Journal of Quality Measurement and Analysis *JQMA* 7(1) 2011, 153-159
Jurnal Pengukuran Kualiti dan Analisis

ON THE COMPUTATIONS OF SOME HOMOLOGICAL FUNCTORS OF 2-ENGEL GROUPS OF ORDER AT MOST 16

(Pengiraan Beberapa Fungtor Homologi bagi Kumpulan Engel-2 Berperingkat Tidak Melebihi 16)

HAZZIRAH IZZATI MAT HASSIM, NOR HANIZA SARMIN,
NOR MUHAINIAH MOHD ALI & MOHD SHAM MOHAMAD

ABSTRACT

The homological functors including $J(G)$, $\nabla(G)$, exterior square, the Schur multiplier, $\Delta(G)$, the symmetric square and $\tilde{J}(G)$ of a group were originated in homotopy theory. The nonabelian tensor square which is a special case of the nonabelian tensor product is vital in the computations of the homological functors of a group. It was introduced by Brown and Loday in 1987. The nonabelian tensor square $G\otimes G$ of a group G is generated by the symbols $g\otimes h$, for all $g,h\in G$ subject to the relations $gg'\otimes h=({}^{g}g'\otimes{}^{g}h)(g\otimes h)$ and $g\otimes hh'=(g\otimes h)({}^{h}g\otimes{}^{h}h')$, for all $g,g',h,h'\in G$ where ${}^{g}g'=gg'g^{-1}$. In this paper, the computations of nonabelian tensor squares and some homological functors of all 2-Engel groups of order at most 16 are done. Groups, Algorithms and Programming (GAP) software has been used to assist and verify the results.

Keywords: Nonabelian tensor square; homological functors; 2-Engel groups; GAP

ABSTRAK

Fungtor homologi termasuk $J(G)$, $\nabla(G)$, kuasa dua peluaran, pendarab Schur, $\Delta(G)$, kuasa dua simetrik dan $\tilde{J}(G)$ bagi suatu kumpulan berasal daripada teori homotopi. Kuasa dua tensor tak Abelan yang merupakan satu kes istimewa bagi hasil darab tensor tak Abelan adalah sangat penting dalam pengiraan fungtor homologi bagi suatu kumpulan. Ianya telah diperkenalkan oleh Brown dan Loday dalam tahun 1987. Kuasa dua tensor tak Abelan $G\otimes G$ bagi suatu kumpulan G adalah dijanakan oleh simbol $g\otimes h$, untuk semua $g,h\in G$ tertakluk kepada hubungan $gg'\otimes h=({}^{g}g'\otimes{}^{g}h)(g\otimes h)$ dan $g\otimes hh'=(g\otimes h)({}^{h}g\otimes{}^{h}h')$ bagi semua $g,g',h,h'\in G$ dengan ${}^{g}g'=gg'g^{-1}$. Dalam makalah ini, pengiraan bagi kuasa dua tensor tak Abelan dan beberapa fungtor homologi untuk semua kumpulan Engel-2 berperingkat paling tinggi 16 telah dilakukan. Perisian *Groups, Algorithms and Programming* (GAP) digunakan untuk membantu dan mengesahkan keputusan yang diperoleh.

Kata kunci: Kuasa dua tensor tak Abelan; fungtor homologi; kumpulan Engel-2; GAP

1. Introduction

The nonabelian tensor square of a group G is generated by symbols $g\otimes h$ where $g,h\in G$ subject to the relations $gg'\otimes h=({}^{g}g'\otimes{}^{g}h)(g\otimes h)$ and $g\otimes hh'=(g\otimes h)({}^{h}g\otimes{}^{h}h')$ for all $g,g',h,h'\in G$ where ${}^{g}g'=gg'g^{-1}$. The group G acts naturally on the tensor products by ${}^{g}(g'\otimes h)={}^{g}g'\otimes{}^{g}h$ and there exists a homomorphism mapping κ, where $\kappa: G\otimes G\to G'$

