

## A NON-LOCAL BOUNDARY VALUE PROBLEM WITH INTEGRAL CONDITIONS FOR A SECOND ORDER HYPERBOLIC EQUATION

(Suatu Masalah Nilai Sempadan Tak-Setempat dengan Syarat Kamiran bagi Suatu Persamaan Hiperbolik Peringkat Kedua)

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### ABSTRACT

In this paper, the classic solution of one-dimensional boundary value problem for a hyperbolic equation with non-classic boundary conditions is investigated. For that, the stated problem is reduced to the not-self-adjoint boundary value problem with equivalent boundary condition. Then, using the method of separation of variables, by means of the known spectral problem the given not self-adjoint boundary value problem is reduced to an integral equation. The existence and uniqueness of the integral equation are proved by means of the contraction mappings principle and it is shown that this solution is unique for a not-adjoint boundary value problem. Finally, using the equivalence, the theorem on the existence and uniqueness of a non-local boundary value problem with integral condition is proved.

*Keywords:* Mixed problem; contracted mappings; fixed point; hyperbolic equation

### ABSTRAK

Dalam makalah ini, penyelesaian klasik bagi masalah nilai sempadan matra satu untuk persamaan hiperbolik dengan syarat sempadan tak-klasik dikaji. Untuk itu masalah tersebut diturunkan kepada masalah nilai sempadan tak-swadampingan dengan syarat sempadan yang setara. Dengan menggunakan kaedah pemisahan pemboleh ubah, melalui masalah spektrum yang diketahui masalah nilai sempadan yang tak-swadampingan tersebut diturunkan kepada suatu persamaan kamiran. Kewujudan dan keunikan bagi persamaan kamiran tersebut dibuktikan dengan menggunakan prinsip pemetaan pengecutan dan ditunjukkan bahawa penyelesaian ini adalah unik bagi masalah nilai sempadan tak-dampingan. Akhir sekali, dengan menggunakan kesetaraan, teorem kewujudan dan keunikan bagi masalah nilai sempadan tak-setempat dibuktikan.

*Kata kunci:* Masalah campuran; pemetaan mengecut; titik tetap; persamaan hiperbolik

## 1. Introduction

Consider the equation

$$u_{tt}(x,t) - u_{xx}(x,t) = q(t)u(x,t) + f(x,t) \quad (1)$$

in the domain  $D_T = \{(x,t) : 0 \leq x \leq 1, 0 \leq t \leq T\}$  and state for it a problem with initial conditions

$$u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \quad (0 \leq x \leq 1) \quad (2)$$

and non-local conditions

$$u(0,t) = \beta u(1,t) \quad (0 \leq t \leq T), \quad (3)$$

$$\int_0^1 u(x,t) dx = 0 \quad (0 \leq t \leq T), \quad (4)$$

where  $\beta \neq \pm 1$  is a given number,  $q(t)$ ,  $f(t,x)$ ,  $\varphi(x)$ ,  $\psi(x)$  are the given functions,  $u(x,t)$  is a desired function.

Earlier, the boundary value problems with non-local integral equations were considered in the papers by Beilin (2001), Bouziani (1997) and Pulkina (2004).

Here, for  $\beta = 0$  we have an Ionkin type boundary condition (Ionkin 1977).

**Definition.** Under the classic solution of problem (1)-(4) we understand the function  $u(x,t)$  continuous in a closed domain  $D_T$  together with all its derivatives contained in equation (1), and satisfying all conditions (1)-(4) in the ordinary sense.

The following lemma is proved similarly (Mehraliyev & Yusifov 2009).

**Lemma 1.** Let  $q(t) \in C[0,T]$ ,  $f(x,t) \in C(D_T)$ ,  $\varphi(x), \psi(x) \in C[0,1]$ ,  $\int_0^1 f(x,t) dx = 0$  and the following agreement conditions be fulfilled:

$$\begin{aligned} \varphi(0) - \beta \varphi(1) = 0, \quad \int_0^1 \varphi(x) dx = 0, \\ \psi(0) - \beta \psi(1) = 0, \quad \int_0^1 \psi(x) dx = 0. \end{aligned} \quad (5)$$

Then the problem on finding the classic solution of problem (1)-(4) is equivalent to the problem on defining of the function  $u(x,t)$  from (1)-(3) and

$$u_x(0,t) = u_x(1,t) \quad (0 \leq t \leq T). \quad (6)$$

## 2. Auxiliary Facts

Now, in order to investigate problem (1)-(3), (6) we cite some known facts.

