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SOME NEW INTEGRAL OPERATORS: SUFFICIENT CONDITIONS FOR THEIR UNIVALENCE

(Beberapa Pengoperasi Baharu Kamiran: Syarat Cukup untuk Keunivalenannya)

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ABSTRACT

In this paper we define the integral operators with the forms:

$$G_{\alpha,M}(z)=\left[\frac{\alpha}{M}\int_0^z (g(u))^{\frac{\alpha}{M}-1}du\right]^{\frac{M}{\alpha}}$$

$$F_{\alpha,M}(z)=\left[\frac{\alpha}{M}\int_0^z u^{\frac{\alpha}{M}-1}\left(\frac{g(u)}{u}\right)du\right]^{\frac{M}{\alpha}}$$

$$H_{\alpha,\beta,M}(z)=\left[\frac{\beta}{M}\int_0^z u^{\frac{\beta}{M}-1}\left(\frac{g(u)}{u}\right)^{\frac{M}{\alpha}}du\right]^{\frac{M}{\beta}}.$$

The operators generalise some integral operators studied by Owa, Pascu and Pescar. The original results contained in the paper give sufficient conditions for the univalence of those integral operators.

Keywords: Analytic function; integral operator; univalent function

ABSTRAK

Dalam makalah ini ditakrifkan pengoperasi kamiran dalam bentuk:

$$G_{\alpha,M}(z)=\left[\frac{\alpha}{M}\int_0^z (g(u))^{\frac{\alpha}{M}-1}du\right]^{\frac{M}{\alpha}}$$

$$F_{\alpha,M}(z)=\left[\frac{\alpha}{M}\int_0^z u^{\frac{\alpha}{M}-1}\left(\frac{g(u)}{u}\right)du\right]^{\frac{M}{\alpha}}$$

$$H_{\alpha,\beta,M}(z)=\left[\frac{\beta}{M}\int_0^z u^{\frac{\beta}{M}-1}\left(\frac{g(u)}{u}\right)^{\frac{M}{\alpha}}du\right]^{\frac{M}{\beta}}.$$

Pengoperasi ini mengitlakkan beberapa pengoperasi kamiran yang dikaji oleh Owa, Pascu dan Pescar. Keputusan asal dalam makalah ini memberi syarat cukup untuk keunivalenan pengoperasi kamiran tersebut.

Kata kunci: Fungsi analisis; pengoperasi kamiran; fungsi univalen

1. Introduction and Preliminaries

Let H be the class of analytic functions in the open unit disc

$$U = \{z \in C: |z| < 1\},$$
$$A_n = \{f \in H: f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots, z \in U\},$$

with $A_1 = A$,

$$S = \{f \in A | f \text{ is univalent in } U\}.$$

In order to prove our main results we shall make use of the following lemma:

Lemma A. (Pascu 1985) *Let* $\alpha \in C$ *with* $\operatorname{Re}\alpha > 0$, *and* $f \in A$. *If*

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \quad \forall z \in U,$$

then the function

$$F_\alpha(z) = \left[\alpha\int_0^z t^{\alpha-1} f'(t)dt\right]^{\frac{1}{\alpha}}$$

is univalent.

Lemma B. (Pascu 1987) *Let* $\alpha \in C$, $\operatorname{Re}\alpha > 0$ *and* $f \in A$. *If*

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \quad \forall z \in U,$$

then $\forall \beta \in C, \operatorname{Re}\beta \ge \operatorname{Re}\alpha$, *the function*

$$F_\beta(z) = \left[\beta\int_0^z t^{\beta-1} f'(t)dt\right]^{\frac{1}{\beta}}$$

is univalent.

2. Main Results

Definition 1. Let $M \geq 1$, α with $\operatorname{Re}\alpha > 0$ be a complex number. For a function $g \in A$ we define the integral operator

$$G_{\alpha,M}(z) = \left[\frac{\alpha}{M}\int_0^z (g(u))^{\frac{\alpha}{M}-1} du\right]^{\frac{M}{\alpha}} \qquad = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots,\ z \in U .$$

This operator generalises the integral operator defined by V. Pescar (Pescar 2006). For this operator we prove the next theorem which contains a sufficient condition for its univalence:

Theorem 1. *Let* $M \geq 1$ *and* α *with* $\operatorname{Re}\alpha > 0$ *be a complex number such that*

$$|\alpha - M| \leq \frac{\operatorname{Re}\alpha}{M+2}, \tag{1}$$

If $g \in A$ *satisfies the conditions*

$$|g(z)| \leq M\ , \tag{2}$$

$$\left|\frac{z^2 g'(z)}{g^2(z)} - 1\right| \leq \frac{1}{M}\ , \tag{3}$$

for all $z \in U$ *, then the function*

$$G_{\alpha,M}(z) = \left[\frac{\alpha}{M}\int_0^z (g(u))^{\frac{\alpha}{M}-1} du\right]^{\frac{M}{\alpha}} \tag{4}$$

is in the class S.