defined by $\kappa(g \otimes h) = [g,h]$, and $[g,h] = ghg^{-1}h^{-1}$. Here, $J(G) = \ker(\kappa)$ and $J(G)$ is a G-trivial subgroup of $G \otimes G$ contained in its center. Also, $\nabla(G)$ denotes the subgroup of $J(G)$ generated by the elements $x \otimes x$ for $x \in G$ while $\Delta(G)$ denotes the subgroup of $J(G)$ generated by the elements $(x \otimes y)(y \otimes x)$ for $x, y \in G$. The definition of exterior square of G is $G \wedge G = (G \otimes G)/\nabla(G)$ while the symmetric square of G is defined as $G \tilde{\otimes} G = (G \otimes G)/\Delta(G)$. Furthermore the Schur multiplier of G is defined as $M(G) = J(G)/\nabla(G)$ and $\tilde{J}(G)$ is defined as $\tilde{J}(G) = J(G)/\Delta(G)$.

The computations of the nonabelian tensor square of some groups had been done by Brown *et al.* (1987). In their research, the nonabelian tensor squares of all groups up to order 30 have been determined. Recently, Erfanian *et al.* (2008) computed the nonabelian tensor square of general linear group. In 2003, Bacon and Kappe determined some homological functors of finite p-groups of nilpotency class two. In this research, the explicit knowledge on the nonabelian tensor squares of these groups has been used in the computations. Meanwhile, Ramachandran *et al.* (2008) found the homological functors of the symmetric group of order six, S_3. The Cayley table of $S_3 \otimes S_3$ has been computed by using the definition of nonabelian tensor square of a group. Then, based on the results on the nonabelian tensor squares, the homological functors of this group was determined. Also in 2008, Mohd Ali found the homological functors of all infinite two-generator groups of nilpotency class two. Recently, in 2011, Rashid *et al.* computed the Schur Multiplier of groups of orders p^3 and p^3q.

Engel groups are certain generalised nilpotent groups which have received considerable attention in recent years. An n-Engel group is a group G such that $[x,{}_n y] = e$ for all $x, y \in G$. Recently, many researches related to Engel groups had been conducted. Kappe and Kappe (1972) investigated on 3-Engel groups. Gupta and Levin (1980) studied on soluble Engel groups and Lie Algebra. Besides, Sarmin and Yusof (2006) determined all 2-Engel groups of order at most 20. In this paper, using the 2-Engel groups of order at most 16 found by Sarmin and Yusof (2006), the nonabelian tensor squares and the homological functors of these groups are determined.

2. Some Preparatory Results

Some preparatory results on nonabelian tensor square of a group, homological functors of a group and 2-Engel groups that are used throughout this research are included in this section.

2.1. *Nonabelian Tensor Square and Homological Functors of a Group*

In this section, some preparatory results on the nonabelian tensor square and homological functors of a group that are used in the computations of some homological functors of all 2-Engel groups of order at most 16 are included.

Theorem 2.1. (Kappe *et al.* 1999)
Let G be a finite nonabelian 2-generator 2-group of class 2. If $G \cong (\langle c \rangle \times \langle a \rangle) \rtimes \langle b \rangle$, *where* $[a,b] = c, [a,c] = [b,c] = 1, |a| = 2^{\alpha}, |b| = 2^{\beta}, |c| = 2^{\gamma}$ *with* $\alpha = \beta = \gamma$, *then*

$$G\otimes G=\langle a\otimes a\rangle\times\langle b\otimes b\rangle\times\langle (a\otimes b)(b\otimes a)\rangle\times\langle a\otimes b\rangle\times\langle (a\otimes b)^2(a\otimes c)\rangle$$
$$\times\langle (a\otimes b)^2(b\otimes c)\rangle \tag{1}$$
$$\cong \mathbb{Z}_{2^\gamma}^3\times\mathbb{Z}_{2^{\gamma+1}}\times\mathbb{Z}_{2^{\gamma-1}}^2.$$

