Journal of Quality Measurement and Analysis *JQMA* 7(1) 2011, 41-51
Jurnal Pengukuran Kualiti dan Analisis

PROBLEMS AND PROPERTIES OF A NEW DIFFERENTIAL OPERATOR

(Masalah dan Sifat-sifat suatu Pengoperasi Pembeza Baharu)

MASLINA DARUS[1] & IMRAN FAISAL[2]

ABSTRACT

In this paper, we introduce and study a new differential operator defined in the open unit disc $U=\{z:z\in C\wedge|z|<1\}$. Using this operator, we then introduce a new subclass of analytic functions $G_n(\mu,\lambda,\alpha,\beta,b)$. Moreover, we discuss coefficient estimates, growth and distortion theorems and inclusion properties for the functions belonging to the class $G_n(\mu,\lambda,\alpha,\beta,b)$.

Keywords: Analytic functions; convex functions; differential operator

ABSTRAK

Dalam makalah ini, pengoperasi pembeza baharu dalam cakera unit $U=\{z:z\in C\wedge|z|<1\}$ diperkenalkan dan dikaji. Dengan menggunakan pengoperasi ini, subkelas baru fungsi analisis $G_n(\mu,\lambda,\alpha,\beta,b)$ diperkenalkan. Malah anggaran pekali, teorem pertumbuhan dan erotan, dan sifat rangkuman untuk kelas $G_n(\mu,\lambda,\alpha,\beta,b)$ turut dibincangkan.

Kata kunci: Fungsi analisis; fungsi cembung; pengoperasi pembeza

1. Introduction and Preliminaries

Let A denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k, \tag{1}$$

which are analytic and normalised (in usual sense) in the open unit disc $U=\{z:z\in C\wedge|z|<1\}\cdot$ For a function f in A, we define the following differential operator:

$$D^0 f(z)=f(z); \tag{2}$$

$$D^1 f(z)=\left(\frac{\alpha-\beta-\lambda}{\alpha}\right)f(z)+\left(\frac{\beta+\lambda}{\alpha}\right)zf'(z); \tag{3}$$

$$\vdots$$

$$D^n f(z)=D_\lambda(D^{n-1}f(z))\cdot \tag{4}$$

If f is given by (1), then from (4) we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\alpha + (\beta + \lambda)(k-1)}{\alpha}\right)^n a_k z^k \tag{5}$$

where $f \in A, n \in N_0$. This generalises many operators as follows.

(i) When $\alpha = 1, \beta = 0,$ we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} (1 + \lambda(k-1))^n a_k z^k$$

the so-called Al-Oboudi (2004) differential operator.

(ii) When $\alpha = 1, \beta = 0$ and $\lambda = 1,$ we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$$

the Sălăgean's (1983) differential operator.

(iii) When $\alpha = 2, \beta = 0$ and $\lambda = 1,$ we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+1}{2}\right)^n a_k z^k$$

a differential operator given by Uralegaddi and Somanatha (1992).

(iv) When $\beta = 1, \lambda = 0$ and replacing α by $\alpha + 1,$ we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{k+\alpha}{\alpha}\right)^n a_k z^k$$

the differential operator of Cho and Srivastava (2003).

(v) When $\beta = 0$ and replacing α by $\alpha + 1,$ we get

$$D^n f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\alpha + \lambda(k-1) + 1}{\alpha + 1}\right)^n a_k z^k$$

a well known differential operator of Aouf *et al*. (2009).

Let $G_n(\mu,\lambda,\alpha,\beta,b)$ denote the subclass of A consisting of functions f which satisfy

$$\operatorname{Re}\{1+\frac{1}{b}[(1-\mu)\frac{D^n f(z)}{z}+\mu(D^n f(z))'-1]\}>0, \tag{6}$$

where $D^n f(z)$ is given by (5).

This implies that it satisfies the following inequality

$$\left|\frac{(1-\mu)\frac{D^n f(z)}{z}+\mu(D^n f(z))'-1}{(1-\mu)\frac{D^n f(z)}{z}+\mu(D^n f(z))'-1+2b}\right|<1 \tag{7}$$

where $z\in U;\mu\geq 0;n\in N_0;b\in C-\{0\}$.

