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INCLUSION PROPERTIES FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS ASSOCIATED WITH A MULTIPLIER TRANSFORMATION

(Sifat Rangkuman bagi beberapa Subkelas Fungsi Analisis Bersekutu dengan Penjelmaan Berganda)

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ABSTRACT

Let f be the normalised analytic function in the open unit disk U. In the present paper, we define a new operator related to generalised Hurwitz–Lerch zeta function which involved the convolution (Hadamard product). Via this operator, we introduce new classes of functions and derive some interesting properties for these classes.

Keywords: Hurwitz–Lerch zeta function; Hadamard product; multiplier transformation; strongly convex and strongly starlike

ABSTRAK

Andaikan f fungsi analisis ternormal dalam cakera unit U. Dalam makalah ini, pengoperasi baru berkenaan dengan fungsi Hurwitz–Lerch zeta teritlak yang melibatkan konvolusi (hasil darab Hadamard) diperkenalkan. Menerusi pengoperasi ini, kelas fungsi baru diperkenalkan dan beberapa sifat menarik diperoleh bagi kelas tersebut.

Kata kunci: fungsi Hurwitz–Lerch zeta; hasil darab Hadamard; penjelmaan multiplier; cembung kuat dan bakbintang kuat

1. Introduction

Let $U = \{z : z \in C,\ |z| < 1\}$ be the open unit disk and A denotes the class of functions f with the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk U and satisfy the condition $f(0) = f'(0) - 1 = 0.$
We denote by $S^*(\eta)$ and $K(\eta)$ for $0 \le \eta < 1$ the familiar subclasses of A consisting of functions which respectively starlike and convex functions are of order η. Thus by definition, we have

$$S^*(\eta) = \left\{ f : f \in A \text{ and } \Re\left(z \frac{f'(z)}{f(z)} \right) > \eta, \quad 0 \le \eta < 1; z \in U \right\}$$

and

$$K(\eta)=\left\{ f: f\in A and\, \Re\left(\frac{zf''(z)}{f'(z)}+1\right)>\eta,\quad 0\le\eta<1; z\in U\right\}.$$

If $f\in A$ satisfies

$$\left|\arg\left(\frac{zf'(z)}{f(z)}-\eta\right)\right|<\frac{\pi}{2}\beta\qquad (0\le\eta<1;\ 0<\beta\le 1; z\in U),$$

then f is said to be strongly starlike of order β and type η in U. If $f\in A$ satisfies

$$\left|\arg\left(1+\frac{zf''(z}{f'(z)}-\eta\right)\right|<\frac{\pi}{2}\beta\qquad (0\le\eta<1;\ 0<\beta\le 1; z\in U),$$

then f is said to be strongly convex of order β and type η in U.

We denote by $S^*(\beta,\eta)$ and $C(\beta,\eta)$, respectively, the subclasses of A consisting of all strongly starlike and strongly convex of order β and type η in U. It is obvious that $f\in A$ belongs to $C(\beta,\eta)$, if and only if $zf'(z)\in S^*(\beta,\eta)$. We also note that $S^*(1,\eta)\equiv S^*(\eta)$ and $C(1,\eta)\equiv C(\eta)$. In particular, the classes $S^*(\beta,0)$ and $C(\beta,0)$ have been extensively studied by Mocanu (1989) and Nunokawa (1993).

Now let us consider the generalised Hurwitz–Lerch Zeta function

$$\phi_\mu(z,m,a)=\sum_{k=0}^{\infty}\frac{(\mu)_k}{k!}\frac{z^k}{(k+a)^m},$$

for $z\in\mathbb{C}, |z|<1, a\in\mathbb{C}/\{0,-1,-2,\ldots\}, \mu,m\in\mathbb{C}$

introduced by Goyal and Laddha (1997). Here $(x)_k$ is Pochhammer symbol (or the shifted factorial, since $(1)_k=k!$) and $(\lambda)_k$ given in terms of the Gamma functions can be written as

$$(x)_k=\frac{\Gamma(x+k)}{\Gamma(x)}\begin{cases}1, & \text{if } k=0 \text{ and } x\in C\setminus\{0\};\\ x(x+1)\ldots(x+k-1), & \text{if } k\in N \text{ and } x\in C.\end{cases}$$