Consider the following spectral problem (Ionkin 1977; Kasumov & Mirzoyev 2007):

$$X''(x) + \lambda X(x) = 0 \quad (0 \leq x \leq 1), \quad (7)$$

$$X(0) = \beta X(1), \quad X'(0) = X'(1) \quad (\beta \neq \pm 1). \quad (8)$$

Boundary value problem (7), (8) is not self-adjoint. The problem

$$Y''(x) + \lambda Y(x) = 0 \quad (0 \leq x \leq 1), \quad (9)$$

$$Y(0) = Y(1), \quad Y'(1) = \beta Y'(0), \quad (10)$$

will be a conjugated problem.

We denote the system of eigen and adjoint functions of problem (10), (11) in the following way (Kasumov & Mirzoyev 2007):

$$X_0(x) = ax + b, \dots, X_{2k-1}(x) = (ax + b) \cos \lambda_k x, X_{2k}(x) = \sin \lambda_k x, \dots, \quad (11)$$

where

$$\lambda_k = 2k\pi \quad (k = 0, 1, 2, \dots), \quad a = (1 - \beta)/(1 + \beta) \neq 0, \quad b = \beta/(1 + \beta). \quad (12)$$

We choose the system of eigen and adjoint functions of the conjugated problem as follows (Kasumov & Mirzoyev 2007):

$$Y_0(x) = 2, \dots, Y_{2k-1}(x) = 4 \cos \lambda_k x, Y_{2k}(x) = 4(1 - b - ax) \sin \lambda_k x, \dots \quad (13)$$

It is directly verified that the biorthogonality conditions

$$(X_i, Y_j) = \int_0^1 X_i(x) Y_j(x) dx = \delta_{ij}$$

are fulfilled.

Here,  $\delta_{ij}$  is the Kronecker symbol.

The following theorem is valid.

**Theorem 1** (Mehraliyev & Yusifov 2009). *The system of functions (11) forms a Riesz basis in the space  $L_2(0,1)$  and the estimates*

$$r \|g(x)\|_{L_2(0,1)} \leq \sum_{k=0}^{\infty} g_k^2 \leq R \|g(x)\|_{L_2(0,1)}, \quad (14)$$

where

$$g_k = (g(x), Y_k(x)) = \int_0^1 g(x) Y_k(x) dx, \quad (k = 0, 1, \dots)$$

$$r = \left\{ \frac{1}{3} \left( \left( a + \frac{3}{2}b \right)^2 + \frac{3}{4}b^2 \right) + \frac{1}{2} \left( 1 + \left\| (ax+b)^2 \right\|_{C[0,1]} \right) \right\}^{-1},$$

$$R = 8 \left( 1 + \left\| (1-b-ax)^2 \right\|_{C[0,1]} \right),$$

are valid for any function  $g(x) \in L_2(0,1)$ .

Under the assumptions

$$g(x) \in C^{2i-1}[0,1], \quad g^{(2i)}(x) \in L_2(0,1),$$

$$g^{(2s)}(0) = \beta g^{(2s)}(1), \quad g^{(2s+1)}(0) = g^{(2s+1)}(1) \quad (s = 0, \overline{i-1}; i \geq 1)$$

we establish the validity of the estimates:

$$\left( \sum_{k=1}^{\infty} (\lambda_k^{2i} g_{2k-1})^2 \right)^{\frac{1}{2}} \leq 2\sqrt{2} \left\| g^{(2i)}(x) \right\|_{L_2(0,1)}, \quad (15)$$

$$\left( \sum_{k=1}^{\infty} (\lambda_k^{2i} g_{2k})^2 \right)^{\frac{1}{2}} \leq 2\sqrt{2} \left\| g^{(2i)}(x)(1-b-ax) - 2aig^{(2i-1)}(x) \right\|_{L_2(0,1)}. \quad (16)$$

Further, under the assumptions

$$g(x) \in C^{2i}[0,1], \quad g^{(2i+1)}(x) \in L_2(0,1),$$

$$g^{(2s)}(0) = \beta g^{(2s)}(1), \quad g^{(2s-1)}(0) = g^{(2s-1)}(1) \quad (i \geq 1; s = \overline{0, i}).$$

