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APPROXIMATE ANALYTICAL SOLUTIONS OF THE KLEIN-GORDON EQUATION BY MEANS OF THE HOMOTOPY ANALYSIS METHOD

(Penyelesaian Analisis Penghampiran bagi Persamaan Klein-Gordon
Menggunakan Kaedah Analisis Homotopi)

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ABSTRACT

In this paper, the homotopy analysis method (HAM) is implemented to give approximate and analytical solutions for the Klein–Gordon equation. The auxiliary parameter $\hbar$ in the HAM solutions has provided a convenient way of controlling the convergent region of series solutions. This problem shows rapid convergence of the sequence constructed by this method to the exact solution. Moreover, this technique reduces the volume of calculations by avoiding discretization of the variables, linearization or small perturbations.

Keywords: Klein–Gordon equation; homotopy analysis method; analytical solutions

ABSTRAK

Dalam makalah ini, kaedah analisis homotopi telah digunakan untuk menghasilkan penyelesaian analisis dan hampiran bagi persamaan Klein-Gordon. Parameter bantu $\hbar$ dalam penyelesaian kaedah analisis homotopi memberikan satu cara yang mudah untuk mengawal rantau penumpuan bagi penyelesaian siri tersebut. Masalah yang dipertimbangkan ini menunjukkan penumpuan yang pantas kepada penyelesaian tepat bagi jujukan yang dibentuk oleh kaedah ini. Di samping itu, teknik ini juga mengurangkan jumlah pengiraan yang banyak tanpa melakukan pendiskretan pemboleh ubah, pelinearan atau usikan kecil.

Kata kunci: persamaan Klein-Gordon; kaedah analisis homotopi; penyelesaian analisis

1. Introduction

Nonlinear phenomena that appear in many areas of scientific fields such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics can be modeled by partial differential equations. A broad class of analytical solutions methods and numerical solutions methods were used to handle these problems (Wazwaz 2006). In this paper, we consider the Klein–Gordon equation

$$u_{tt} - u_{xx} + b_1 u + b_2 g(u) = f(x,t), \tag{1}$$

where u is a function of x and t, g is a nonlinear function, and f is a known analytic function. The Klein–Gordon and sine-Gordon equations model many problems in classical and quantum mechanics, solitons and condensed matter physics (Abbasbandy 2006; Caudrey *et al*. 1975; Cveticanin 2005; Deeba & Khuri 2005; Dodd *et al*. 1982; El-Sayed 2003; Ismail & Sarie 1989).

The approximate analytical solution to (1) was presented by Deeba and Khuri (2005) using the analytic Adomian decomposition method (ADM). Wazwaz (2006) presented modified Adomian decomposition method (MADM) to solve non-linear Klein-Gordon equation.

Abbasbandy (2006) used variational iteration method (VIM) to get the solution of Klein-Gordon equation. Recently, Chowdhury and Hashim (2007) used homotopy-perturbation method (HPM) to obtain the solution of the equation.

Another powerful analytical method, called the homotopy analysis method (HAM), was first envisioned by Liao (2003). Recently this method has been successfully employed to solve many types of problems in science and engineering (Ayub *et al*. 2003; Bataineh *et al*. 2007; Liao 1995, 1997, 2004, 2005; Hayat *et al*. 2004; Hayat & Sajid 2007). Homotopy analysis method contains an auxiliary parameter $\hbar$ which provides us with a simple way to adjust and control the convergence region and the rate of convergence of the series solution.

The aim of the present work is to effectively employ the HAM to establish exact or approximate solutions for the Klein-Gordon equation. The examples illustrated in this present paper have not been exactly solved before using HAM. Comparison of the present method and the ADM is also presented in this paper.