Lemma 2.1. (Dummit & Foote 2004)
Let A and B be groups with $M\lhd A$ and $N\lhd B$. Then

$$(A\times B)\big/(M\times N)\cong A/M\times B/N. \tag{2}$$

Lemma 2.2. (Dummit & Foote 2004)
Let $A=\langle a\rangle$ and $B=\langle a^h\rangle$ be groups. If $\langle a\rangle\cong C_\infty$, then the quotient of cyclic group is cyclic, i.e. $A/B\cong C_h$. If $\langle a\rangle\cong C_k$, then $\left|a^h\right|=\dfrac{k}{\gcd(h,k)}$. Furthermore, $\langle a^h\rangle\cong C_t$ where $t=\dfrac{k}{\gcd(h,k)}$. Thus, $A/B=\langle a\rangle\big/\langle a^h\rangle\cong C_{k/t}=C_{\gcd(h,k)}$.

Proposition 2.1. (Bacon & Kappe 2003)
Given a group G of nilpotency class 2 and a generating set X for G, we have

$$G\otimes G=\langle u\otimes v,u\otimes[v,w]\,|\,u,v,w\in X\rangle, \tag{3}$$
$$\nabla(\mathrm{G})=\langle u\otimes u,(u\otimes v)(v\otimes u)\,|\,u,v\in X\rangle, \tag{4}$$
$$\Delta(\mathrm{G})=\langle (u\otimes u)^2,(u\otimes v)(v\otimes u)\,|\,u,v\in X\rangle. \tag{5}$$

2.2. *The 2-Engel Groups of Order at Most 16*

Eight groups have been determined as 2-Engel groups of order at most 16 (Sarmin & Yusof 2006). They are:

(1) The Dihedral group of order 8, $D_4=\langle a,b:a^4=b^2=e,ba=a^3b\rangle$,

(2) The Quaternion group of order eight, $Q=\langle a,b:a^4=b^4=e,a^2=b^2,ba=a^3b\rangle$,

(3) $G_{4,4}=\langle a,b:a^4=b^4=abab=e,ab^3=ba^3\rangle$,

(4) $\mathbb{M}=\langle a,b:a^4=b^4=e,ab=ba^3\rangle$,

(5) $\text{The Modular-16}=\langle a,b:a^8=b^2=1,ab=ba^5\rangle$,

(6) The direct product of the Dihedral group of order eight, with cyclic group of order two, $D_4\times\mathbb{Z}_2=\langle a,b,c:a^4=b^2=c^2=e,ac=ca,bc=cb,bab=a^{-1}\rangle$,

(7) The direct product of the Quaternion group of order eight, with cyclic group of order two, $Q\times\mathbb{Z}_2=\langle a,b,c:a^4=b^4=c^2=e,b^2=a^2,ba=a^3b,ac=ca,bc=cb\rangle$,

(8) $\mathbb{Y} = \langle a,b,c : a^4 = b^2 = c^2 = e,\ cbc = ba^2, bab = a, ac = ca \rangle$.

3. The Computation of the Nonabelian Tensor Square and The Homological Functors of the Dihedral Group of Order Eight

In this research, the hand computations of the nonabelian tensor square of the Dihedral group of order eight, D_4 are done.

Theorem 3.1.
Let $G = D_4$ *with the presentation* $\langle a,b : a^4 = b^2 = e, ba = a^3 b \rangle$. *Then*

$$G \otimes G = \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (a \otimes b)(b \otimes a) \rangle \times \langle a \otimes b \rangle \cong \mathbb{Z}_2^3 \times \mathbb{Z}_4, \quad (6)$$

$$\nabla(G) = \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (a \otimes b)(b \otimes a) \rangle \cong \mathbb{Z}_2^3, \quad (7)$$

