Journal of Quality Measurement and Analysis *JQMA 4(1) 2008, 205-218*
Jurnal Pengukuran Kualiti dan Analisis

VEHICLE ROUTING PROBLEM: MODELS AND SOLUTIONS

(*Masalah Perjalanan Kenderaan: Model dan Penyelesaian*)

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ABSTRACT

The Vehicle Routing Problem (VRP) is a well known problem in operational research where customers of known demands are supplied by one or several depots. The objective is to find a set of delivery routes satisfying some requirements or constraints and giving minimal total cost. The VRP has drawn enormous interests from many researchers during the last decades because of its vital role in planning of distribution systems and logistics in many sectors such as garbage collection, mail delivery, snow ploughing and task sequencing. The VRP is divided into many types. The important problems are VRP with Time Windows, VRP with Pick-Up and Delivery and Capacitated VRP. Recently many exact methods have been used to solve the VRP such as exact algorithms based on linear programming techniques and guided local search. Besides that, heuristic techniques have received wide interests in researchers' effort to solve large scale VRPs. Among the recently applied heuristic techniques are genetic algorithm, evolution strategies and neural networks.

Keywords: Vehicle routing problem; VRP with time windows; VRP with pick-up and delivery; capacitated VRP; exact algorithms; heuristic methods

ABSTRAK

Masalah perjalanan kenderaan (MPK) merupakan suatu masalah yang terkenal dalam penyelidikan operasi yang para pelanggan dengan permintaan tertentu disalurkan keperluannya dari satu atau beberapa depot. Objektifnya adalah untuk mencari suatu set perjalanan yang memenuhi keperluan atau kekangan tertentu dengan jumlah kos yang minimum. MPK telah menarik perhatian ramai penyelidik dalam beberapa dekad yang lalu kerana peranan pentingannya dalam perancangan sistem pengedaran dan logistic dalam banyak sector seperti pengutipan sampah-sarap, penyerahan mel, penenggalaan salji dan penjadualan kerja. MPK terbahagi kepada banyak jenis. Antara yang penting adalah MPK dengan Tetingkap Masa, MPK dengan Pengambilan dan Penghantaran, dan MPK dengan Kapasiti. Kebelakangan ini banyak kaedah tepat telah diguna untuk menyelesaikan MPK seperti kaedah berasaskan teknik pengaturcaraan linear dan carian setempat berpandu. Di samping itu kaedah heuristik pula telah menambat minat ramai penyelidik untuk menyelesaikan masalah MPK berskala besar. Antara kaedah heuristik yang digunakan kebelakangan ini adalah al-Khwarizmi genetik, strategi evolusi dan rangkaian neural.

Kata kunci: Masalah perjalanan kenderaan; MPK dengan tetingkap masa; MPK dengan pengambilan dan penghantaran; MPK dengan kapasiti; kaedah tepat; kaedah heuristik

1. Introduction

The Vehicle Routing Problem (VRP) can be defined as a problem of finding the optimal routes of delivery or collection from one or several *depots* to a number of *cities* or *customers,* while satisfying some constraints. Collection of household waste, gasoline delivery trucks, goods distribution, snowplough and mail delivery are the most used applications of the VRP. The VRP plays a vital role in distribution and logistics. Huge research efforts have been devoted to studying the VRP since 1959 where Dantzig and Ramser have described the problem as a generalised problem of Travelling Salesman Problem (TPS). Thousands of papers have been written on several VRP variants such as Vehicle Routing Problem with Time Windows (VRPTW), Vehicle Routing Problem with Pick-Up and Delivery (VRPPD) and Capacitated Vehicle Routing Problem (CVRP).

The VRP is an important combinatorial optimisation problem. Toth and Vigo have reported in 2002 (Maffioli 2003) that the use of computerised methods in distribution processes often results in savings ranging from 5% to 20% in transportation costs. Barker (2002) describe several case studies where the application of VRP algorithms has led to substantial cost savings.

In this paper the definition of VRP is presented. The organisation of the paper is as follows: section 2 presents the Classical VRP, section 3 gives the definition of the CVRP, and section 4 is devoted to the presentation of the VRPTW, while section 5 is for the VRPPD presentation. Section 6 presents the models of the VRP and its variants. Algorithms used to solve the VRP and its variants are presented in section 7, and finally the last section gives the concluding remarks.