Then

$$z\phi_\mu(z,m,a)=\sum_{k=1}^{\infty}\frac{(\mu)_{k-1}}{(k-1)!}\frac{z^k}{(k+a)^m},$$

where $z\in\mathbb{C}, |z|<1, a\in\mathbb{C}/\{-1,-2,\ldots\}, \mu,m\in\mathbb{C}$

Now we define the function $G(z,m,a)$ by

$$G(z,m,a)=(1+a)^m[z\phi_\mu(z,m,a)]=\sum_{k=1}^{\infty}\left(\frac{1+a}{k+a}\right)^m\frac{(\mu)_{k-1}}{(k-1)!}z^k$$

and obtain the following operator

$$D_{\mu,a}^m f(z)=G(z,m,a)*f(z)$$

then

$$D_{\mu,a}^m f(z)=z+\sum_{k=2}^{\infty}\left(\frac{1+a}{k+a}\right)^m\frac{(\mu)_{k-1}}{(k-1)!}a_k z^k, \qquad (1)$$

$$(z\in U,\ a\in\mathbb{C}\setminus\mathbb{Z}^-,\ \mu,m\in\mathbb{C}).$$

It is clear that $D_{\mu,a}^m$ are multiplier transformations. For $m=1$ and $\mu=1$ the operator $D_{\mu,a}^m$ is the integral operator studied by Owa and Srivastava (1986), for any nonnegative real number m and $\mu=1, a=1$ the operator $D_{\mu,a}^m$ is the integral operator studied by Jung et al. (1993), for any negative integer number m and $\mu=1, a=1$ the operator $D_{\mu,a}^m$ is the differential operator defined by Salagean (1983), for $m\in\mathbb{Z}$ and $\mu=2, a\geq 0$ the operator $D_{2,a}^m$ is the multiplier transformation defined by Cho and Kim (2003). In Particular, we note that $D_{1,a}^0=f(z)$ and $D_{1,0}^{-1}=D_{2,0}^0=zf'(z)$.

In view of (1), we obtain

$$z(D_{\mu,a}^{m+1}f(z))'=(a+1)D_{\mu,a}^m f(z)-aD_{\mu,a}^{m+1}f(z) \qquad (2)$$

$$z(D_{\mu,a}^m f(z))'=\mu D_{\mu+1,a}^m f(z)-(\mu-1)D_{\mu,a}^m f(z). \qquad (3)$$

The relations (2) and (3) play important and significant roles in obtaining our results.

Using the operator $D_{\mu,a}^m$, we now introduce the following classes:

$$ST_{\mu,a}^m(\beta,\eta)=\left\{f(z)\in A: D_{\mu,a}^m f(z)\in S^*(\beta,\eta),\ \frac{z\left(D_{\mu,a}^m f(z)\right)'}{D_{\mu,a}^m f(z)}\neq\eta,\ z\in U\right\}$$

$$CV_{\mu,a}^m(\beta,\eta)=\left\{f(z)\in A: D_{\mu,a}^m f(z)\in C(\beta,\eta),\ \frac{\left(z\left(D_{\mu,a}^m f(z)\right)'\right)'}{\left(D_{\mu,a}^m f(z)\right)'}\neq\eta,\ z\in U\right\}.$$

It is obvious that $f\in CV_{\mu,a}^m(\beta,\eta)$ if and only if $zf'\in ST_{\mu,a}^m(\beta,\eta)$.

In the present paper, we investigate some properties of the classes $ST^{m}_{\mu,a}(\beta,\eta)$ and $CV^{m}_{\mu,a}(\beta,\eta)$. The integral preserving properties in connection with the operator $D^{m}_{\mu,a}$ defined by (1) are also considered.

The basic tools of our investigation are the following lemmas**.**

Lemma 1 (Nunokawa 1993). *Let* p *is analytic in* U *with* $p(0)=1$ *and* $p(z)\neq 0$ in U. *Suppose that there exists a point* $z_0 \in U$ *such that*

$$\left|\arg p(z)\right| < \frac{\pi}{2}\alpha \quad \textit{for } |z|<|z_0| \;\; \textit{and } \left|\arg p(z_0)\right| = \frac{\pi}{2}\alpha \quad \textit{for } (0<\alpha\leq 1). \tag{4}$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha, \tag{5}$$

where

$$k \geq \frac{1}{2}\left(b+\frac{1}{b}\right) \quad \textit{when} \quad \arg p(z_0) = \frac{\pi}{2}\alpha, \tag{6}$$