We prove the validity of the estimates:

$$\left( \sum_{k=1}^{\infty} (\lambda_k^{2i+1} g_{2k-1})^2 \right)^{\frac{1}{2}} \leq 2\sqrt{2} \left\| g^{(2i+1)}(x) \right\|_{L_2(0,1)}, \quad (17)$$

$$\left( \sum_{k=1}^{\infty} (\lambda_k^{2i+1} g_{2k})^2 \right)^{\frac{1}{2}} \leq 2\sqrt{2} \left\| g^{(2i+1)}(x)(1-b-ax) - a(2i+1)g^{(2i)}(x) \right\|_{L_2(0,1)}. \quad (18)$$

Now, denote by  $B_{2,T}^\alpha$  (Khudaverdiyev & Azizbekov 2002) an aggregate of all the functions of the form

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) X_k(x),$$

considered in  $D_T$ , where each of the functions from  $u_k(t)$ , ( $k = 0, 1, 2, \dots$ ) is continuous on  $[0, T]$  and

$$J(u) \equiv \|u_0(t)\|_{C[0,T]} + \left( \sum_{k=1}^{\infty} (\lambda_k^\alpha \|u_{2k-1}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} + \left( \sum_{k=1}^{\infty} (\lambda_k^\alpha \|u_{2k}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} < +\infty,$$

where  $\alpha \geq 0$ . The norm in this set is defined as follows:

$$\|u(x,t)\|_{B_{2,T}^\alpha} = J(u).$$

It is known that  $B_{2,T}^\alpha$  is a Banach space.

### 3. Existence and Uniqueness of the Solution of the Boundary Value Problem

Since the system (11) forms a Riesz basis in  $L_2(0,1)$  and systems (11), (13) form a system of functions biorthogonal in  $L_2(0,1)$ , each solution of problem (1)-(3), (6) has the form:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t) X_k(x), \quad (19)$$

where

$$u_k(t) = \int_0^1 u(x,t) Y_k(x) dx, \quad (k = 0, 1, \dots). \quad (20)$$

Moreover,  $X_k(x)$  and  $Y_k(x)$  are defined by relations (11) and (13) respectively.

Applying the method of separation of variables for determining the desired functions  $u_k(t)$  ( $k = 0, 1, \dots$ ), from (1), (2) we have:

$$u_0''(t) = q(t)u_0(t) + f_0(t), \quad (0 \leq t \leq T) \quad (21)$$

$$u_{2k-1}''(t) + \lambda_k^2 u_{2k-1}(t) = q(t)u_{2k-1}(t) + f_{2k-1}(t), \quad (k = 1, 2, \dots; 0 \leq t \leq T), \quad (22)$$

$$u_{2k}''(t) + \lambda_k^2 u_{2k}(t) = q(t)u_{2k}(t) + f_{2k}(t) - 2a\lambda_k u_{2k-1}(t), \quad (k = 1, 2, \dots; 0 \leq t \leq T), \quad (23)$$

$$u_k(0) = \varphi_k, \quad u_k'(0) = \psi_k, \quad (k = 0, 1, \dots), \quad (24)$$

where

$$f_k(t) = \int_0^1 f(x,t)Y_k(x)dx, \quad \varphi_k = \int_0^1 \varphi(x)Y_k(x)dx, \quad \psi_k = \int_0^1 \psi(x)Y_k(x)dx, \quad (k=0,1,\dots).$$

Solving problem (21)-(24), we have:

$$u_0(t) = \varphi_0 + t\psi_0 + \int_0^t (t-\tau)F_0(\tau;u)d\tau \quad (0 \leq t \leq T), \quad (25)$$

$$u_{2k-1}(t) = \varphi_{2k-1} \cos \lambda_k t + \frac{1}{\lambda_k} \psi_{2k-1} \sin \lambda_k t + \frac{1}{\lambda_k} \int_0^t F_{2k-1}(\tau;u) \sin \lambda_k (t-\tau) d\tau, \quad (k=1,2,\dots; 0 \leq t \leq T), \quad (26)$$

$$u_{2k}(t) = \varphi_{2k} \cos \lambda_k t + \frac{1}{\lambda_k} \psi_{2k} \sin \lambda_k t + \frac{1}{\lambda_k} \int_0^t F_{2k}(\tau;u) \sin \lambda_k (t-\tau) d\tau - at\varphi_{2k-1} \sin \lambda_k t - \frac{a}{\lambda_k} \left( \frac{1}{\lambda_k} \sin \lambda_k t - t \cos \lambda_k t \right) \psi_{2k-1} - \frac{2a}{\lambda_k} \int_0^t \left( \int_0^\tau F_{2k-1}(\xi;u) \sin \lambda_k (\tau-\xi) d\xi \right) \sin \lambda_k (t-\tau) d\tau, \quad (k=1,2,\dots; 0 \leq t \leq T), \quad (27)$$

