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ESTIMATE ON THE SECOND HANKEL FUNCTIONAL FOR FUNCTIONS WHOSE DERIVATIVE HAS A POSITIVE REAL PART

(Anggaran Fungsian Hankel Kedua bagi Fungsi dengan Terbitan Bahagian Nyata Positif)

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ABSTRACT

Denote S to be the class of functions which are analytic, normalised and univalent in the open unit disc $D=\{z:|z|<1\}$. Also denote R to represent the subclass of S whose derivative has a positive real part. Next, write $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ where a_n is a complex constant and denote the qth Hankel determinant for f as $H_q(n)$ for $q\geq 1, n\geq 1$. Our intention is to seek sharp upper bounds for $H_2(n)$, however, we begin by first looking at $H_2(2)$. In this paper, we give the upper bound for the second Hankel determinant for this particular class of functions.

Keywords: Hankel determinant; upper bound

ABSTRAK

Lambangkan S sebagai kelas fungsi yang terdiri daripada fungsi-fungsi analisis, ternormal dan univalen di dalam cakera unit terbuka $D=\{z:|z|<1\}$. Lambangkan juga R sebagai subkelas bagi S yang terdiri daripada fungsi-fungsi dengan terbitannya mempunyai bahagian nyata yang positif. Seterusnya dapat diungkapkan bahawa $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ dengan a_n adalah pemalar kompleks dan lambangkan penentu Hankel ke-q bagi f sebagai $H_q(n)$ untuk $q\geq 1, n\geq 1$. Makalah ini bertujuan untuk mendapatkan batas atas terbaik bagi $H_2(n)$, namun, ianya dimulakan dengan melihat $H_2(2)$. Dalam makalah ini, batas atas bagi penentu Hankel kedua diberikan.

Kata kunci: penentu Hankel; batasan atas

1. Introduction

Let S denote the class of normalised analytic univalent functions f of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1}$$

where $z\in D=\{z:|z|<1\}$. In Noonan and Thomas (1976), the qth Hankel determinant for $q\geq 1$ and $n\geq 1$ is stated by Noonan and Thomas as

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & \cdots & & \vdots \\ \vdots & & & \\ a_{n+q-1} & \cdots & & a_{n+2q-2} \end{vmatrix}.$$

This determinant has also been considered by several authors. For example, Noor (1983) determined the rate of growth of $H_q(n)$ as $n\to\infty$ for functions f given by Eq. (1) with bounded boundary. Ehrenborg (2000) studied the Hankel determinant of exponential polynomials and in (Layman 2001), the Hankel transform of an integer sequence is defined and some of its properties discussed by Layman.

Easily, one can observe that the Fekete and Szegö functional is $H_2(1)$. Fekete and Szegö then further generalised the estimate $|a_3 - \mu a_2^2|$ where μ is real and $f\in S$. For our discussion in this paper, we consider the Hankel determinant in the case of $q=2$ and $n=2$,

$$H_2(2)=\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

We seek upper bound for the functional $|a_2a_4-\mu a_3^2|$ where μ is real for functions f belongs to the class R. The class R is defined as follows:

Definition 1.1 Let f be given by Eq. (1). Then $f\in R$ if and only if

$$\operatorname{Re}\{f'(z)\}>0, \quad (z\in D). \tag{2}$$

We first state some preliminary lemmas, required for proving our result.

2. Preliminary results

Let P be the family of all functions p analytic in D for which $\operatorname{Re} p(z)>0$ and

$$p(z)=1+c_1z+c_2z^2+... \tag{3}$$

for $z\in D$.

Lemma 2.1 (Pommerenke (1975)) *If $p\in P$ then $|c_k|\le 2$ for each k.*

Lemma 2.2 (Grenander & Szegö (1958)) *The power series for p given in Eq.* (3) *converges in D to a function in P if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \dots & c_n \\ c_{-1} & 2 & c_1 & \dots & c_{n-1} \\ M & & & & \\ c_{-n} & c_{-n+1} & c_{-n+2} & \dots & 2 \end{vmatrix}, \quad n = 1, 2, 3, \dots \tag{4}$$

and $c_{-k} = \bar{c}_k$, *are all non negative. They are strictly positive except for* $p(z) = \sum_{k=1}^{m} \rho_k p_0\left(e^{it_k z}\right)$, $\rho_k > 0$, t_k *real and* $t_k \neq t_j$ *for* $k \neq j$; *in this case* $D_n > 0$ *for* $n < m-1$ *and* $D_n = 0$ *for* $n \geq m$.