$$k \leq -\frac{1}{2}\left(b+\frac{1}{b}\right) \quad \textit{when} \quad \arg p(z_0) = -\frac{\pi}{2}\alpha. \tag{7}$$

and

$$p(z_0)^{\frac{1}{\alpha}} = \pm ib \;\; (b>0). \tag{8}$$

Lemma 2 (Eenigenburg *et al.* 1981; 1983)**.** *Let* σ, ν *be complex numbers. Let* $h(z)$ *be convex univalent in* U *with* $h(0)=1$ *and* $\Re[\sigma h(z)+\nu]>0, z\in U$. *If* $p(z)=1+p_1 z+p_2 z^2+\ldots$ *is analytic in* U *with* $p(0)=1$ *then,*

$$p(z)+\frac{zp'(z)}{\sigma\, p(z)+\nu} \prec h(z) \Rightarrow p(z)\prec h(z).$$

Our first inclusion theorem is stated as

Theorem 1. *For* $m,\mu \in \mathbb{C}$and $a \in \mathbb{C} \setminus \mathbb{Z}$, $ST^{m}_{\mu,a}(\beta,\eta) \subset ST^{m+1}_{\mu,a}(\beta,\eta)$

Proof. Let $f(z) \in ST^{m}_{\mu,a}(\beta,\eta)$ Define the function $p(z)$ by

$$\frac{z\left(D^{m+1}_{\mu,a} f(z)\right)'}{D^{m+1}_{\mu,a} f(z)} = \gamma + (1-\gamma) p(z), \tag{9}$$

where $p(z) = 1 + p_1 z + p_2 z^2 + ...$ is analytic in U and $p(z) \neq 0$ for all $z \in U$.
Using the identity

$$z(D^{m+1}_{\mu,a} f(z))' = (a+1) D^{m}_{\mu,a} f(z) - a D^{m+1}_{\mu,a} f(z)$$

we get

$$(1+a)\frac{D^{m}_{\mu,a} f(z)}{D^{m+1}_{\mu,a} f(z)} = \gamma + a + (1-\gamma) p(z) \tag{10}$$

Differentiating both sides of (10) logarithmically and multiplying by z, we obtain

$$\frac{z\left(D^{m}_{\mu,a} f(z)\right)'}{D^{m}_{\mu,a} f(z)} - \gamma = (1-\gamma) p(z) + \frac{(1-\gamma) z p'(z)}{(1-\gamma) p(z) + \gamma + a}.$$

Suppose now that there exists a point $z_0 \in U$ such that

$$\left|\arg p(z)\right| < \frac{\pi}{2}\alpha \quad for\ |z| < |z_0| \ \text{ and } \ \left|\arg p(z_0)\right| = \frac{\pi}{2}\alpha.$$

Then, by applying Lemma 1, we can write that $z_0\, p'(z_0)/p(z_0) = ik\alpha$ and $p(z_0)^{\frac{1}{\alpha}} = \pm ib\ \ (b>0).$

Therefore, if $\arg p(z_0) = -(\pi/2)\alpha$, then

$$\begin{aligned} \frac{z_0\left(D^{m}_{\mu,a} f(z_0)\right)'}{D^{m}_{\mu,a} f(z_0)} - \gamma &= (1-\gamma) p(z_0)\left[1 + \frac{z_0 p'(z_0)/p(z_0)}{(1-\gamma) p(z_0) + \gamma + a}\right] \\ &= (1-\gamma) b^{\alpha} e^{-i\pi\alpha/2}\left[1 + \frac{ik\alpha}{(1-\gamma) b^{\alpha} e^{-i\pi\alpha/2} + \gamma + a}\right]. \end{aligned}$$

Thus we have

$$\arg\left[\frac{z_0\left(D_{\mu,a}^m f(z_0)\right)'}{D_{\mu,a}^m f(z_0)}-\gamma\right]=-\frac{\pi}{2}\alpha+\arg\left[1+\frac{ik\alpha}{(1-\gamma)b^\alpha e^{-i\pi\alpha/2}+\gamma+a}\right]$$

$$=-\frac{\pi}{2}\alpha+\tan^{-1}\left[\frac{k\alpha[\gamma+a+(1-\gamma)b^\alpha\cos(\pi\alpha/2)]}{(\gamma+a)^2+2(\gamma+a)(1-\gamma)b^\alpha\cos(\pi\alpha/2)+(1-\gamma)^2b^{2\alpha}-k\alpha(1-\gamma)b^\alpha\sin(\pi\alpha/2)}\right]$$