where

$$F_k(t;u) = f_k(t) + q(t)u_k(t), \quad (k=0,1,2,\dots).$$

After substitution of expressions  $u_0(t)$ ,  $u_{2k-1}(t)$ ,  $u_{2k}(t)$  of (25), (26), (27), respectively in (19) we have:

$$u(x,t) = \left( \varphi_0 + t\psi_0 + \int_0^t (t-\tau)F_0(\tau;u)d\tau \right) X_0(x) + \sum_{k=1}^{\infty} \left( \varphi_{2k-1} \cos \lambda_k t + \frac{1}{\lambda_k} \psi_{2k-1} \sin \lambda_k t + \frac{1}{\lambda_k} \int_0^t F_{2k-1}(\tau;u) \sin \lambda_k (t-\tau) d\tau \right) X_{2k-1}(x) + \sum_{k=1}^{\infty} \left( \varphi_{2k} \cos \lambda_k t + \frac{1}{\lambda_k} \psi_{2k} \sin \lambda_k t + \frac{1}{\lambda_k} \int_0^t F_{2k}(\tau;u) \sin \lambda_k (t-\tau) d\tau - at\varphi_{2k-1} \sin \lambda_k t - \frac{a}{\lambda_k} \left( \frac{1}{\lambda_k} \sin \lambda_k t - t \cos \lambda_k t \right) \psi_{2k-1} - \frac{2a}{\lambda_k} \int_0^t \left( \int_0^\tau F_{2k-1}(\xi;u) \sin \lambda_k (\tau-\xi) d\xi \right) \sin \lambda_k (t-\tau) d\tau \right) X_{2k}(x)$$

$$-\frac{2a}{\lambda_k} \cdot \int_0^t \left( \int_0^\tau F_{2k-1}(\xi; u) \sin \lambda_k(\tau - \xi) d\xi \right) \sin \lambda_k(t - \tau) d\tau \Big) X_{2k}(x). \quad (28)$$

Now, proceeding from definition of the solution of problem (1)-(3), (6) similar to (Khudaverdiyev & Azizbekov 2002), the following lemma is proved.

**Lemma 2.** *If  $u(x, t) = \sum_{k=0}^{\infty} u_k(t) X_k(x)$  is any solution of problem (1)-(3), (6), the functions*

$u_k(t)$ , ( $k = 0, 1, 2, \dots$ ) *satisfy system (25)-(27).*

**Theorem 2.** *Let*

(1)  $q(t) \in C[0, T]$ ,  $\beta \neq \pm 1$ ;

(2)  $\varphi(x) \in C^2[0, 1]$ ,  $\varphi'''(x) \in L_2(0, 1)$ ,  $\varphi(0) = \beta\varphi(1)$ ,  $\varphi'(0) = \varphi'(1)$ ,  $\varphi''(0) = \beta\varphi''(1)$ ;

(3)  $\psi(x) \in C^1[0, 1]$ ,  $\psi''(x) \in L_2(0, 1)$ ,  $\psi(0) = \beta\psi(1)$ ,  $\psi'(0) = \psi'(1)$ ;

(4)  $f(x, t), f_x(x, t) \in C(D_T)$ ,  $f_{xx}(x, t) \in L_2(D_T)$ ,

$$f(0, t) = \beta f(1, t), f_x(0, t) = f_x(1, t), (0 \leq t \leq T).$$

*Then problem (1)-(3), (6) under small values of  $T$  has a unique classic solution.*

**Proof.** Denoting

$$P u = \sum_{k=0}^{\infty} P_k(t; u) X_k(x),$$

where  $P_0(t; u)$ ,  $P_{2k-1}(t; u)$ ,  $P_{2k}(t; u)$  equal the right hand sides of (25), (26), (27), respectively and we write equation (28) in the form:

$$u = P u. \quad (29)$$

We will study equation (29) in the space  $B_{2,T}^3$ .