This necessary and sufficient condition is due to Carathéodory and Toeplitz and can be found in (Grenander & Szegö (1958)).

3. Main results

Theorem 3.1 *Let* $f \in R$. *Then*

$$\left|a_2 a_4 - \mu a_3^2\right| \leq \begin{cases} \dfrac{(27-16\mu)^2}{144(9-8\mu)} - \dfrac{4\mu}{9}, & \text{if} \quad \mu \leq 0, \\ \dfrac{(27-32\mu)^2}{144(9-8\mu)} + \dfrac{4\mu}{9}, & \text{if} \quad 0 \leq \mu \leq \dfrac{27}{32}, \\ \dfrac{4\mu}{9}, & \text{if} \quad \dfrac{27}{32} \leq \mu \leq \dfrac{27}{16}, \\ \dfrac{(16\mu-27)^2}{144(8\mu-9)} + \dfrac{4\mu}{9}, & \text{if} \quad \mu \geq \dfrac{27}{16}. \end{cases}$$

Proof: Since $f \in R$, it follows from Eq. (2) that

$$f'(z) = p(z) \tag{5}$$

for some $z \in D$. Equating coefficients in Eq. (5), yields

$$\left.\begin{aligned} 2a_2 &= c_1 \\ 3a_3 &= c_2 \\ 4a_4 &= c_3 \end{aligned}\right\}. \tag{6}$$

From Eq. (6),

$$\left|a_2a_4-\mu a_3^2\right|=\left|\frac{c_1c_3}{8}-\frac{\mu c_2^2}{9}\right|. \tag{7}$$

Lemma 2.2 can then be used to obtain the proper bound on Eq. (7). We may assume without restriction that $c_1 \geq 0$. Rewriting Eq. (4) for the cases $n=2$ and $n=3$, result in

$$D_2=\begin{vmatrix} 2 & c_1 & c_2 \\ c_1 & 2 & c_1 \\ \overline{c}_2 & c_1 & 2 \end{vmatrix}=8+2\ \mathrm{Re}\left\{c_1^2c_2\right\}-2\left|c_2\right|^2-4c_1^2\geq 0,$$

which is equivalent to

$$2c_2=c_1^2+x\left(4-c_1^2\right) \tag{8}$$

for some $x, |x|\leq 1$.

Further, $D_3 \geq 0$ is equivalent to

$$\left|\left(4c_3-4c_1c_2+c_1^3\right)\left(4-c_1^2\right)+c_1\left(2c_2-c_1^2\right)^2\right|\leq 2\left(4-c_1^2\right)^2-2\left|2c_2-c_1^2\right|^2;$$

and this, with Eq. (8), provides the relation

$$4c_3=c_1^3+2\left(4-c_1^2\right)c_1x-c_1\left(4-c_1^2\right)x^2+2\left(4-c_1^2\right)\left(1-\left|x\right|^2\right)z, \tag{9}$$

for some value of $z, |z|\leq 1$.

Suppose now that $c_1=c$ and $0\leq c\leq 2$. Using Eq. (8) along with Eq. (9), we obtain

$$\left|\frac{c_1c_3}{8}-\frac{\mu c_2^2}{9}\right|$$

$$=\left|\frac{(9-8\mu)c^4}{288}+\frac{(9-8\mu)\left(4-c^2\right)c^2x}{144}+\frac{\left(4-c^2\right)\left(1-|x|^2\right)cz}{16}-\frac{\left(4-c^2\right)x^2\left(c^2(9-8\mu)+32\mu\right)}{288}\right|$$