$$G \wedge G = \langle a \wedge b \rangle \cong \mathbb{Z}_4, \quad (8)$$

$$\Delta(G) = \left\langle (a \otimes a)^2 \right\rangle \times \left\langle (b \otimes b)^2 \right\rangle \times \langle (a \otimes b)(b \otimes a) \rangle \cong \mathbb{Z}_2, \quad (9)$$

$$G \tilde{\otimes} G = \langle a \tilde{\otimes} a \rangle \times \langle b \tilde{\otimes} b \rangle \times \langle a \tilde{\otimes} b \rangle \cong \mathbb{Z}_2^2 \times \mathbb{Z}_4. \quad (10)$$

Proof:
We first observe that (6) follows from Theorem 2.1 by letting $\alpha = \beta = \gamma = 1$. Note that

$$|a \otimes a| = |b \otimes b| = |(a \otimes b)(b \otimes a)| = 2 \text{ and } |a \otimes b| = 4.$$

Since G is a group of nilpotency class two, it follows by (4) in Proposition 2.1 that

$$\nabla(G) = \langle a \otimes a \rangle \times \langle b \otimes b \rangle \times \langle (a \otimes b)(b \otimes a) \rangle .$$

By restrictions on the generators, it follows $\nabla(G) \cong \mathbb{Z}_2^3$. Thus, (7) holds.

Similarly, by (5) in Proposition 2.1, we have

$$\Delta(G) = \left\langle (a \otimes a)^2 \right\rangle \times \langle (a \otimes b)(b \otimes a) \rangle \times \left\langle (b \otimes b)^2 \right\rangle.$$

From Lemma 2.2, we have $\left|(a \otimes a)^2\right| = \left|(b \otimes b)^2\right| = \dfrac{2}{\gcd(2,2)} = 1$. By these and the order restrictions on the generators, we obtain $\Delta(G) \cong \mathbb{Z}_2$. Thus, (9) holds.

Next, we prove (8). By Lemma 2.1, Lemma 2.2 and observing that $\nabla(G) \lhd (G \otimes G)$,

$$
\begin{aligned}
G \wedge G &= {(G \otimes G)}/{\nabla(G)} \\
&= {\langle a \otimes a\rangle}/{\langle a \otimes a\rangle} \times {\langle b \otimes b\rangle}/{\langle b \otimes b\rangle} \times {\langle (a \otimes b)(b \otimes a)\rangle}/{\langle (a \otimes b)(b \otimes a)\rangle} \times {\langle a \otimes b\rangle}/{\langle 1_{\otimes}\rangle} \\
&= \langle (a \otimes b)\nabla(G)\rangle \\
&= \langle a \wedge b\rangle
\end{aligned}
$$

where $1_{\otimes}$ is the identity tensor. From the order restrictions on the generators, it follows that $G \wedge G \cong \mathbb{Z}_4$, the desired result.

To prove (10), note that $\langle (a \otimes a)^2\rangle$ is a proper subgroup of index 2 in $\langle a \otimes a\rangle$ and $\langle (b \otimes b)^2\rangle$ is a proper subgroup of index 2 in $\langle b \otimes b\rangle$. Thus, ${\langle a \otimes a\rangle}/{\langle (a \otimes a)^2\rangle} \cong {\langle b \otimes b\rangle}/{\langle (b \otimes b)^2\rangle} \cong \mathbb{Z}_2$. Together with Lemma 2.1, Lemma 2.2 and observing $\Delta(G) \lhd (G \otimes G)$, we obtain

$$
\begin{aligned}
G \tilde{\otimes} G &= {(G \otimes G)}/{\Delta(G)} \\
&= {\langle a \otimes a\rangle}/{\langle (a \otimes a)^2\rangle} \times {\langle b \otimes b\rangle}/{\langle (b \otimes b)^2\rangle} \times {\langle (a \otimes b)(b \otimes a)\rangle}/{\langle (a \otimes b)(b \otimes a)\rangle} \\
&\quad \times {\langle a \otimes b\rangle}/{\langle 1_{\otimes}\rangle} \\
&= \langle (a \otimes a)\nabla(G)\rangle \times \langle (b \otimes b)\nabla(G)\rangle \times \langle (a \otimes b)\nabla(G)\rangle \\
&= \langle a \tilde{\otimes} a\rangle \times \langle b \tilde{\otimes} b\rangle \times \langle a \tilde{\otimes} b\rangle.
\end{aligned}
$$