It is easy to see that

$$\|P_0(t; u)\|_{C[0,T]} \leq |\phi_0| + T |\psi_0| + T \sqrt{T} \left( \int_0^T |f_0(\tau)|^2 d\tau + T^2 \|q(t)\|_{C[0,T]} \|u_0(t)\|_{C[0,T]} \right).$$

$$\begin{aligned}
 \left( \sum_{k=1}^{\infty} (\lambda_k^3 \|\mathbf{P}_{2k-1}(t; u)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} &\leq 2 \left( \sum_{k=1}^{\infty} (\lambda_k^3 |\phi_{2k-1}|)^2 \right)^{\frac{1}{2}} + 2 \left( \sum_{k=1}^{\infty} (\lambda_k^2 |\psi_{2k-1}|)^2 \right)^{\frac{1}{2}} + \\
 &\quad + 2\sqrt{T} \left( \int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 |f_{2k-1}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} + 2T \|q(t)\|_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^2 \|u_{2k-1}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}}, \\
 \left( \sum_{k=1}^{\infty} (\lambda_k^3 \|\mathbf{P}_{2k}(t; u)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} &\leq 2\sqrt{2} \left( \sum_{k=1}^{\infty} (\lambda_k^3 |\phi_{2k}|)^2 \right)^{\frac{1}{2}} + 2\sqrt{2} \left( \sum_{k=1}^{\infty} (\lambda_k^3 |\psi_{2k}|)^2 \right)^{\frac{1}{2}} + \\
 &\quad + 2\sqrt{2T} \left( \int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 |f_{2k}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} + 2\sqrt{2}T \|q(t)\|_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^2 \|u_{2k}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} + \\
 &\quad + 2\sqrt{2}aT \left( \sum_{k=1}^{\infty} (\lambda_k^3 |\phi_{2k-1}|)^2 \right)^{\frac{1}{2}} + 2\sqrt{2}a(1+T) \left( \sum_{k=1}^{\infty} (\lambda_k^2 |\psi_{2k-1}|)^2 \right)^{\frac{1}{2}} + \\
 &\quad + 4\sqrt{2}aT\sqrt{T} \left( \int_0^T \sum_{k=1}^{\infty} (\lambda_k^2 |f_{2k-1}(\tau)|)^2 d\tau \right)^{\frac{1}{2}} + \\
 &\quad + 4\sqrt{2}aT^2 \|q(t)\|_{C[0,T]} \left( \sum_{k=1}^{\infty} (\lambda_k^2 \|u_{2k-1}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}}.
 \end{aligned}$$

Here, allowing for (15)-(18), we have:

$$\begin{aligned}
 \|\mathbf{P}_0(t; u)\|_{C[0,T]} &\leq a\|\varphi(x)\|_{L_2(0,1)} + aT\|\psi(x)\|_{L_2(0,1)} + \\
 &\quad + T\sqrt{T}a\|f(x, t)\|_{L_2(D_T)} + T^2\|q(t)\|_{C[0,T]}\|u(x, t)\|_{B_{2,T}^3}, \quad (30)
 \end{aligned}$$

$$\begin{aligned}
 \left( \sum_{k=1}^{\infty} (\lambda_k^3 \|\mathbf{P}_{2k-1}(t; u)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} &\leq 4\sqrt{2}\|\varphi'''(x)\|_{L_2(0,1)} + 4\sqrt{2}\|\psi''(x)\|_{L_2(0,1)} + \\
 &\quad + 4\sqrt{2T}\|f_{xx}(x, t)\|_{L_2(D_T)} + 2T\|q(t)\|_{C[0,T]}\|u(x, t)\|_{B_{2,T}^3}, \quad (31)
 \end{aligned}$$

$$\left( \sum_{k=1}^{\infty} (\lambda_k^3 \|\mathbf{P}_{2k}(t; u)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq 8\|\varphi'''(x)(1-b-ax) - 3a\varphi''(x)\|_{L_2(0,1)} +$$



$$\begin{aligned}
 & + 8\|\psi''(x)(1-b-ax) - 2a\psi'(x)\|_{L_2(0,1)} + \\
 & + 8\sqrt{T}\|f_{xx}(x,t)(1-b-ax) - 2af_x(x,t)\|_{L_2(D_T)} + \\
 & + 8aT\|\phi'''(x)\|_{L_2(0,1)} + 8a(1+T)\|\psi''(x)\|_{L_2(0,1)} + \\
 & + 8aT\sqrt{T}\|f_{xx}(x,t)\|_{L_2(D_T)} + \\
 & + 2\sqrt{2}T(1+aT)\|q(t)\|_{C[0,T]}\|u(x,t)\|_{B_{2,T}^3}, \tag{32}
 \end{aligned}$$