2. Basic Idea of HAM

In this paper, we apply the HAM to the four problems to be discussed. In order to show the basic idea of HAM, consider thefollowing differential equation:

$$N[u(x,t)] = 0,$$

where N is a nonlinear operator, x and t denote the independent variables and u is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of the HAM, we first construct the so-called zeroth-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = q\hbar N[\phi(x,t;q)], \tag{2}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is an auxiliary parameter, L is an auxiliary linear operator, $\phi(x,t;q)$ is an unknown function and $u_0(x,t)$ is an initial guess of $u(x,t)$. It is obvious that when the embedding parameter $q = 0$ and $q = 1$, Eq. (2) becomes

$$\phi(x,t;0) = u(x,0), \qquad \phi(x,t;1) = u(x,t),$$

respectively. Thus as q increases from 0 to 1, the solution $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t;q)$ in Taylor series with respect to q, one has

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t) q^m, \tag{3}$$

where

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m}\bigg|_{q=0}. \tag{4}$$

The convergence of the above series depends upon the auxiliary parameter $\hbar$. If it is convergent at $q=1$, one has

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{+\infty}u_m(x,t)\,,$$

which must be one of the solutions of the original nonlinear equation, as proven by Liao (2003). Define the vectors

$$\overrightarrow{u_n}(x,t)=\{u_0(x,t),u_1(x,t),\cdots,u_n(x,t)\}\,.$$

Differentiating the zeroth-order deformation equation *m*-times with respect to *q* and then dividing them by *m*! and finally setting $q=0$, we get the following *m*th-order deformation equation:

$$L[u_m(x,t)-\chi_m u_{m-1}(x,t)]=H(x,t)\hbar R_m(\vec{u}_{m-1})\,, \tag{5}$$

where

$$R_m(\vec{u}_{m-1})=\frac{1}{(m-1)!}\frac{\partial^{m-1}N[\phi(x,t;q)]}{\partial q^{m-1}}\bigg|_{q=0}\,, \tag{6}$$

and

$$\chi_m=\begin{cases}0, & m\le 1,\\ 1, & m>1.\end{cases}$$

It should be emphasized that $u_m(x,t)$ for $m\ge 1$ is governed by the linear *m*th-order deformation equation with linear boundary conditions that come from the original problem, which can be solved by the symbolic computation softwares such as Maple and Mathematica.

3. Applications

In this part, we will apply the HAM to the linear and nonlinear Klein-Gordon equation. In the first three examples, we define the linear operator as

$$L[\phi(x,t;q)]=\frac{\partial^2\phi(x,t;q)}{\partial t^2}\,, \tag{7}$$

with the property

$$L[C_1(x)t+C_2(x)]=0\,,$$

where C_i $(i=1,2)$ are integral constants.

Example 1
Consider the linear form of Klein-Gordon equation

$$u_{tt} - u_{xx} = u\ , \tag{8}$$

subject to the initial conditions

$$u(x,0) = 1 + \sin(x), \qquad u_t(x,0) = 0. \tag{9}$$

To solve Eqs. (8) and (9) by means of the homotopy analysis method, we choose the initial approximation

$$u_0(x,t) = u(x,0) = 1 + \sin(x)\,.$$

Further, Eq. (8) suggests that we define the nonlinear operator as

$$N[\phi(x,t;q)] = \frac{\partial^2 \phi(x,t;q)}{\partial t^2} - \frac{\partial^2 \phi(x,t;q)}{\partial x^2} - \phi(x,t;q).$$

Using the above definition, we construct the *zeroth-order deformation equation*

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = q\hbar N[\phi(x,t;q)]. \tag{10}$$

Obviously, when $q = 0$ and $q = 1$,

$$\phi(x,t;0) = u(x,0), \qquad \phi(x,t;1) = u(x,t).$$

Therefore, as the embedding parameter q increases from 0 to 1, $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x, t)$. Expanding $\phi(x,t;q)$ in Taylor series with respect to q one has

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t) q^m\ ,$$

where

$$u_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m}\bigg|_{q=0}\,.$$

If the auxiliary linear operator, the initial guess and the auxiliary parameter h are properly chosen, the above series is convergent at $q = 1$, then one has

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t) ,$$

which must be one of the solutions of the original nonlinear equation, as proven by Liao (2003). Define the vectors

$$\overrightarrow{u_n}(x,t) = \{u_0(x,t), u_1(x,t), \cdots, u_n(x,t)\}.$$

Differentiating the zeroth-order deformation equation m-times with respect to q and then dividing them by $m!$ and finally setting q=0, we get the following mth-order deformation equation:

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\vec{u}_{m-1}), \qquad (12)$$

with the boundary conditions

$$u_m(x,0) = 0, \quad u_{m_t}(x,0) = 0,$$

where

$$R_m(\vec{u}_{m-1}) = \frac{\partial^2 u_{m-1}}{\partial t^2} - \frac{\partial^2 u_{m-1}}{\partial x^2} - u_{m-1}.$$

Now, the solution of the mth-order deformation equation (12) for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})].$$

We now successively obtain

$$u_1(x,t) = -\frac{\hbar t^2}{2}$$

$$u_2(x,t) = \frac{1}{24}\hbar^2 t^4 - \frac{1}{2}\hbar t^2 - \frac{1}{2}\hbar^2 t^2$$

$$u_3(x,t) = -\frac{1}{720}\hbar^3 t^6 + \frac{1}{12}\hbar^2 t^4 + \frac{1}{12}\hbar^3 t^4 - \frac{1}{2}\hbar t^2 - \hbar^2 t^2 - \frac{1}{2}\hbar^3 t^2 .$$

Then the series solution expression can be written in the form

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \dots \ .$$

The first four terms of the series solution when $\hbar = -1$ are

$$u_0(x,t) := 1 + \sin(x)$$

$$u_1(x,t) := \frac{t^2}{2}$$

$$u_2(x,t) := \frac{t^4}{24}$$

$$u_3(x,t) := \frac{t^6}{720}$$

Finally, the approximate solution in a series form is

$$u(x,t) \simeq \sin(x) + 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots,$$

and this will, in the limit of infinitely many terms, yield the closed-form solution

$$u(x,t) \simeq \sin(x) + \cosh(t),$$

which is the exact solution.

Example 2

Consider the linear nonhomogeneous Klein–Gordon equation

$$u_{tt} - u_{xx} - 2u = -2\sin(x)\sin(t), \tag{13}$$

subject to initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = \sin(x). \tag{14}$$

To solve Eqs. (13) and (14) by means of the homotopy analysis method, we choose the initial approximation

$$u_0(x,t) = u(x,0) = t\sin(x).$$

Further, Eq. (13) suggests that we define the nonlinear operator as

$$N[\phi(x,t;q)] = \frac{\partial^2\phi(x,t;q)}{\partial t^2} - \frac{\partial^2\phi(x,t;q)}{\partial x^2} - 2\phi(x,t;q) + 2\sin(x)\sin(t).$$

We apply the *m*th-order deformation equation with the boundary conditions

$$u_m(x,0) = 0, \quad u_{m_t}(x,0) = 0,$$

where

$$R_m(\vec{u}_{m-1}) = \frac{\partial^2 u_{m-1}}{\partial t^2} - \frac{\partial^2 u_{m-1}}{\partial x^2} - 2u_{m-1} + 2\sin(x)\sin(t).$$

Now, the solution of the *m*th-order deformation equation for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})].$$

We now successively obtain

$$u_1(x,t) = -\frac{1}{6}\hbar\sin(x)t^3 - 2\hbar\sin(x)\sin(t) + 2\hbar t\sin(x),$$

$$u_2(x,t) = -\frac{1}{6}\hbar\sin(x)t^3 - 2\hbar\sin(x)\sin(t) - \frac{1}{2}\hbar^2\sin(x)t^3 - 4\hbar^2\sin(x)\sin(t) + \frac{1}{120}\hbar^2\sin(x)t^5 + 2\hbar t\sin(x) + 4\hbar^2 t\sin(x),$$

$$u_2(x,t) = -\frac{1}{6}\hbar\sin(x)t^3 - 2\hbar\sin(x)\sin(t) - \hbar^2\sin(x)t^3 - 8\hbar^2\sin(x)\sin(t) + \frac{1}{60}\hbar^2\sin(x)t^5 - \frac{7}{6}\hbar^3\sin(x)t^3 - 8\hbar^3\sin(x)\sin(t) + \frac{1}{30}\hbar^3\sin(x)t^5 - \frac{1}{5040}\hbar^3\sin(x)t^7 + 2\hbar t\sin(x) + 8\hbar^2 t\sin(x) + 8\hbar^3 t\sin(x),$$