$$\leq-\frac{\pi}{2}\alpha \quad \left(\text{where } k\leq-\frac{1}{2}\left(b+\frac{1}{b}\right)\leq-1\right),$$

and this contradicts the condition $f\in ST_{\mu,a}^m(\beta,\eta)$.
Similarly, if $\arg p(z_0)=(\pi/2)\alpha$, then we have

$$\arg\left[\frac{z_0\left(D_{\mu,a}^m f(z_0)\right)'}{D_{\mu,a}^m f(z_0)}-\gamma\right]\geq\frac{\pi}{2}\alpha$$

which contradicts the condition $f(z)\in ST_{\mu,a}^m(\beta,\eta)$.

Thus the function $p(z)$ has to satisfy $\left|\arg p(z)\right|<(\pi/2)\alpha \quad (z\in U)$, which leads us to the following

$$\left|\arg\left(\frac{z(D_{\mu,a}^m f(z))'}{D_{\mu,a}^m f(z)}-\eta\right)\right|<\frac{\pi}{2}\beta.$$

This evidently completes the proof of Theorem 1.

Theorem 2. *For* $m,\mu\in\mathbb{C}$ and $a\in\mathbb{C}\setminus\mathbb{Z}^-$, $ST_{\mu+1,a}^m(\beta,\eta)\subset ST_{\mu,a}^m(\beta,\eta)$

Proof. Let

$$\frac{z\left(D_{\mu,a}^m f(z)\right)'}{D_{\mu,a}^m f(z)}=\gamma+(1-\gamma)p(z), \tag{11}$$

where $p(z)=1+p_1z+p_2z^2+\ldots$ is analytic in U and $p(z)\neq 0$ for all $z\in U$.
Using the identity

$$z(D_{\mu,a}^m f(z))'=\mu D_{\mu+1,a}^m f(z)-(\mu-1)D_{\mu,a}^m f(z)$$

Then

$$\mu\frac{D^m_{\mu+1,a}f(z)}{D^m_{\mu,a}f(z)}=\gamma+(1-\gamma)p(z)+(\mu-1). \tag{12}$$

Differentiating both sides of (12) logarithmically and multiplying by z, we obtain

$$\frac{z\left(D^m_{\mu+1,a}f(z)\right)'}{D^m_{\mu+1,a}f(z)}-\gamma=(1-\gamma)p(z)+\frac{(1-\gamma)zp'(z)}{(1-\gamma)p(z)+\gamma+\mu-1}.$$

The remaining part of the proof in Theorem 2 is similar to that of Theorem 1 and so we omit the details of the proofs. We next state

Theorem 3. *For* $m,\mu\in C$ and $a\in\mathbb{C}\setminus\mathbb{Z}^-$, $CV^m_{\mu,a}(\beta,\eta)\subset CV^{m+1}_{\mu,a}(\beta,\eta)$.

Proof.

$$f(z)\in CV^m_{\mu,a}(\beta,\eta)\Leftrightarrow D^m_{\mu,a}f(z)\in C(\beta,\eta)\Leftrightarrow z\left(D^m_{\mu,a}f(z)\right)'\in S^*(\beta,\eta)$$

$$\Leftrightarrow D^m_{\mu,a}\left(zf'(z)\right)\in S^*(\beta,\eta)\Leftrightarrow zf'(z)\in ST^m_{\mu,a}(\beta,\eta)$$

$$\Rightarrow zf'(z)\in ST^{m+1}_{\mu,a}(\beta,\eta)\Leftrightarrow D^{m+1}_{\mu,a}\left(zf'(z)\right)\in S^*(\beta,\eta)$$

$$\Leftrightarrow z\left(D^{m+1}_{\mu,a}f(z)\right)'\in S^*(\beta,\eta)\Leftrightarrow D^{m+1}_{\mu,a}f(z)\in C(\beta,\eta)\Leftrightarrow f(z)\in CV^{m+1}_{\mu,a}(\beta,\eta).$$

Theorem 4. $CV^m_{\mu+1,a}(\beta,\eta)\subset CV^m_{\mu,a}(\beta,\eta)$.

Proof. Similar proving as in Theorem 3.