2. Classical VRP

In classical VRP, the customers are known in advance. Moreover, the driving time between the customers and the service times at each customer are used to be known (Madsen *et al.* 1995). The classical VRP can be defined as follow (Laporte 1992):

Let $G = (V, A)$ be a graph where $V = \{1 \ldots n\}$ is a set of vertices representing *cities* with the *depot* located at vertex 1, and A is the set of arcs. With every arc (i, j) $i \neq j$ is associated a non-negative distance matrix $C = (c_{ij})$. In some contexts, c_{ij} can be interpreted as a *travel cost* or as a *travel time.* When C is symmetrical, it is often convenient to replace A by a set E of undirected edges. In addition, assume there are m available vehicles based at the depot, where $m_L < m < m_U$. When $m_L = m_U$, m is said to be *fixed.* When $m_L = 1$ and $m_U = n - 1$, m is said to be *free.* When m is not fixed, it often makes sense to associate a fixed cost f on the use of a vehicle. The VRP consists of designing a set of least-cost vehicle routes in such a way that:

(i) each city in $V\backslash\{1\}$ is visited exactly once by exactly one vehicle;

(ii) all vehicle routes start and end at the depot;

(iii) some side constraints are satisfied.

The VRP has been reported as *NP*-Hard which pushed researchers to use heuristics, see Chiang and Russell (1996), Braysy *et al.* (2004), Nagy and Salhi (2007) and Choi and Tcha (2007). However, exact algorithms were also applied for VRP. There have been many contributions to the subject, including various extensions to the basic problem described above. Laporte (1992) gives a survey, and an extensive bibliography has been compiled by Laporte and Osman (1995).

Taillard (1993) and Rochat and Taillard (1995) have applied Tabu Search (TS) to many VRP variants, where the best known results to benchmark VRPs were obtained. Various authors have reported similar results, obtained using TS, or Simulated Annealing (SA) (Baker & Ayechew 2003). However, it has been reported by Renaud *et al.* (1996) that such heuristics require considerable computing times and several parameter settings.

Baker & Ayechew (2003) have reported a several applications of Genetic Algorithms (GAs) to VRPs since GAs have seen extensive use, most recently, amongst modern metaheuristics. Applications of GAs have also been reported for a variant of VRP (Potvin *et* *al.* 1996b), for a multi-depot routing problem (Salhi *et al.* 1998), and a school bus routing problem (Thangiah & Nygard 1992).

Potvin *et al.* (1996b) have used a hybrid approach to VRP using Neural Networks (NNs) and GAs. Baker & Ayechew (2003) reported that the GAs do not appear to have made a great impact so far on the basic VRP. They add that, a hybrid heuristic which incorporates neighbourhood search into a basic GA has given, for benchmark problems, some of the well-known results obtained using TS and SA.

Ant Colony (AC) optimisation is another recent approach to difficult combinatorial problems with a number of successful applications reported, including the VRP. With a 2-optimal heuristic incorporated to improve individual routes produced by artificial ants, this approach also has given results which are only slightly inferior to those from TS (Bullnheimer *et al.* 1999).

3. Capacitated Vehicle Routing Problem (CVRP)

The Capacitated Vehicle Routing Problem (CVRP) can be described as follows:

Let $G = (V', E)$ an undirected graph is given where $V' = \{0, 1, . . . , n\}$ is the set of $n + 1$ vertices and E is the set of edges. Vertex 0 represents the depot and the vertex set $V = V' \backslash \{0\}$ corresponds to n customers. A nonnegative cost d_{ij} is associated with each edge $\{i, j\} \in E$. the q_i units are supplied of from depot 0 (we assume $q_0 = 0$). A set of m identical vehicles of capacity Q is stationed at depot 0 and must be used to supply the customers. A *route* is defined as a least cost simple cycle of graph G passing through depot 0 and such that the total demand of the vertices visited does not exceed the vehicle capacity.

The practical importance of the CVRP provides the motivation for the effort involved in the development of heuristic algorithms (Baldacci *et al.* 2007). Survey covering exact algorithms was given by Laporte (1992). The chapters of Toth and Vigo (Maffioli 2003) have surveyed the most effective exact methods proposed in the literature up to 2002. A recent survey of the CVRP, covering both exact and heuristic algorithms, can be found in the chapter of Cordeau *et al.* (2001) in the book edited by Barnhart and Laporte (Baldacci *et al.* 2007). The most promising exact algorithms for the symmetric CVRP which have been published since then are due to Baldacci *et al.* (2004), Lysgaard *et al.* (2004) and Fukasawa *et al.* (2006).