Then the series solution expression can be written in the form

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \cdots.$$

The first four terms of the series solution when $\hbar = -1$ are

$$u_1(x,t) := \frac{1}{6}\sin(x)\,t^3 + 2\sin(x)\sin(t) - 2\,t\sin(x)$$

$$u_2(x,t) := \frac{1}{120}\sin(x)\,t^5 - 2\sin(x)\sin(t) - \frac{1}{3}\sin(x)\,t^3 + 2\,t\sin(x)$$

$$u_3(x,t) := \frac{1}{5040}\sin(x)\,t^7 + 2\sin(x)\sin(t) - \frac{1}{60}\sin(x)\,t^5 + \frac{1}{3}\sin(x)\,t^3 - 2\,t\sin(x)$$

$$u_4(x,\, t) := \frac{1}{362880}\sin(x)\, t^9 - 2\sin(x)\sin(t) - \frac{1}{2520}\sin(x)\, t^7 + \frac{1}{60}\sin(x)\, t^5 - \frac{1}{3}\sin(x)\, t^3 + 2\, t\sin(x)$$

Hence, the 5-term approximate series solution can be written as

$$u(x,t) \simeq \sin(x)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} \right),$$

and this will, in the limit of infinitely many terms, yield the closed-form solution

$$u(x,t) = \sin(x)\sin(t).$$

Example 3

Consider the nonlinear Klein–Gordon equation

$$u_{tt} - u_{xx} = -u^2, \tag{15}$$

subject to initial conditions

$$u(x,0) = 1 + \sin(x), \qquad u_t(x,0) = 0. \tag{16}$$

To solve Eqs. (15) and (16) by means of the homotopy analysis method, we choose the initial approximation

$$u_0(x,t) = u(x,0) = 1 + \sin(x).$$

Further, Eq. (15) suggests that we define the nonlinear operator as

$$N[\phi(x,t;q)] = \frac{\partial^2 \phi(x,t;q)}{\partial t^2} - \frac{\partial^2 \phi(x,t;q)}{\partial x^2} + \phi^2(x,t;q).$$

We apply the *m*th-order deformation equation with the boundary conditions

$$u_m(x,0) = 0, \qquad u_{m_t}(x,0) = 0,$$

where

$$R_m(\vec{u}_{m-1}) = \frac{\partial^2 u_{m-1}}{\partial t^2} - \frac{\partial^2 u_{m-1}}{\partial x^2} + \sum_{j=0}^{m-1} u_j u_{m-1-j}.$$

Now, the solution of the *m*th-order deformation equation for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})].$$

We now successively obtain

$$u_1(x,t) = \frac{1}{2}t^2\hbar(3\sin(x) + 2 - \cos(x)^2)$$

$$u_2(x,t) = -\frac{1}{24}t^2\hbar(-13\sin(x)t^2\hbar + 2\sin(x)t^2\hbar\cos(x)^2 - 36\sin(x) - 36\hbar\sin(x)$$
$$-24 + 12t^2\hbar\cos(x)^2 + 12\cos(x)^2 + 12\hbar\cos(x)^2 - 24\hbar - 12t^2\hbar)$$

$$u_3(x,t) = -\frac{1}{720}t^2\hbar(120\sin(x)t^2\hbar\cos(x)^2 + 120t^2\hbar^2\sin(x)\cos(x)^2$$
$$+82t^4\hbar^2\sin(x)\cos(x)^2 - 139t^4\hbar^2\sin(x) - 2160\hbar\sin(x) - 780t^2\hbar^2\sin(x)$$
$$-1080\sin(x) - 780\sin(x)t^2\hbar - 1080\hbar^2\sin(x) + 720\hbar\cos(x)^2 + 360\cos(x)^2$$
$$+720t^2\hbar^2\cos(x)^2 - 720 - 1440\hbar - 720t^2\hbar^2 - 152t^4\hbar^2 - 720\hbar^2 - 720t^2\hbar$$
$$-10\hbar^2t^4\cos(x)^4 + 180t^4\hbar^2\cos(x)^2 + 720t^2\hbar\cos(x)^2 + 360\hbar^2\cos(x)^2).$$