We note that

(*i*) $G_0(\mu,\lambda,\alpha,\beta,b)=G(\lambda,b)$

$$\operatorname{Re}\{f\in A:\operatorname{Re}\{1+\frac{1}{b}[(1-\mu)\frac{f(z)}{z}+\mu(f(z))'-1]\},z\in U\}>0,$$

(*ii*) $G_n(0,1,1,0,b)=G_n(b)$

$$\operatorname{Re}\{f\in A:\operatorname{Re}\{1+\frac{1}{b}[\frac{D^n f(z)}{z}-1]\},z\in U\}>0,$$

(*iii*) $G_n(1,1,1,0,b)=R_n(b)$

$$\operatorname{Re}\{f\in A:\operatorname{Re}\{1+\frac{1}{b}[(D^n f(z))'-1]\},z\in U\}>0,$$

(*iv*) $G_0(0,1,1,0,b)=G(b)$

$$\operatorname{Re}\{f\in A:\operatorname{Re}\{1+\frac{1}{b}[\frac{f(z)}{z}-1]\},z\in U\}>0,$$

(*v*) $G_0(1,1,1,0,b)=R(b)$

$$\operatorname{Re}\{f\in A:\operatorname{Re}\{1+\frac{1}{b}[(f(z))'-1]\},z\in U\}>0,$$

$(vi)\ G_0(0,1,1,0,1-\alpha)=G_\alpha$

$\mathrm{Re}\{f\in A:\mathrm{Re}\frac{f(z)}{z}>\alpha,\ 0\le\alpha<1,\ z\in U\}>0,$

$(vii)\ G_0(1,1,1,0,1-\alpha)=R_\alpha$

$\mathrm{Re}\{f\in A:\mathrm{Re}(f(z))'>\alpha,\ 0\le\alpha<1,\ z\in U\}>0\cdot$

The class $R(b)$ was studied by Halim (1999), the class G_α by Chen (1974; 1975) and whereas the class R_α by Ezrohi (1965).

2. Coefficient Inequalities

In this section we find the coefficient inequality for the class $G_n(\mu,\lambda,\alpha,\beta,b)$.

Theorem 1. *Let the function* f *defined by* (1) *satisfies the condition*

$$\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n\,|a_k|\le|b|\ (\mu\ge0,n\in N_0,\alpha>0,\beta+\lambda>0)\cdot \tag{8}$$

Then $f\in G_n(\mu,\lambda,\alpha,\beta,b)\cdot$

Proof. Suppose that the inequality (8) holds. Then we have for $z\in U$

$$|(1-\mu)\frac{D^n f(z)}{z}+\mu(D^n f(z))'-1|-$$

$$|(1-\mu)\frac{D^n f(z)}{z}+\mu(D^n f(z))'+2b-1|$$

$$=|\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n\,|a_k|\,z^{k-1}|-$$

$$|2b+\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n\,|a_k|\,z^{k-1}|$$

$$\le\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n\,|a_k||z^{k-1}|-$$

$$2|b|-\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n|a_k||z^{k-1}|$$
$$\leq\{\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n|a_k||z^{k-1}|$$
$$-|b|\}\leq 0.$$

where $D^n f(z)$ is given by (5).

This implies

$\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n|a_k|\leq|b|,$ which shows that $f\in G_n(\mu,\lambda,\alpha,\beta,b)\cdot$

Corollary 1. *Let the function* f *defined by* (1) *be in the class* $G_n(\mu,\lambda,\alpha,\beta,b)\cdot$ *Then we have*

$$|a_k|\leq\frac{|b|}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n},\ k\geq 2.$$

Corollary 2. *Let the hypotheses of Theorem* 2.1 *be satisfied. Then for* $\beta=\lambda=0$ *and* $\mu=1$ *we have*

$$|a_k|\leq\frac{|b|}{k},\ k\geq 2.$$

3. Growth and Distortion Theorems

A growth and distortion property for function *f* to be in the class $G_n(\mu,\lambda,\alpha,\beta,b)$ is given as follows:

Theorem 2. *If the function f defined by* (1) *is in the class* $G_n(\mu,\lambda,\alpha,\beta,b)$*, then for* $|z|<1,$ *we have*

$$|f(z)|\leq|r|+\frac{|b||r|^k}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n},$$
$$|f(z)|\geq|r|-\frac{|b||r|^k}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n}.$$

Proof. Let $f\in G_n(\mu,\lambda,\alpha,\beta,b),$ then by Theorem 1. We have

$$\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^{n}\,|a_k|\leq|b|$$

$\Longrightarrow$

$$\sum_{k=2}^{\infty}|a_k|\leq\frac{|b|}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^{n}}.$$

From equation (1) we have

$$|f(z)|=\left|z+\sum_{k=2}^{\infty}a_k z^k\right|\leq|z|+\sum_{k=2}^{\infty}|a_k||z^k|.$$

Which implies

$$|f(z)|\leq|z|+\sum_{k=2}^{\infty}|a_k||z^k|,$$

$$|f(z)|\leq r+\frac{|b|}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^{n}}r^k.$$

Similarly we can prove that

$$|f(z)|\geq r-\frac{|b|}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^{n}}r^k.$$