Baldacci *et al.* (2004) have described a branch-and-cut algorithm that is based on a two commodity network flow formulation of the CVRP. Lysgaard *et al.* (2004) have proposed a branch-and-cut algorithm that is an enhancement of the method proposed by Augerat *et al.* (1995). They used a variety of valid inequalities, including capacity, framed capacity, comb, partial multistar, hypotour and classical Gomory mixed integer cuts. Baldacci *et al.* (2007) have reported that the algorithms of Augerat *et al.* (1995), Baldacci *et al.* (2004) and of Lysgaard *et al.* (2004) were able to solve a 135-customer instance which is the largest non-trivial CVRP instance solved to date. They added, the best exact method currently available for the CVRP has been proposed by Fukasawa *et al.* (2006). This method combines the branch-and-cut of Lysgaard *et al.* (2004) with the Set Partitioning (SP) approach. Besides the well-known capacity constraints, these authors also use framed capacity, strengthened comb, multistar, partialmultistar, generalised multistar and hypotour inequalities, all presented in Lysgaard *et al.* (2004). The columns of the SP correspond to the set of q-routes that contains the set of valid CVRP routes. Since the resulting formulation has an exponential number of both columns and rows, this leads to column and cut generation for computing the lower bound and to a branch-and-cut-and-price algorithm for solving the CVRP. The computational results indicate that the new bounding procedure obtains lower bounds that are superior to those given by previous methods. However, this procedure is time consuming as the LP-relaxation of the master problem is usually highly degenerate and degeneracy implies alternative optimal dual solutions. Consequently, the generation of new columns and their associated variables may not change the value of the objective function of the master problem, the master problem may become large, and the overall method may become slow computationally. Moreover, in some CVRP instances, the increase in the lower bound with respect to the one achieved by the pure branch-and-cut method is very small and is not worth the computing time required by the additional SP approach. The exact algorithm presented by Fukasawa *et al.* (2006) decides at the root node, according to the best balance between running time and bound quality, either to use the branch-and-cut method of Lysgaard *et al.* (2004) or the new proposed branch-and-cut-and-price strategy. The computational results reported by Fukasawa *et al.* (2006) have shown that this algorithm is very consistent on solving instances from the literature with up to 135 customers.

4. Vehicle Routing Problem with Time Windows (VRPTW)

The VRPTW is a generalisation of the well-known VRP. It can be reviewed as a combined vehicle routing and scheduling problem which often arises in many real-world applications. It is to optimise the use of a fleet of vehicles that must make a number of stops to serve a set of customers, and to specify which customers should be served by each vehicle and in what order to minimise the cost, subject to vehicle capacity and service time restrictions (Ellabib *et al.* 2002). The problem involves assignment of vehicles to trips such that the assignment cost and the corresponding routing cost are minimal.

The VRPTW can be defined as follows: Let $G = (V, E)$ be a connected digraph consisting of a set of $n + 1$ nodes, each of which can be reached only within a specified time interval or time window, and a set E of arcs with non-negative weights representing travel distances and associated travel times. Let one of the nodes be designated as the depot. Each node i, except the depot, requests a service of size q_i.

The VRPTW has been the subject of intensive research efforts for both heuristic and exact optimisation approaches. Early surveys of solution techniques for the VRPTW can be found in Golden and Assad (1986), Desrochers *et al.* (1988), and Solomon and Desrosiers (Chiang & Russell 1996). The main focus in Desrosiers *et al.* (1995) and Cordeau *et al.* (2001) were exact solution techniques. Further details on these exact methods can be found in Larsen (1999) and Cook and Rich (1999). Because of the high complexity level of the VRPTW and its wide applicability to real-life situations, solution techniques capable of producing high-quality solutions in limited time, i.e. heuristics, are of prime importance.

Fleischmann (1990) and Taillard *et al.* (1996) have used heuristic for VRP without time windows. In Taillard *et al.* (1996), different solutions to the classical vehicle routing problem have been generated using a TS heuristic. The routes obtained are then combined to produce workdays for the vehicles by solving a bin packing problem, an idea previously introduced in Fleischmann (1990). A recent work in Compbell and Savelsbergh (2004) has reported about insertion heuristics that can efficiently handle different types of constraints including time windows and multiple uses of vehicles. Compbell and Savelsbergh (2005) have introduced the home delivery problem, which is more closely related to real-world applications. The probability of occurrence and the revenue have been associated with each potential customer. When a new request occurs, a decision to accept or reject must be taken in real-time, and a time window for service is determined. Although vehicle routes are generated and used to decide about the acceptance or rejection of a particular request, the ''real'' routes are executed.