So

$$u(x,t) \simeq u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t).$$

Example 4
Finally, we consider the nonlinear nonhomogeneous Klein–Gordon equation

$$u_{tt} - u_{xx} + u^2 = -x\cos(t) + x^2\cos^2(t)\,, \qquad (17)$$

subject to initial conditions

$$u(x,0) = x, \qquad u_t(x,0) = 0. \qquad (18)$$

To solve Eqs. (17) and (18) by means of homotopy analysis method, from the freedom to choosing the linear operator in HAM, we choose the linear poperator as

$$L[\phi(x,t;q)] = \frac{\partial^2\phi(x,t;q)}{\partial t^2} + \phi(x,t;q)\,, \qquad (19)$$

with the property

$$L[C_1(x)\sin(t) + C_2(x)\cos(t)] = 0\,.$$

Further, Eq. (17) suggests that we define the nonlinear operator as

$$N[\phi(x,t;q)] = \frac{\partial^2 \phi(x,t;q)}{\partial t^2} - \frac{\partial^2 \phi(x,t;q)}{\partial x^2} + \phi^2(x,t;q) + x\cos(t) - x^2\cos^2(t).$$

Also we can solve the zeroth-order deformation equation (2) using the linear operator (19) under initial conditions

$$u_0(x,t) = x\cos(t), \qquad u_{0t}(x,t) = 0.$$

Now we succefully have

$$u_0(x,t) = x\cos(t).$$

We apply the *m*th-order deformation equation with the boundary conditions

$$u_m(x,0) = 0, \qquad u_{m_t}(x,0) = 0,$$

where

$$R_m(\vec{u}_{m-1}) = \frac{\partial^2 u_{m-1}}{\partial t^2} - \frac{\partial^2 u_{m-1}}{\partial x^2} + \sum_{j=0}^{m-1} u_j u_{m-1-j} + x\cos(t) - x^2\cos^2(x).$$

Now, the solution of the *m*th-order deformation equation for $m \geq 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1}[R_m(\vec{u}_{m-1})].$$

We now successively obtain

$$u_1(x,t) = 0, \qquad u_2(x,t) = 0, \qquad u_3(x,t) = 0.$$

Similarly, higher order solutions are also zero which then yields the exact solution

$$u(x,t) = x\cos(t).$$

4. Comparison and Discussion

In this part we plot the $\hbar$-curves as presented in Figures 1 to 3 for Examples 1 to 3, respectively. In all of these examples we show that $\hbar$ = -1 is in the convergent region, also in this case, when $\hbar$ = -1, we have the ADM and HPM solutions for Examples 1 and 2; which means the ADM and HPM solutions are special cases of the HAM solution. For Example 3, comparison between ADM and HAM is done in Figures 4 and 5. In Example 4, we found that HAM is an effective method to obtain the initial guess to get the exact solution.

