a_n - b| \leq \lambda$, which implies the coefficient inequality $\sum_{n=2}^{\infty} |a_n - b| \leq \frac{\lambda}{2}$. Also since $f \in \mathcal{A}(\alpha, \beta, t, m)$, we have from (11),

$$\sum_{n=2}^{\infty} a_n \leq \frac{\alpha - \beta}{(me^{-m} - t)(1-\beta) + t(\alpha - \beta)},$$

so that

$$\begin{aligned}\left|\frac{g(z)}{f(z)} - 1\right| < \left|\frac{\sum_{n=2}^{\infty}|a_n - b_n|z^n}{z - \sum_{n=2}^{\infty} a_n z^n}\right| &< \frac{\sum_{n=2}^{\infty}|a_n - b_n|}{1 - \sum_{n=2}^{\infty} a_n} \\ &\leq \frac{\lambda}{2} \frac{(me^{-m} - t)(1-\beta) + t(\alpha - \beta)}{(me^{-m} - t)(1-\beta) + t(\alpha - \beta) - (\alpha - \beta)} \\ &= 1 - \theta.\end{aligned}$$

Thus by definition, $g \in \mathcal{A}^{\theta}(\alpha, \beta, t, m)$, for θ given by (18) □
.
Theorem 11. *$\mathcal{A}(\alpha, \beta, t, m)$ is a convex set.*

Proof. We must show that if the functions $f_k(z)$, $k = 1,2,\dots,m$, be in the class $\mathcal{A}(\alpha,\beta,t,m)$, then $h(z) = \sum_{k=1}^{m}\sigma_k f_k(z)$ is also in $\mathcal{A}(\alpha,\beta,t,m)$, where $\sigma_k \geq 0$ and $\sum_{k=1}^{m}\sigma_k = 1$. By the definition of $h(z)$, we have

$$h(z) = \sum_{k=1}^{m}\sigma_k\left(z - \sum_{n=2}^{\infty}a_{n,k}z^n\right)$$
$$= z - \sum_{n=2}^{\infty}\left(\sum_{k=1}^{m}\sigma_k a_{n,k}\right)z^n = z - \sum_{n=2}^{\infty}A_n z^n,$$

where $A_n = \sum_{k=1}^{m}\sigma_k a_{n,k}$. But from Theorem 1, we have

$$\sum_{n=2}^{\infty}\left\{\left[\frac{e^{-m}m^{n-1}}{(n-1)!} - t\right](1-\beta) + t(\alpha-\beta)\right\}A_n$$
$$= \sum_{k=1}^{m}\sigma_k\sum_{n=2}^{\infty}\left\{\left(\left[\frac{e^{-m}m^{n-1}}{(n-1)!} - t\right](1-\beta) + t(\alpha-\beta)\right)a_{n,k}\right\}$$
$$\leq \sum_{k=1}^{m}\sigma_k\,(\alpha-\beta)$$
$$= (\alpha-\beta)\sum_{k=1}^{m}\sigma_k = (\alpha-\beta),$$

which completes the proof. □

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