Current VRPTW heuristics can be categorised as follows: (i) construction heuristics, (ii) improvement heuristics and (iii) metaheuristics. Construction heuristics are sequential or parallel algorithms aiming at designing initial solutions to routing problems that can be improved upon by improvement heuristics or metaheuristics. Sequential algorithms build a route for each vehicle, one after another, using decision functions for the selection of the customer to be inserted in the route and the insertion position within the route. Parallel algorithms build the routes for all vehicles in parallel, using a pre-computed estimate of the number of routes. Different variants of construction heuristics for the VRPTW can be found in Solomon (1987), Potvin and Rousseau (1993), Bramel and Simchi-Levi (1996), and Dullaert and Braysy (2003).

5. Vehicle Routing Problem with Pick-Up and Delivery (VRPPD)

The problems that need to be solved in real-life situations are usually much more complicated than the classical VRP. One complication that arises in practice is that goods not only need to be brought from the depot to the customers, but also must be picked up at a number of customers and brought back to the depot. This problem is well known as VRP with Pick-Up and Delivery (VRPPD). In the literature, the VRPPD is also called VRP with Backhauls (VRPB) (Ropke & Pisinger 2006; Bianchessi & Righini 2007). The problem can be divided into two independent CVRPs (Ropke & Pisinger 2006); one for the delivery (linehaul) customers and one for the pickup (backhaul) customers, such that some vehicles would be designated to linehaul customers and others to backhaul customers.

6. Mathematical Formulation of the VRP

The formulation of the VRP can be presented as follow (Laporte 1992):

Let x_{ij} be an integer variable which may take value $\{0, 1\}$, $\forall\{i, j\} \in E\backslash\{\{0, j\} : j \in V\}$ and value $\{0, 1, 2\}$, $\forall\{0, j\} \in E, j \in V$. Note that $x_{0j} = 2$ when a route including the single customer j is selected in the solution.

The VRP can be formulated as the following integer program:

$$\textit{Minimise} \sum_{i \neq j} d_{ij}\, x_{ij} \tag{1}$$

Subject to :

$$\sum_{j} x_{ij} = 1, \forall i \in V, \tag{2}$$

$$\sum_{i} x_{ij} = 1, \forall j \in V, \tag{3}$$

$$\sum_{i} x_{ij} \geq |S| - v(S), \{S : S \subseteq V\backslash\{1\}, |S| \geq 2\}, \tag{4}$$

$$x_{ij} \in \{0, 1\}, \forall\{i, j\} \in E ; i \neq j \tag{5}$$

In this formulation, (1), (2), (3) and (5) define a modified assignment problem (i.e. assignments on the main diagonal are prohibited). Constraints (4) are sub-tour elimination constraints: $v(S)$ is an appropriate lower bound on the number of vehicles required to visit all vertices of S in the optimal solution.

6.1. *Mathematical Formulation of the VRPTW*

In the literature, the VRPTW can be stated mathematically as (Azi *et al.* 2007):

We have a single vehicle of capacity Q delivering perishable goods from a depot to a set of customer nodes $N = \{1, 2, \ldots, n\}$ in a complete directed graph with arc set A. A distance d_{ij} and a travel time t_{ij} are associated with every arc $(i, j) \in A$. Each customer $i \in N$ is characterised by a demand q_i, a service or dwell time s_i and a time window $[a_i,b_i]$, where a_i is the earliest time to begin service and bi the latest time. Accordingly, the vehicle must wait if it arrives at customer i before time a_i.