Now, consider the operator  $\mathbf{P}$  in the sphere

$$K = K_R(\|u\|_{B_{2,T}^3} \leq A(T)+1) \text{ from } B_{2,T}^3.$$

where

$$\begin{aligned}
 A(T) = & a\|\phi(x)\|_{L_2(0,1)} + aT\|\psi(x)\|_{L_2(0,1)} + T\sqrt{T}a\|f(x,t)\|_{L_2(D_T)} + 4(\sqrt{2} + 2aT)\|\phi'''(x)\|_{L_2(0,1)} + \\
 & + 4(\sqrt{2} + 2a(1+T))\|\psi''(x)\|_{L_2(0,1)} + 4\sqrt{T}(\sqrt{2} + aT)\|f_{xx}(x,t)\|_{L_2(D_T)} + \\
 & + 8\|\phi'''(x)(1-b-ax) - 3a\phi''(x)\|_{L_2(0,1)} + 8\|\phi''(x)(1-b-ax) - 2a\phi'(x)\|_{L_2(0,1)} + \\
 & + 8\sqrt{T}\|f_{xx}(x,t)(1-b-ax) - 2af_x(x,t)\|_{L_2(D_T)}. \tag{33}
 \end{aligned}$$

It is seen from (30)-(32) that for any  $u, u_1, u_2 \in K_R$  the estimates :

$$\|\mathbf{P} u\|_{B_{2,T}^3} \leq A(T) + B(T)\|u(x,t)\|_{B_{2,T}^3}, \tag{34}$$

$$\|\mathbf{P} u_1 - \mathbf{P} u_2\|_{B_{2,T}^3} \leq B(T)\|u(x,t)\|_{B_{2,T}^3}, \tag{35}$$

where

$$B(T) = T((1 + 2\sqrt{2}a)T + 2(1 + \sqrt{2}))\|q(t)\|_{C[0,T]}, \tag{36}$$

are valid.

Then it follows from estimates (34), (35) that under sufficiently small values of  $T$  the operator  $\mathbf{P}$  acts in the sphere  $K = K_R$  from  $B_{2,T}^3$  and it is contractive. Therefore, in the sphere  $K = K_R$  the operator  $\mathbf{P}$  has a unique fixed point  $\{u\}$ , that is a solution of equation (29).

The function  $u(x, t)$ , as an element of the space  $B_{2,T}^3$ , is continuous and has continuous derivatives  $u_x(x, t)$ ,  $u_{xx}(x, t)$  on  $D_T$ . Now, prove that  $u_t(x, t)$  and  $u_{tt}(x, t)$  are continuous in  $D_T$ . From (25)-(27) we have:

$$u'_0(t) = \psi_0 + \int_0^t F_0(\tau; u) d\tau, \quad (25')$$

$$u'_{2k-1}(t) = -\lambda_k \varphi_{2k-1} \sin \lambda_k t + \psi_{2k-1} \cos \lambda_k t + \int_0^t F_{2k-1}(\tau; u) \cos \lambda_k (t - \tau) d\tau, \quad (26')$$

$$u'_{2k}(t) = -\lambda_k \varphi_{2k} \sin \lambda_k t + \psi_{2k} \cos \lambda_k t + \int_0^t F_{2k}(\tau; u) \cos \lambda_k (t - \tau) d\tau - at\varphi_{2k-1}\lambda_k \cos \lambda_k t - a \left( \frac{1}{\lambda_k} \cos \lambda_k t + t \sin \lambda_k t \right) \psi_{2k-1} - 2a \int_0^t \left( \int_0^\tau F_{2k-1}(\xi; u) \sin \lambda_k (\tau - \xi) d\xi \right) \cos \lambda_k (t - \tau) d\tau. \quad (27')$$

$$u''_0(t) = F_0(\tau; u), \quad (25'')$$

$$u''_{2k-1}(t) = -\lambda_k^2 \varphi_{2k-1} \cos \lambda_k t - \lambda_k \psi_{2k-1} \sin \lambda_k t - \lambda_k \int_0^t F_{2k-1}(\tau; u) \sin \lambda_k (t - \tau) d\tau + F_{2k-1}(t; u), \quad (26'')$$

$$u''_{2k}(t) = -\lambda_k^2 \varphi_{2k} \cos \lambda_k t - \lambda_k \psi_{2k} \sin \lambda_k t - \lambda_k \int_0^t F_{2k-1}(\tau; u) \sin \lambda_k (t - \tau) d\tau + F_{2k-1}(t; u) + at\varphi_{2k-1}\lambda_k^2 \sin \lambda_k t - a\lambda_k \left( -\frac{1}{\lambda_k} \sin \lambda_k t + t \cos \lambda_k t \right) \psi_{2k-1} - 2a\lambda_k \int_0^t \left( \int_0^\tau F_{2k-1}(\xi; u) \sin \lambda_k (\tau - \xi) d\xi \right) \sin \lambda_k (t - \tau) d\tau - 2a \int_0^t F_{2k-1}(\tau; u) \sin \lambda_k (t - \tau) d\tau. \quad (27'')$$