Triangle inequality gives

$$\begin{aligned}&\left|\frac{c_1c_3}{8}-\frac{\mu c_2^2}{9}\right|\\ &\leq \frac{|9-8\mu|\,c^4}{288}+\frac{c\left(4-c^2\right)}{16}+\frac{|9-8\mu|\,c^2\left(4-c^2\right)\rho}{144}+\frac{\left(4-c^2\right)\left(|9-8\mu|c^2+32|\mu|-18c\right)\rho^2}{288}\\ &=F(\rho)\end{aligned} \tag{10}$$

with $\rho=|x|\leq 1$. This gives rise to

$$F'(\rho)=\begin{cases}\dfrac{(9-8\mu)\left(4-c^2\right)c^2}{144}+\dfrac{\left(4-c^2\right)\left\{(9-8\mu)c^2-32\mu-18c\right\}\rho}{144}, & if \quad \mu\leq 0,\\[2ex] \dfrac{(9-8\mu)\left(4-c^2\right)c^2}{144}+\dfrac{\left(4-c^2\right)\left\{(9-8\mu)c^2+32\mu-18c\right\}\rho}{144}, & if \quad 0\leq\mu\leq\dfrac{9}{8},\\[2ex] \dfrac{(8\mu-9)\left(4-c^2\right)c^2}{144}+\dfrac{\left(4-c^2\right)\left\{(8\mu-9)c^2+32\mu-18c\right\}\rho}{144}, & if \quad \mu\geq\dfrac{9}{8}.\end{cases}$$

and again for all the cases above, $F'(\rho)>0$ for $\rho>0$; implying that F is an increasing function with $Max_{\rho\leq 1}F(\rho)=F(1)$.

Now let

$$G(c)=F(1)=\frac{|9-8\mu|\,c^4}{288}+\frac{c\left(4-c^2\right)}{16}+\frac{|9-8\mu|\,c^2\left(4-c^2\right)}{144}+\frac{\left(4-c^2\right)\left\{|9-8\mu|\,c^2+32|\mu|-18c\right\}}{288}. \tag{11}$$

(i) First, let us consider the case $\mu\leq 0$.

Eq. (11) gives

$$G'(c)=\frac{c}{36}\left\{-(9-8\mu)c^2+27-16\mu\right\}.$$

Elementary calculation reveals that G attains its maximum value at $c=\sqrt{\dfrac{27-16\mu}{9-8\mu}}$.

The upper bound for Eq. (10) corresponds to $\rho=1$ and $c=\sqrt{\dfrac{27-16\mu}{9-8\mu}}$, in which case

$$\left|\frac{c_1c_3}{8}-\frac{\mu c_2^2}{9}\right| \le \frac{(27-16\mu)^2}{144(9-8\mu)}-\frac{4\mu}{9}.$$

(ii) Secondly, we consider the case $0 \le \mu \le \frac{27}{32}$.

This gives

$$G'(c)=\frac{c}{36}\left\{-(9-8\mu)c^2+27-32\mu\right\}$$

where G attains its maximum value at $c=\sqrt{\frac{27-32\mu}{9-8\mu}}$. Hence, we obtain

$$\left|\frac{c_1c_3}{8}-\frac{\mu c_2^2}{9}\right| \le \frac{(27-32\mu)^2}{144(9-8\mu)}+\frac{4\mu}{9}.$$

(iii) To prove the third result, we consider two cases.

First, assume that $\frac{27}{32} \le \mu \le \frac{9}{8}$.

In this case, G attains its maximum value at $c=\sqrt{\frac{32\mu-27}{9-8\mu}}$.

Next, consider the case $\frac{9}{8} \le \mu \le \frac{27}{16}$. For this case, Eq. (11) gives rise to

$$G'(c)=\frac{c}{36}\left\{-(8\mu-9)\,c^2+16\mu-27\right\},$$

where G attains its maximum value at $c=\sqrt{\frac{27-16\mu}{8\mu-9}}$.

In both cases, the upper bound is attained as

$$\left|\frac{c_1c_3}{8}-\frac{\mu c_2^2}{9}\right| \le \frac{4\mu}{9}.$$

(iv) Finally, consider $\mu \ge \frac{27}{16}$.

Here, G attains its maximum value at $c=\sqrt{\frac{16\mu-27}{8\mu-9}}$. Hence,

$$\left|\frac{c_1c_3}{8}-\frac{\mu c_2^2}{9}\right|\leq\frac{\left(16\mu-27\right)^2}{144\left(8\mu-9\right)}+\frac{4\mu}{9}.$$

This completes the proof of theorem.□

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