$$Minimise \sum_{e\in K}\sum_{(i,j)\in A} d_{ij}x_{ij}^r \tag{6}$$

subject to:

$$\sum_{j\in N^+} x_{ij}^r = y_i^r, \quad i \in N, j \in K, \tag{7}$$

$$\sum_{r\in K} y_i^r = 1, \quad i \in N, \tag{8}$$

$$\sum_{i\in N^+} x_{ih}^r - \sum_{j\in N^+} x_{hj}^r = 0, \quad h \in N, r \in K, \tag{9}$$

$$\sum_{i\in N^+} x_{0i}^r = 1, \; r \in K, \tag{10}$$

$$\sum_{i\in N^+} x_{i(n+1)}^r = 1, \; r \in K, \tag{11}$$

$$\sum_{i\in N} q_i y_i^r \le Q, \; r \in K, \tag{12}$$

$$t_i^r + s_i + t_{ij} - M(1 - x_{ij}^r) \le t_j^r, \quad (i,j) \in A^+, \mathrm{r} \in K, \tag{13}$$

$$a_i y_i^r \le t_i^r \le b_i y_i^r, \; i \in N, r \in K, \tag{14}$$

$$t_0^1 \ge \sigma^1, \tag{15}$$

$$t_{n+1}^r + \sigma^{r+1} \le t_0^{r+1}, \quad r = 1, \ldots, k-1, \tag{16}$$

$$\sigma^r = \beta \sum s_i y_i^r, \; r \in K, \tag{17}$$

$$\mathrm{t_i^r} \le \mathrm{t_0^r} + t_{\max}, \; i \in N, r \in K, \tag{18}$$

$$x_{ij}^r \text{ binary}, (i,j) \in A^+, r \in K, \tag{19}$$

$$y_i^r \text{ binary}, i \in N,, r \in K, \tag{20}$$

where

x_{ij}^r is 1 if arc $(i, j) \in A^+$ is in route r, 0 otherwise; note that $x_{0,n+1}^r$ is 1 if route r is empty;

y_i^r is 1 if customer i is in route r, 0 otherwise;

$\mathrm{t_i^r}$ is the time of beginning of service at customer i in route r;

$\mathrm{t_0^r}$ is the start time of route r;

$\mathrm{t_{n+1}^r}$ is the end time of route r.

In this formulation, Equation (8) states that every customer should be visited exactly once. Equations (9), (10) and (11) are flow conservation constraints that describe the vehicle path. Equation (12) states that the total demand on a route should not exceed the vehicle capacity. Equations (13) to (16) ensure feasibility of the time schedule while Equation (17) defines the vehicle setup time as the sum of service times of all customers in a route, multiplied by parameter β. Finally, Equation (18) corresponds to the deadline constraint for serving a customer.

6.2. *Mathematical Formulation of the VRPPD*

The VRPPD can be formulated using the Generalised Assignment Procedure (GAP), which addresses the capacity constraints (Fisher & Jaikumar 1981). GAP is used to find the minimum cost assignment of *v* vehicles to *n* clusters such that each vehicle is assigned to exactly one cluster, subject to its available capacity. While applying GAP, each cluster is treated as a node. Since the number of clusters depends upon the number of vehicles, the problem will be relatively small and can be solved by the extended GAP in reasonable computing time. The extended GAP is similar to the heuristic of Fisher and Jaikumar (1981) for VRP. The problem can be formulated as follows:

$v = \{1, 2, \ldots, V\}$ a set of vehicles
$n = \{1, 2, \ldots, N\}$ be a set of clusters
C_n is the cost of assigning a vehicle to cluster n; $\forall\, n \in N$
u_n the maximum load that will have to be carried in cluster n
t_v remaining capacity of each partially loaded vehicle v

$$X_{vn} = \begin{cases} 1 \text{ if vehicle } v \text{ assigned to cluster } n \\ 0 \text{ otherwise.} \end{cases}$$

The mathematical formulation of the GAP is:

$$\textit{Minimise} \sum_{v \in V} \sum_{n \in N} C_n X_{vn} \tag{21}$$

Subject to

$$\sum_{v \in V} X_{vn} = 1 \quad \text{for } n = 1, \ldots, N, \tag{22}$$

$$\sum_{n \in N} u_n X_{vn} \le t_v \quad \text{for } v = 1, \ldots, V, \tag{23}$$

$$X_{vn} \in \{0,1\} \quad \text{for } n = 1, \ldots, N \quad \text{and} \quad v = 1, \ldots, V, \tag{24}$$

The constraint (22) ensures that each cluster is assigned to exactly one vehicle while the constraint (23) ensures that the maximum load in a cluster does not exceed the capacity of the vehicle assigned to that cluster.