Proof. From (4), we have

$$G_{\alpha,M}(z) = \left[\frac{\alpha}{M}\int_0^z u^{\left(\frac{\alpha}{M}-1\right)}\left[\frac{g(u)}{u}\right]^{\left(\frac{\alpha}{M}-1\right)} du\right]^{\frac{M}{\alpha}},\ \ z \in U . \tag{5}$$

We consider

$$f(z) = \int_0^z \left[\frac{g(u)}{u}\right]^{\left(\frac{\alpha}{M}-1\right)} du. \tag{6}$$

The function f is regular in U.

Differentiating (6), we obtain

$$f'(z)=\left[\frac{g(z)}{z}\right]^{\frac{\alpha}{M}-1},$$

$$f''(z)=\frac{\alpha-M}{M}\left[\frac{g(z)}{z}\right]^{\frac{\alpha-M}{M}-1}\cdot\frac{g'(z)z-g(z)}{z^2},\quad z\in U,$$

and

$$\frac{zf''(z)}{f'(z)}=\frac{\alpha-M}{M}\left[\frac{zg'(z)}{g(z)}-1\right],\quad z\in U. \tag{7}$$

Using (1), (2), (3) and (7), we calculate

$$\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\left|\frac{zf''(z)}{f'(z)}\right|=\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\frac{|\alpha-M|}{M}\cdot\left|\frac{zg'(z)}{g(z)}-1\right| \tag{8}$$

$$\le\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\frac{|\alpha-M|}{M}\left[\left|\frac{zg'(z)}{g(z)}\right|+1\right]$$

$$\le\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\frac{|\alpha-M|}{M}\left[\left|\frac{z^2g'(z)}{g^2(z)}\right|\cdot\left|\frac{g(z)}{z}\right|+1\right]$$

$$\le\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\frac{|\alpha-M|}{M}\left[\left|\frac{z^2g'(z)}{g^2(z)}\right|\cdot M+1\right]$$

$$\le\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\frac{|\alpha-M|}{M}\left[\left|\frac{z^2g'(z)}{g^2(z)}-1\right|\cdot M+M+1\right]$$

$$\le\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\frac{|\alpha-M|}{M}(M+2)$$

$$\leq \frac{|\alpha - M|}{\operatorname{Re}\alpha}(M+2) \leq 1, \text{ for all } z \in U.$$

From (8), using Lemma A, we have that $G_{\alpha,M}$ is in the class *S*.

Remark 1. For $M = 1$, the result was obtained in Pescar (2006).

Remark 2. For $M = 1$, the condition (3) expresses a sufficient condition for univalence of function *g* and this result can be found in Ozaki and Nunokawa (1972, Lemma C).

Definition 2. Let $1 \leq M \leq 2$ and α be a complex number such that $\operatorname{Re}\alpha \geq 3M$. For a function $g \in A$ we define the integral operator

$$F_{\alpha,M}(z) = \left[\frac{\alpha}{M}\int_0^z u^{\frac{\alpha}{M}-1}\left(\frac{g(u)}{u}\right)du\right]^{\frac{M}{\alpha}} = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots, \; z \in U.$$

This operator generalises the integral operator defined by Owa *et al.* (1999). For this operator we prove the next theorem which contains a sufficient condition for its univalence:

Theorem 2. *Let* $1 \leq M \leq 2$ *and* α *be a complex number such that*

$$\operatorname{Re}\alpha \geq 3M. \tag{9}$$

If $g \in A$ *satisfies the conditions*

$$|g(z)| \leq M, \tag{10}$$

$$\left|\frac{z^2 g'(z)}{g^2(z)} - 1\right| \leq \frac{2-M}{M}, \tag{11}$$

for all $z \in U$*, then the function*

$$F_{\alpha,M}(z) = \left[\frac{\alpha}{M}\int_0^z u^{\frac{\alpha}{M}-1}\left(\frac{g(u)}{u}\right)du\right]^{\frac{M}{\alpha}} \tag{12}$$

is in the class S.