Now, with the help of Lemma 2, we obtain the following:

Theorem 5. *Let* $h(z)$ *be convex univalent in* U *with* $h(0)=1$ *and* $\Re h(z)\geq 0$. *If* $f\in A$ *satisfies the condition*

$$\frac{1}{1-\eta}\left(\frac{z(D^m_{\mu+1,a}f(z))'}{D^m_{\mu+1,a}f(z)}-\eta\right)\prec h(z)\qquad(0\leq\eta<1\,;z\in U)$$

Then

$$\frac{1}{1-\eta}\left(\frac{z(D^m_{\mu,a}f(z))'}{D^m_{\mu,a}f(z)}-\eta\right)\prec h(z)\qquad(0\leq\eta<1\,;z\in U)$$

Proof. Let

$$p(z)=\frac{1}{1-\eta}\left(\frac{z(D_{\mu,a}^{m}f(z))'}{D_{\mu,a}^{m}f(z)}-\eta\right),$$

where $p(z)$ is analytic function with $p(0)=1$. By using the identity

$$z(D_{\mu,a}^{m}f(z))'=\mu D_{\mu+1,a}^{m}f(z)-(\mu-1)D_{\mu,c}^{m}f(z)$$

we get

$$(1-\eta)p(z)+\eta+\mu-1=\mu\frac{D_{\mu+1,a}^{m}f(z)}{D_{\mu,a}^{m}f(z)}. \tag{13}$$

Taking logarithmic derivatives in both sides of (13) and multiplying by z, we have

$$p(z)+\frac{zp'(z)}{(1-\eta)p(z)+\eta+\mu-1}=\frac{1}{1-\eta}\left(\frac{z(D_{\mu+1,a}^{m}f(z))'}{D_{\mu+1,a}^{m}f(z)}-\eta\right)$$

Applying Lemma 2, it follows that $p\prec h,$ that is

$$\frac{1}{1-\eta}\left(\frac{z(D_{\mu,a}^{m}f(z))'}{D_{\mu,a}^{m}f(z)}-\eta\right)\prec h(z),\quad (z\in U).$$

Theorem 6. *Let $h(z)$ be convex univalent in U with $h(0)=1$ and $\Re h(z)\geq 0$. If a function $f\in A$ satisfies the condition*

$$\frac{1}{1-\eta}\left(\frac{z(D_{\mu,a}^{m}f(z))'}{D_{\mu,a}^{m}f(z)}-\eta\right)\prec h(z)\ \ (0\leq\eta<1; z\in U),$$

Then

$$\frac{1}{1-\eta}\left(\frac{z(D_{\mu,a}^{m}\Psi(z))'}{D_{\mu,a}^{m}\Psi(z)}-\eta\right)\prec h(z)\quad (0\leq\eta<1; z\in U),$$

where Ψ be the integral operator introduced by Bernardi(1969) and defined by

$$\Psi(z)=\frac{c+1}{z^{c}}\int_{0}^{z}t^{c-1}f(t)dt\ \ (c>-1). \tag{14}$$

Proof. Let

$$p(z)=\frac{1}{1-\eta}\left(\frac{z(D_{\mu,a}^{m}\Psi(z))'}{D_{\mu,a}^{m}\Psi(z)}-\eta\right).$$

where $p(z)\in p$. From (14), we have

$$z(D_{\mu,a}^{m}\Psi(z))'=(c+1)D_{\mu,a}^{m}f(z)-cD_{\mu,a}^{m}\Psi(z). \tag{15}$$

Then by using (15), we get

$$(1-\eta)p(z)+c+\eta=\frac{(c+1)D_{\mu,a}^{m}f(z)}{D_{\mu,a}^{m}\Psi(z)}. \tag{16}$$

Taking logarithmic derivatives in both sides of (16), we obtain

$$p(z)+\frac{zp'(z)}{c+\eta+(1-\eta)p(z)}=\frac{1}{1-\eta}\left(\frac{z(D_{\mu,a}^{m}f(z))'}{D_{\mu,a}^{m}f(z)}-\eta\right).$$

Finally, by using Lemma 2, we obtain that

$$\frac{1}{1-\eta}\left(\frac{z(D_{\mu,a}^{m}\Psi(z))'}{D_{\mu,a}^{m}\Psi(z)}-\eta\right)\prec h(z)\quad(0\le\eta<1;z\in U).$$

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