6.3. *Mathematical Formulation of the CVRP*

The CVRP formulation can be presented as follow (Fukasawa *et al.* 2004):

Let $H = (N,A)$, d, q and Q define a CVRP instance having vertex 0 as the depot and the remaining vertices in N as clients.

$$\textit{Minimise} \quad \sum_{e=(u,v)\in A} d(e)x_e \tag{25}$$

Subject to:

$$\sum_{e\in\delta(\{\upsilon\})} x_e = 2 \quad ,\forall\, u \in N\backslash\{0\}, \tag{26}$$

$$\sum_{e\in\delta(\{0\})} x_e \geq 2\,k^*, \tag{27}$$

$$\sum_{e\in\delta(S)} x_e \geq 2\,k(S) \;,\forall\; S \in N\backslash\{0\}, \tag{28}$$

$$x_e \leq 1, \quad \forall e \in A \setminus \delta(\{0\}), \tag{29}$$

$$\sum_{l=1}^{p} q_l^e \lambda_l - x_e = 0 \;,\forall e \in A, \tag{30}$$

$$x_e \in \{0,1,2\}, \quad \forall e \in A, \tag{31}$$

$$\lambda_l \geq 0 \quad \forall\, l \in \{1,\ldots,p\}. \tag{32}$$

where :
x_e represents the number of times that edge e is traversed by a vehicle. This variable can assume value 2 if e is adjacent to the depot, corresponding to a route with a single client. λ_l variables would ideally be associated with valid routes. This would imply having a strongly *NP*-hard column generation problem. λ_l variables are associated to the set of all *q-routes* satisfying the vehicle capacity constraint. A *q-route* is a walk that starts at the depot, traverses a sequence of clients with total demand at most Q, and returns to the depot. Clients may appear more than once in a *q-route* and its demand considered for each time. Each variable λ_l is therefore associated to one of the p possible *q*-routes.

Degree constraints (26) states that each client vertex is served by exactly one vehicle. Constraint (27) requires that at least K^* vehicles leave and return to the depot. This number, representing the minimum number of vehicles to service all clients, is calculated by solving a Bin-Packing Problem. The *rounded capacity* constraints stated in (28) use $k(S) = [\sum_{u\in S} q(u)/Q]$ as a lower bound on the minimum number of vehicles necessary to service the clients in set $S \subset N$. Constraints (30) oblige x to be a linear combination of *q*-routes. The total constraints complete the formulation.

7. Algorithms for VRP

Since the first VRP presented by Dantzig and Ramser in 1959 (Kallehauge 2006), many algorithms have been proposed for solving either the classical VRP or its variants. Exact algorithms were proposed as well as heuristics. In this paper we reviewed some algorithms.

Branch and bound (Laporte & Nobert 1983; Fischetti *et al.* 1994; Lau *et al.* 1997; Toth & Vigo 2001), Branch and cut (Augerat *et al.* 1995; Bard *et al.* 2002; Lysgaard *et al.* 2004) Branch and cut and price (Fukasawa *et al.* 2004; 2006) were the most widely used exact algorithms for solving different variants of VRP. Meanwhile, many Heuristic algorithms applied for the VRP. The nearest neighbour algorithm, insertion algorithms and tour improvement procedures were applied to CVRPs (Laporte 1992). In this section, we present some of both exact and heuristic algorithms used to solve the VRP and its variants.

Baldacci *et al.* (2007) have presented an exact algorithm for the Capacitated Vehicle Routing Problem (CVRP) based on the set partitioning formulation with additional cats that correspond to capacity and clique inequalities. The exact algorithm has used a bounding procedure that found a near optimal dual solution of the LP-relaxation of the resulting mathematical formulation by combining three dual ascent heuristics. The first dual heuristic has been based on the *q*-routes relaxation of the set partitioning formulation of the CVRP. The second one combined Lagrangean relaxation, pricing and cut generation. While the third attempted to close the duality gap left by the first two procedures using a classical pricing and cut generation technique. The final dual solution is used to generate a reduced problem containing only the routes whose reduced costs were smaller than the gap between an upper bound and the lower bound achieved.

Azi *et al.* (2007) have described an exact algorithm for solving a problem where the same vehicle performs several routes to serve a set of customers with time windows. A two phases method has been proposed based on an elementary shortest path algorithm with resource constraints. In the first phase, all non-dominated feasible routes have been generated, while in the second phase, some routes have been selected and sequenced to form the vehicle workday.