Using the order restrictions on the generators, this leads to $G \tilde{\otimes} G \cong \mathbb{Z}_2^2 \times \mathbb{Z}_4$. □

4. The Computations of the Nonabelian Tensor Squares and Some Homological Functors Using GAP

In this section, the nonabelian tensor squares and some homological functors are determined for all 2-Engel groups of order at most 16 using GAP. Unfortunately, up to date, GAP is only capable to determine three of those functors which are $J(G)$, exterior square and Schur Multiplier.

Example of the GAP coding used to determine the nonabelian tensor square, $J(G)$, exterior square and Schur Multiplier of $D_4 = \langle a, b : a^4 = b^2 = e, ba = a^3 b\rangle$ is given as follows:

Algorithm A

```
gap> f:=FreeGroup("a","b");
<free group on the generators [ a, b ]>
gap> a:=f.1;b:=f.2;
a
b
gap> r:=[a^4,b^2,(b*a)^-1*a^3*b];
[ a^4, b^2, a^-1*b^-1*a^3*b ]
gap> G:=f/r;
<fp group on the generators [ a, b ]>
gap> IdGroup(G);
[ 8, 3 ]
gap> k:=NilpotentQuotient(G);
Pcp-group with orders [ 2, 2, 2 ]
gap> ts:=NonAbelianTensorSquare(k);
Pcp-group with orders [ 2, 2, 2, 2, 2 ]
gap> AbelianInvariants(ts);
[ 2, 2, 2, 4 ]
gap> beta:=NonAbelianTensorSquareEpimorphism(k);
[ g7*g11*g13, g8*g12, g9, g10, g11 ] -> [ g3*g7, g6, id, id, id ]
gap> phi:=Range(beta)!.epimorphism;
[ g3*g7, g6 ] -> [ g3, id ]
gap> JG:=Kernel(beta*phi);
Pcp-group with orders [ 2, 2, 2, 2 ]
gap> AbelianInvariants(JG);
[ 2, 2, 2, 2 ]
gap> es:=NonAbelianExteriorSquare(k);
Pcp-group with orders [ 2, 2 ]
gap> AbelianInvariants(es);
[ 4 ]
gap> SchurMultiplicator(k);
[ [ 2, 1 ] ]
```

The results are summarised in Table 1.

Table 1: The Nonabelian Tensor Squares and The Homological Functors of 2-Engel Groups of Order at Most 16

Group	$G \otimes G$	$J(G)$	$G \wedge G$	$M(G)$
$D_4 = \langle a,b : a^4 = b^2 = e, ba = a^3b \rangle$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_4$	$\mathbb{Z}_2$
$Q = \langle a,b : a^4 = b^4 = e, a^2 = b^2, ba = a^3b \rangle$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	$\mathbb{Z}_2 \times \mathbb{Z}_4^2$	$\mathbb{Z}_2$	$\mathbb{Z}_1$
$G_{4,4} = \langle a,b : a^4 = b^4 = abab = e, ab^3 = ba^3 \rangle$	$\mathbb{Z}_2^3 \times \mathbb{Z}_4^2$	$\mathbb{Z}_2^4 \times \mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\mathbb{Z}_2^2$
$\mathbb{M} = \langle a,b : a^4 = b^4 = e, ab = ba^3 \rangle$	$\mathbb{Z}_2 \times \mathbb{Z}_4^3$	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$	$\mathbb{Z}_4$	$\mathbb{Z}_2$

Table 1: (Continued)