Here, by (15)-(18) we have:

$$\|u'_0(t)\|_{C[0,T]} \leq 2\|\psi(x)\|_{L_2(0,1)} + 2\sqrt{T}\|f(x, t) + q(t)u(x, t)\|_{L_2(D_T)}, \quad (37)$$

$$\left( \sum_{k=1}^{\infty} (\lambda_k^2 \|u'_{2k-1}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq 2\sqrt{6}\|\varphi'''(x)\|_{L_2(0,1)} + 2\sqrt{6}\|\psi''(x)\|_{C[0,T]} +$$

$$+ 2\sqrt{6}\sqrt{T}\|f_{xx}(x,t) + q(t)u_{xx}(x,t)\|_{L_2(D_T)}, \quad (38)$$

$$\begin{aligned} & \left( \sum_{k=1}^{\infty} (\lambda_k^2 \|u'_{2k}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq 4\sqrt{3}\|\varphi'''(x)(1-b-ax) - 3a\varphi''(x)\|_{L_2(0,1)} + \\ & + 4\sqrt{3}\|\psi''(x)(1-b-ax) - 2a\psi'(x)\|_{L_2(0,1)} + \\ & + 4\sqrt{3T}\|(f_{xx}(x,t) + q(t)u_{xx}(x,t))(1-b-ax) - 3a(f_x(x,t) + q(t)u_x(x,t))\|_{L_2(D_T)} + \\ & + 4\sqrt{3aT}\|\varphi'''(x)\|_{L_2(0,1)} + 4\sqrt{3a(1+T)}\|\psi''(x)\|_{L_2(0,1)} + \\ & + 8\sqrt{3aT}\|f_{xx}(x,t) + q(t)u_{xx}(x,t)\|_{L_2(D_T)}. \end{aligned} \quad (39)$$

$$\|u'_0(t)\|_{C[0,T]} \leq 2\|f(x,t) + q(t)u(x,t)\|_{C[0,T]} \Big|_{L_2(0,1)}, \quad (40)$$

$$\begin{aligned} & \left( \sum_{k=1}^{\infty} (\lambda_k \|u''_{2k-1}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq 4\sqrt{2}\|\varphi'''(x)\|_{L_2(0,1)} + 4\sqrt{2}\|\psi''(x)\|_{L_2(0,1)} + \\ & + 4\sqrt{2T}\|f_{xx}(x,t) + q(t)u_{xx}(x,t)\|_{L_2(D_T)} + 4\sqrt{2}\|f_x(x,t) + q(t)u_x(x,t)\|_{C[0,T]} \Big|_{L_2(0,1)}, \end{aligned} \quad (41)$$

$$\begin{aligned} & \left( \sum_{k=1}^{\infty} (\lambda_k \|u''_{2k}(t)\|_{C[0,T]})^2 \right)^{\frac{1}{2}} \leq 8\|\varphi'''(x)(1-b-ax) - 3a\varphi''(x)\|_{L_2(0,1)} + \\ & + 8\|\psi''(x)(1-b-ax) - 2a\psi'(x)\|_{L_2(0,1)} + \\ & + 8\sqrt{T}\|(f_{xx}(x,t) + q(t)u_{xx}(x,t))(1-b-ax) - 2a(f_x(x,t) + q(t)u_x(x,t))\|_{L_2(D_T)} + \\ & + 8\|(f_x(x,t) + q(t)u_x(x,t))(1-b-ax)\|_{C[0,T]} \Big|_{L_2(D_T)} + 8aT\|\varphi'''(x)\|_{L_2(0,1)} + \\ & + 8a(1+T)\|\psi''(x)\|_{L_2(0,1)} + 16aT\sqrt{T}\|f_{xx}(x,t) + q(t)u_{xx}(x,t)\|_{L_2(D_T)} + \\ & + 16a\sqrt{T}\|f_x(x,t) + q(t)u_x(x,t)\|_{L_2(D_T)}. \end{aligned} \quad (42)$$

It follows from estimates (37)-(39) that  $u_t(x,t)$  is continuous in  $D_T$ , and from the estimates (40)-(42) that  $u_{tt}(x,t)$  is continuous in  $D_T$ .