A new exact algorithm for the VRP has been presented by Fukasawa *et al.* (2006). This method has been judged as the best exact method (Baldacci *et al.* 2007). The main idea is to combine the branch-and-cut approach with the *q*-routes approach (which is interpreted as *Column Generation Algorithm*, described blow, instead of the original Lagrangean relaxation) to derive superior lower bounds. Thus, the branch and cut and price algorithm. The idea of combining column and cut generation to improve lower bounds has rarely been used, since new dual variables corresponding to separated cuts may have the undesirable effect of changing the structure of the pricing sub-problem. However, if cuts are expressed in terms of variables from a suitable original formulation, they can be incorporated into the column generation process without disturbing the pricing. It refers to branch-and-bound procedures based on such formulations as *robust branch-and-cut-and-price* algorithms. The pricing sub-problem of finding the *q*-routes yielding a variable with minimum reduced cost is *NP*-hard (it contains the capacitated shortest path problem), but can be solved in pseudo-polynomial $O(n^2C_k)$ time. Since the reduced costs of the arcs that are not leaving the depot are independent of k, it is possible to price *q*-routes for every capacity C_k by making a single call (with $C_k = C$) to this dynamic programming algorithm.

Column Generation Algorithm

Step 1. Generate a matrix T' containing a small subset of promising columns from T .

Step 2. Solve the model using the dual simplex algorithm replacing T by T'. If this LP is feasible, then **STOP.** The columns corresponding to the nonzero components of λ, the current solution, comprise the set D.

Step 3. Otherwise, let r be the row in which the dual unboundedness condition was discovered, and let $(a,-\beta)$ be the r^{th} row of B^{-1}. Solve the model with cost vector c defined by :

$$c_e = \begin{cases} M & \text{if } x_e = 0; \\ -M & \text{if } x_e = 1; \\ a_e & \end{cases}$$

otherwise
$\forall e \in E$, M is chosen large enough to ensure that the conditions are met. Let t be the incidence vector of the result. If $at < \beta$, then t is a column eligible to enter the basis. Add t to T' and go to 1. Otherwise, impose the appropriate Farkas inequality.
End of Algorithm

Ropke and Pisinger (2006) have developed a unified model capable of handling most variants of VRP. The unified model can be seen as a Rich Pickup and Delivery Problem with Time Windows, which can be solved through an improved version of the large neighbourhood search heuristic.

Fugenschuh (2006) has given a mixed-integer programming formulation for the VRPCTW. He has solved it using a new meta-heuristic that combines classical construction aspects with mixed-integer pre-processing techniques, and improving hit-and-run, a randomised search strategy from global optimisation.

Cordeau *et al.* (2001) have introduced a simple Tabu Search procedure for VRPTW and two of its extensions, namely Periodic VRPTW (PVRPTW) and Multi depot VRPTW (MDVRPTW). An important feature of the approach is the possibility of exploring infeasible solutions during the search.

Braysy and Gendreau (2002) have reported a classical two-phase mechanism proposed by De Backer and Furnon (1997) to solve VRP and VRPTW. The initial solution is first generated using the savings heuristic of Clarke and Wright (1964). Then, intra-route local searches (2-opt and Or-opt) and three inter-route operators guided with TS is used to refine the solution.

Potvin and Bengio (1996) have proposed a GA called GENEROUS that directly applies genetic operators to solutions, thus avoiding coding issues. They used Solomon's method (cheapest insertion heuristic) to create the initial population. The fitness values of the proposed approach has been based on the number of vehicles and the total route time of each solution. The selection process is stochastic and highly biased towards the best solutions. For this purpose, a linear ranking scheme has been used. The linear ranking scheme prevented individuals, with significantly better fitness values than average, from dominating the selection process.

Baker (1985) has suggested deriving the probability of an individual being selected for mating from its rank within the population, instead of calculating it directly from the objective value. During the recombination phase, two parent solutions have been merged into a single one, to guaranty the feasibility of the new solution. Two types of crossover operators were used, namely a route-based and a sequence-based crossover. The route-based crossover replaced one route of parent solution $P2$ by a route of parent solution $P1$ whereas in the sequence-based crossover only a randomly defined end part in a route of parent-solution $P1$ has been replaced by a set of customers served by a route of parent solution $P2$. A repair operator has been used to remove duplicates and insert missing customers into the solution. Mutation operators have been aimed to reducing the number of routes by trying to insert the customers of a randomly selected short route into other routes, either directly or by first removing some customer from the target route and inserting it into some other route to make room for the new customer. Finally, in order to locally optimise the solution, a mutation operator based on Or-opt exchanges (Or 1976) has been used.