The Modular-16 $=\langle a,b : a^8=b^2=e, ab=ba^5\rangle$	$\mathbb{Z}_2^3\times\mathbb{Z}_8$	$\mathbb{Z}_2^2\times\mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_1$
$D_4\times\mathbb{Z}_2=\left\langle \begin{matrix} a,b,c : a^4=b^2=c^2=e, ac=ca, \\ bc=cb, bab=a^{-1} \end{matrix}\right\rangle$	$\mathbb{Z}_2^8\times\mathbb{Z}_4$	$\mathbb{Z}_2^9$	$\mathbb{Z}_2^2\times\mathbb{Z}_4$	$\mathbb{Z}_2^3$
$Q\times\mathbb{Z}_2=\left\langle \begin{matrix} a,b,c : a^4=b^4=c^2=e, b^2=a^2, \\ ba=a^3b, ac=ca, bc=cb \end{matrix}\right\rangle$	$\mathbb{Z}_2^7\times\mathbb{Z}_4^2$	$\mathbb{Z}_2^6\times\mathbb{Z}_4^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2$
$\mathbb{Y}=\left\langle \begin{matrix} a,b,c : a^4=b^2=c^2=e, cbc=ba^2, \\ bab=a, ac=ca \end{matrix}\right\rangle$	$\mathbb{Z}_2^9$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2$

5. Conclusion

In this paper, the nonabelian tensor squares and the homological functors of all 2-Engel groups of order at most 16 have been determined using definition and GAP.

References

Bacon M. & Kappe L.C. 2003. On capable p-group of nilpotency class two. *Illinois Journal of Math*. **47**: 49-62

Brown R., Johnson D.L. & Robertson E.F. 1987. Some computations of non-abelian tensor products of groups. *Journal of Algebra* **111**: 177-202.

Dummit & Foote R.M. 2004. *Abstract Algebra*. Hoboken, NJ: John Wiley & Son.

Erfanian A, Razaei R. & Jafari H. 2008. Computing the nonabelian tensor square of general linear groups. *Italian Journal of Pure and Applied Mathematics* **24**: 203-210.

Gupta N. & Levin F. 1980. On soluble engel groups and lie algebras. *Arch. Math.* **34**: 289-295.

Kappe L.C. & Kappe W.P. 1972. On three Engel groups. *Bulletin of the Australian Mathematical Society* **7**(3): 391-406.

Kappe L.C., Visscher M.P. & Sarmin N.H. 1999. Two-generator two-groups of class two and their nonabelian tensor squares. *Glasgow Math. J.* **41**: 417-430.

Mohd Ali N.M. 2008. Capability and Homological Functors of Infinite Two-Generator Groups of Nilpotency Class Two. PhD. Thesis, Universiti Teknologi Malaysia.

Ramachandran V.A.R., Sarmin N.H., Mohd Ali N.M. 2008. The Nonabelian Tensor Square and Homological Functors of S_3. Proceedings of Regional Annual Fundamental Science Seminar in Ibnu Sina Institute for Fundamental Science Studies, Universiti Teknologi Malaysia, Johor Bahru.

Rashid S., Sarmin N.H., Erfanian A. & Mohd Ali N.M. 2011. The multiplicator and nonabelian tensor square of groups of orders p^3 and p^3q. Proceedings of The First Biennial International Group Theory Conference 2011 (BIGTC 2011) at Ibnu Sina Institute for Fundamental Science Studies, Universiti Teknologi Malaysia, Johor Bahru.

Sarmin N.H. & Yusof Y. 2006. On 2-Engel Groups of Order at Most 20. Technical Report of Department of Mathematics, Universiti Teknologi Malaysia. LT/M Bil.5.

Department of Mathematics, Faculty of Science
and Ibnu Sina Institute for Fundamental Science Studies
Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, MALAYSIA
*E-mails: hazzirah_hassim@yahoo.com, nhs@fs.utm.my**

*Corresponding author