Further, it follows from systems (25)-(27) and (25')-(27') that

$$u(x,0) = \sum_{k=0}^{\infty} u_k(0)X_k(x) = \sum_{k=0}^{\infty} \varphi_k X_k(x) = \varphi(x) \quad (0 \leq x \leq 1),$$

$$u_t(x,0) = \sum_{k=0}^{\infty} u'_k(0)X_k(x) = \sum_{k=0}^{\infty} \psi_k X_k(x) = \psi(x) \quad (0 \leq x \leq 1),$$

since by the given theorem

$$\sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_{2k-1}|)^2 < +\infty, \quad \sum_{k=1}^{\infty} (\lambda_k^3 |\varphi_{2k}|)^2 < +\infty,$$

$$\sum_{k=1}^{\infty} (\lambda_k^2 |\psi_{2k-1}|)^2 < +\infty, \quad \sum_{k=1}^{\infty} (\lambda_k^2 |\psi_{2k}|)^2 < +\infty,$$

and the more so,  $\sum_{k=1}^{\infty} |\varphi_k| < +\infty$ ,  $\sum_{k=1}^{\infty} |\psi_k| < +\infty$ .

Thus, conditions (2) are fulfilled.

It is obvious that conditions (3) is fulfilled for the function

$$u(x,t) = \sum_{k=0}^{\infty} u_k(t)X_k(x) = u_0(t)X_0(x) + \sum_{k=1}^{\infty} u_{2k-1}(t)X_{2k-1}(x) + \sum_{k=1}^{\infty} u_{2k}(t)X_{2k}(x).$$

It is easy to see that

$$\begin{aligned} u_{tt}(x,t) - u_{xx}(x,t) &= u_0''(t) + \sum_{k=1}^{\infty} [u_{2k-1}''(t) + \lambda_k^2 u_{2k-1}(t)]X_{2k-1}(x) + \\ &+ \sum_{k=1}^{\infty} [u_{2k}''(t) + \lambda_k^2 u_{2k}(t) + 2a\lambda_k u_{2k-1}(t)]X_{2k}(x) \quad (k=1,2,\dots; 0 \leq t \leq T). \end{aligned} \quad (43)$$

Now, if we use systems (25)-(27) and (25'')-(27''), equality (43) takes the form:

$$u_{tt}(x,t) - u_{xx}(x,t) = F_0(t;u) + \sum_{k=1}^{\infty} F_{2k-1}(t;u)X_{2k-1}(x) +$$

$$+ \sum_{k=1}^{\infty} F_{2k}(t;u) X_{2k}(x) = \sum_{k=0}^{\infty} F_k(t;u) X_k(x), \quad (44)$$

where the functions  $X_k(x)$  ( $k = 0,1,2,\dots$ ) are determined by relation (11), and

$$F_k(t;u) = f_k(t) + q(t)u_k(t) \quad (k = 0,1,2,\dots).$$

Under the conditions of the theorem it is obvious that

$$\sum_{k=0}^{\infty} |F_k(t;u)| < +\infty. \quad (45)$$

Then it follows from (45) that for any fixed  $t \in [0, T]$ :

$$\sum_{k=0}^{\infty} F_k(t;u) X_k(x) = F(u(x,t)) \equiv q(t)u(x,t) + f(x,t) \quad \forall x \in [0,1]. \quad (46)$$

Thus, relations (44) and (46) yield

$$u_{tt}(x,t) - u_{xx}(x,t) = q(t)u(x,t) + f(x,t)$$

Consequently, the function  $u(x,t)$  satisfies equation (1) everywhere in  $D_T$ .

So,  $u(x,t)$  is a solution of problem (1)-(3), (6), and by lemma 2 it is unique. The theorem is proved.

By means of lemma 1 we prove the following:

**Theorem 3.** *Let all the conditions of theorem 2 and agreement conditions (5) be fulfilled. Then for sufficiently small values of  $T$ , problem (1)-(3) has a unique classic solution.*

#### 4. Conclusion

The following results have been obtained:

- (1) The existence of the solution of a not self-adjoint boundary value problem for a second order hyperbolic equation has been proved;
- (2) The uniqueness of the solution of a not self-adjoint boundary value problem for a second order hyperbolic equation has been shown;
- (3) The existence of the classic solution of a non-classic boundary value problem with integral boundary for a second order hyperbolic equation has been proved;
- (4) The uniqueness of the classic solution of a non-classic boundary value problem with integral boundary for a second order hyperbolic equation has been shown.

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