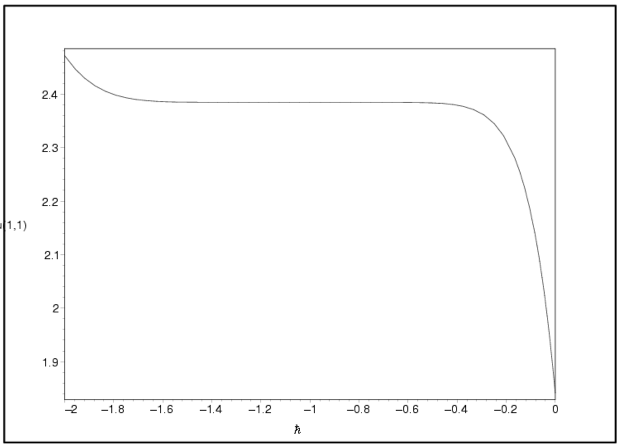

Figure 1: The $\hbar$ -curve of 10-approximation for Example 1

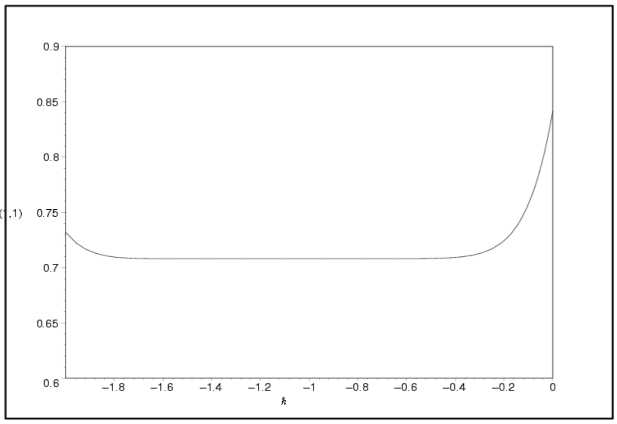

Figure 2: The $\hbar$ -curve of 10-approximation for Example 2

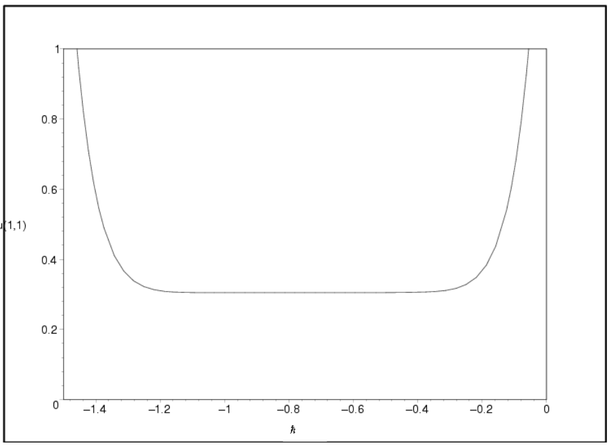

Figure 3: The $\hbar$ -curve of 10-approximation for Example 3

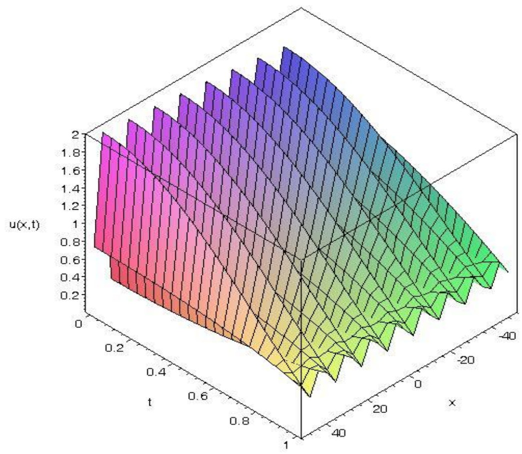

Figure 4: The ADM solution for Example 3 under 4th-order approximation

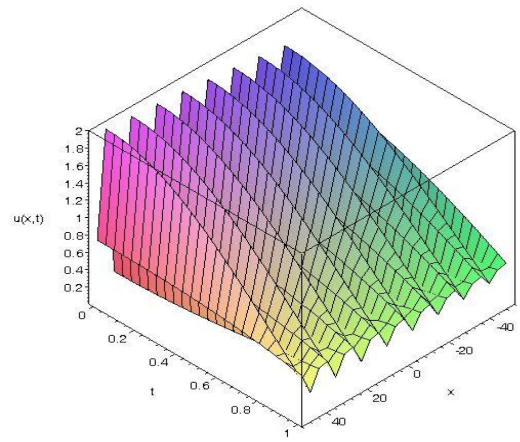

Figure 5: The HAM solution for Example 3 under 4th-order approximation

5. Conclusions

In this paper, the standard homotopy analysis method (HAM) has been successfully employed to obtain the approximate analytical solutions of the Klein–Gordon equation. In comparison to the Adomian decomposition method (ADM), HAM avoids the difficulties arising in finding the Adomian polynomials. In addition, the calculations involved in HAM are simple and straightforward. It is shown that the HAM is a promising tool for both linear and nonlinear partial differential equations.

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