Proof. Let us consider the function

$$f(z) = \int_0^z \frac{g(u)}{u}du. \tag{13}$$

Differentiating (13), we obtain

$$f'(z)=\frac{g(z)}{z},\quad f''(z)=\frac{zg'(z)-g(z)}{z^2},\quad z\in U,$$

and

$$\frac{zf''(z)}{f'(z)}=\frac{zg'(z)}{g(z)}-1. \tag{14}$$

Using (9), (10), (11), (14), we calculate

$$\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\left|\frac{zf''(z)}{f'(z)}\right|=\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\left|\frac{zg'(z)}{g(z)}-1\right| \tag{15}$$

$$\le\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\left[\left|\frac{zg'(z)}{g(z)}\right|+1\right]$$

$$=\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\left[\left|\frac{z^2g'(z)}{g^2(z)}\right|\cdot\left|\frac{g(z)}{z}\right|+1\right]$$

$$\le\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\left[\left|\frac{z^2g'(z)}{g^2(z)}\right|\cdot M+1\right]$$

$$\le\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\left[\left|\frac{z^2g'(z)}{g^2(z)}-1\right|\cdot M+M+1\right]$$

$$\le\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\left[\frac{2-M}{M}\cdot M+M+1\right]$$

$$\le\frac{1-|z|^{2\operatorname{Re}\alpha}}{\frac{\operatorname{Re}\alpha}{M}}\cdot 3M\le\frac{3M}{\operatorname{Re}\alpha}\le 1.$$

From (14) and using Lemma A we have that $F_{\alpha,M}$ is in the class *S*.

Remark 3. For $M=1$, the result was obtained in Owa *et al.* (1999).

Remark 4. For $M=1$, the condition (11) expresses a sufficient condition for univalence of function *g* (Ozaki & Nunokawa 1972, Lemma C).

Definition 3. Let $M\geq 1$ and let α be a complex number such that

$\operatorname{Re}\alpha \geq \dfrac{M^5+M^4+M^3}{|\alpha|}$. For a function $g\in A$ we define the integral operator

$$H_{\alpha,\beta,M}(z)=\left[\frac{\beta}{M}\int_0^z u^{\frac{\beta}{M}-1}\left(\frac{g(u)}{u}\right)^{\frac{M}{\alpha}}du\right]^{\frac{M}{\beta}}=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\dots,\ z\in U .$$

This operator generalises the integral operator defined by V. Pescar (Pescar 2006). For this operator we prove the next theorem which contains a sufficient condition for its univalence:

Theorem 3. *Let* $M\geq 1$, α *a complex number such that*

$$\operatorname{Re}\alpha \geq \frac{M^5+M^4+M^3}{|\alpha|} . \tag{16}$$

If $g\in A$ *satisfies*

$$|g(z)|\leq M , \tag{17}$$

$$\left|\frac{z^2 g'(z)}{g^2(z)}-1\right|\leq M^2 , \tag{18}$$

for all $z\in U$ *, then for every complex number* β *, with* $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$ *, the function*

$$H_{\alpha,\beta,M}(z)=\left[\frac{\beta}{M}\int_0^z u^{\frac{\beta}{M}-1}\left(\frac{g(u)}{u}\right)^{\frac{M}{\alpha}}du\right]^{\frac{M}{\beta}} \tag{19}$$

is in the class S.

Proof. Let us consider the function

$$h(z)=\int_0^z\left[\frac{g(u)}{u}\right]^{\frac{M}{\alpha}}du ,\quad z\in U . \tag{20}$$

The function *h* is regular in *U*. From (20), we have

$$h'(z)=\left[\frac{g(z)}{z}\right]^{\frac{M}{\alpha}},\quad h''(z)=\frac{M}{\alpha}\left[\frac{g(z)}{z}\right]^{\frac{M}{\alpha}-1}\frac{zg'(z)-g(z)}{z^2},$$

and

$$\frac{zh''(z)}{h'(z)}=\frac{M}{\alpha}\left(\frac{zg'(z)}{g(z)}-1\right),\ \text{for all}\ \ z\in U. \tag{21}$$