Berger *et al.* (2003) have presented an approach where two populations are evolved in parallel. The first population has been used to minimise the total distance and the second population has tried to minimise violations of time window constraints. The initial population has been created using a random sequential insertion heuristic. The first of the two-recombination operators has been the same as in Berger *et al.* (1998). The second has extended the first operator by also removing illegally routed customers and by using the insertion procedure proposed in Liu and Shen (1999) instead of Solomon's (1987) heuristic in the reinsertion phase.

In Braysy *et al.* (2004), the heuristic search methods that hybridise ideas of evolutionary computation with some other search techniques, such as Tabu Search (TS) or Simulated Annealing (SA) have also been used for solving VRPs. Most of the hybrid mothods presented have been use local search mutation instead of the random mutation operators. In the first phase, an initial solution has been created by either the cheapest insertion heuristic or the sectoring based genetic algorithm GIDEON. The second phase has applied one of the following search procedures that use the λ-interchange mechanism: a local search descent procedure, a SA algorithm or a hybrid SA and TS, where TS is combined with the SA-based acceptance criterion to decide which moves to accept from the candidate list. The main feature of the local search procedures is that infeasible solutions with penalties have been allowed if considered attractive (Braysy *et al.* 2004).

Potvin *et al.* (1996a) have used the competitive Neural Network of Potvin and Robillard (Braysy *et al.* 2004) to select the seed customers for the modification of Solomon's insertion heuristic (Potvin & Rousseau 1993) where several routes have been constructed simultaneously. The algorithm required a value for three parameters, $\alpha 1$, $\alpha 2$ and μ. The first two constants determine the importance of distance and travel time in the cost function for inserting each unrouted customer. The third parameter has been used to weigh the distance savings. A Genetic Algorithm is used to find values for these three constants.

Créput *et al.* (2007) have presented an evolutionary algorithm embedding self-organizing maps (SOM) as operators to address the vehicle routing problem with time windows (VRPWT). The approach has extended and improved SOM based neural networks applications to the VRPWT. From the point of view of Neural Networks, the evolutionary framework introduced a level of supervision but it corresponded to a selection principle at a higher level with the aim to allow simplicity and flexibility and favour further parallel implantations. Operators have a similar structure based on closest point findings and simple moves performed in the Euclidean plane.

Le Bouthillier and Crainic (2005) have presented a parallel co-operative methodology in which several agents communicate through a pool of feasible solutions. The agents consist of simple construction and local search algorithms and four different metaheuristic methods, namely two evolutionary algorithms and two Tabu Searches. The evolutionary algorithms have used a probabilistic mutation and the well-known edge recombination and order crossovers, while the TS procedures are adaptations of the TABUROUTE method of Gendreau *et al.* (1994) and unified TS of Cordeau *et al.* (2001). The fitness value of solutions is based on the number of vehicles, distance and waiting times. The pool is initialised with a set of four simple construction heuristics: least successor, double-ended nearest neighbour, multiple fragments (which adds sequentially the shortest arcs) and shortest arc hybridising (probabilistic version of the previous). The created initial and final solutions are post-optimised with an ejection chain procedure and well-known 2-opt, 3-opt and Or-opt improvement heuristics.

8. Conclusion

The Vehicle Routing Problem lies at the heart of distribution management. There exists several variants of the VRP and a wide variety of exact and approximate algorithms have been proposed for solving them. Exact algorithms can only solve relatively small problems. The *branch-and-cut-and-price* algorithm used by Fukasawa *et al.* (2006) has brought a new idea by combining the classical *branch-and-cut* and *pricing* methods. As for the heuristics, a number of approximate algorithms have proved very satisfactory for large problems. From the simple basic heuristics such as tabu search and simulated annealing, to the hybrid multi-phases ones, many researcher still trying to find the best heuristic which give a very good approximate solution in a proper running time. Nowadays, evolutionary algorithms like Genetic Algorithm, Ant Colony and Neural Network are the main interests of many researchers. In this paper, the VRP and its variants, and the different models of the VRP have been reviewed. Many works on the VRP and its variants have also been reported.

Acknowledgements

The authors would like to acknowledge and thank the University and the Ministry of Higher Education, Malaysia for the Research Grant No. UKM-ST-02-FRGS0013-2006.

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