Using (16), (17), (18), (21), we calculate

$$\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\left|\frac{zh''(z)}{h'(z)}\right|=\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\frac{M}{|\alpha|}\cdot\left|\frac{zg'(z)}{g(z)}-1\right| \tag{22}$$

$$\leq\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\frac{M^2}{|\alpha|}\cdot\left[\left|\frac{zg'(z)}{g(z)}\right|+1\right]$$

$$=\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\frac{M^2}{|\alpha|}\left[\left|\frac{z^2g'(z)}{g^2(z)}\right|\cdot\left|\frac{g(z)}{z}\right|+1\right]$$

$$\leq\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\frac{M^2}{|\alpha|}\cdot\left[\left|\frac{z^2g'(z)}{g^2(z)}\right|\cdot M+1\right]$$

$$\leq\frac{1-|z|^{\frac{2\operatorname{Re}\alpha}{M}}}{\frac{\operatorname{Re}\alpha}{M}}\cdot\frac{M^2}{|\alpha|}\cdot\left[\left|\frac{z^2g'(z)}{g^2(z)}-1\right|M+M+1\right]$$

$$\leq\frac{M^2}{\operatorname{Re}\alpha|\alpha|}\left[M^3+M^2+M\right]\leq 1.$$

Using (22) , we apply Lemma A. Combining this result with the condition $\operatorname{Re}\beta\geq\operatorname{Re}\alpha$, we apply Lemma B and we have that $H_{\alpha,\beta,M}$ is in the class *S*.

Remark 5. For $M=1$, the result was obtained in Pescar (2006).

Remark 6. For $M=1$, the condition (18) expresses a sufficient condition for univalence of function g (Ozaki & Nunokawa 1972, Lemma C).

Example 1. For $M=1, \frac{2-M}{M}=1, \alpha=4+3i$ and $g \in A$,

$$g(z)=z+\frac{1}{2}z^2, \quad z \in U,$$

with $\operatorname{Re}\alpha = 4 \geq 3$ we obtain

$$|g(z)|=\left|z+\frac{1}{2}z^2\right| \leq |z|+\frac{1}{2}\left|z^2\right| \leq 1+\frac{1}{2}=\frac{3}{2}<\frac{8}{5},$$

$$\left|\frac{z^2(1+z)}{z^2\left(1+\frac{1}{2}z\right)^2}-1\right|=\left|\frac{1+z-1-z-\frac{1}{4}z^2}{\left(1+\frac{1}{2}z\right)^2}\right|=\frac{\frac{1}{4}\left|z^2\right|}{\left|\left(1+\frac{1}{2}z\right)^2\right|}\leq\frac{\frac{1}{4}\left|z^2\right|}{\frac{1}{4}}\leq|z|^2\leq 1.$$

Using Theorem 2, we have

$$F_{(4+3i),1}(z)=\left[(4+3i)\int_0^z u^{3+3i}\left(1+\frac{1}{2}u\right)du\right]^{\frac{4-3i}{25}}$$

$$=\left[z^{4+3i}+\frac{29+3i}{68}z^{5+3i}\right]^{\frac{4-3i}{25}}, \quad z \in U,$$

is in the class *S*.

Example 2. For $M=1, \alpha=4+3i, \beta=5-i$ and $g \in A$,

$$g(z)=z+\frac{1}{2}z^2, \quad z \in U,$$

with $\operatorname{Re}\alpha=4$, $\operatorname{Re}\beta=5, \operatorname{Re}\beta \geq \operatorname{Re}\alpha$, we obtain

$$|g(z)|=\left|z+\frac{1}{2}z^2\right| \leq |z|+\frac{1}{2}\left|z^2\right| \leq 1+\frac{1}{2}=\frac{3}{2}<\frac{8}{5},$$

$$\left|\frac{z^2(1+z)}{z^2\left(1+\frac{1}{2}z\right)^2}-1\right|=\left|\frac{1+z-1-z-\frac{1}{4}z^2}{\left(1+\frac{1}{2}z\right)^2}\right|=\frac{\frac{1}{4}\left|z^2\right|}{\left|\left(1+\frac{1}{2}z\right)^2\right|}\le\frac{\frac{1}{4}\left|z^2\right|}{\frac{1}{4}}\le|z|^2\le 1.$$

Using Theorem 3, we have

$$H_{(4+3i),(5-i),1}(z)=\left[(5-i)\int_0^z u^{4-i}\left(1+\frac{1}{2}u\right)du\right]^{\frac{1}{5-i}}$$

$$=\left[z^{5-i}+\frac{31-i}{74}z^{6-i}\right]^{\frac{5+i}{26}},\quad z\in U,$$

is in the class *S*.

Remark 7. Similar results for other univalent integral operators can also be seen in Oros (2009), Oros *et al.* (2008), and Pescar and Breaz (2008).

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