Theorem 3. *Let the hypotheses of Theorem* 1 *be satisfied, then for* $|z|<1$,

$$|f(z)|\leq|r|+\frac{\alpha^n|b||r|^2}{[1+\mu][\alpha+\beta+\lambda]^n}$$

$$|f(z)|\geq|r|-\frac{\alpha^n|b||r|^2}{[1+\mu][\alpha+\beta+\lambda]^n}.$$

Proof. From Theorem 1 we have $f\in G_n(\mu,\lambda,\alpha,\beta,b)$ and hence

$$\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^{n}\,|a_k|\leq|b|.$$

Since

$$[1+\mu][\frac{\alpha+\beta+\lambda}{\alpha}]^n \sum_{k=2}^{\infty} |a_k| \leq \sum_{k=2}^{\infty} [1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n |a_k| \leq |b|$$

we have

$$[1+\mu][\frac{\alpha+\beta+\lambda}{\alpha}]^n \sum_{k=2}^{\infty} |a_k| \leq |b|.$$

From (1) we have

$$|f(z)| \leq |z| + \sum_{k=2}^{\infty} |a_k| |z^k| \leq |z| + \sum_{k=2}^{\infty} |a_k| |z^2|,$$

$$|f(z)| \leq |z| + \sum_{k=2}^{\infty} |a_k| |z^2|.$$

Which proves that

$$|f(z)| \leq |r| + \frac{\alpha^n |b| |r|^2}{[1+\mu][\alpha+\beta+\lambda]^n}.$$

Similarly

$$|f(z)| = \left| z + \sum_{k=2}^{\infty} a_k z^k \right| \geq |z| - \sum_{k=2}^{\infty} |a_k| |z^k|$$

$$|f(z)| \geq |z| - \sum_{k=2}^{\infty} |a_k| |z^2|$$

shows that

$$|f(z)| \geq |r| - \frac{\alpha^n |b| |r|^2}{[1+\mu][\alpha+\beta+\lambda]^n}.$$

Corollary 3. *Let the hypotheses of Theorem* 1 *be satisfied, if* $\alpha=\lambda=\mu=1,\ \beta=0$ *then for* $|z|<1,$ *we have*

$$|f(z)|\leq |r|+\frac{|b||r|^2}{2^{n+1}}$$

$$|f(z)|\geq |r|-\frac{|b||r|^2}{2^{n+1}}.$$

Theorem 4. *If the function f defined by* (1) *is in the class* $G_n(\mu,\lambda,\alpha,\beta,b)$ *, then for* $|z|<1,$ *we have*

$$|f'(z)|\leq 1+\frac{k|b||r|^{k-1}}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n}$$

$$|f'(z)|\geq 1-\frac{k|b||r|^{k-1}}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n}.$$

Proof. Let $f\in G_n(\mu,\lambda,\alpha,\beta,b),$ then by using Theorem 1 we have

$$\sum_{k=2}^{\infty}|a_k|\leq\frac{|b|}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n}.$$

Also

$$|f'(z)|=\left|1+\sum_{k=2}^{\infty}ka_kz^{k-1}\right|\leq 1+k\sum_{k=2}^{\infty}|a_k||z^{k-1}|$$

$$|f'(z)|\leq 1+k\sum_{k=2}^{\infty}|a_k||r^{k-1}|.$$

This shows that

$$|f'(z)|\leq 1+\frac{k|b||r|^{k-1}}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n}.$$

Similarly we can prove that

$$|f'(z)| \geq 1 - \frac{k|b||r|^{k-1}}{[1+\mu(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n}.$$

3. Inclusion properties

The inclusion properties for the class $G_n(\mu,\lambda,\alpha,\beta,b)$ are given by the following theorem.

Theorem 5. *Let the hypotheses of Theorem* 1 *be satisfied. Then*

$$G_n(\mu_2,\lambda,\alpha,\beta,b) \subseteq G_n(\mu_1,\lambda,\alpha,\beta,b)$$
$$G_n(\mu,\lambda_2,\alpha,\beta,b) \subseteq G_n(\mu,\lambda_1,\alpha,\beta,b)$$
$$G_n(\mu,\lambda,\alpha_1,\beta,b) \subseteq G_n(\mu,\lambda,\alpha_2,\beta,b)$$
$$G_n(\mu,\lambda,\alpha,\beta_2,b) \subseteq G_n(\mu,\lambda,\alpha,\beta_1,b)$$

where

$$\alpha_2 \geq \alpha_1, \beta_2 \geq \beta_1, \mu_2 \geq \mu_1 \text{ and } \lambda_2 \geq \lambda_1 \cdot$$

Proof. Let $f \in G_n(\mu_2,\lambda,\alpha,\beta,b)$. Then by using Theorem 1 we have

$$\sum_{k=2}^{\infty}[1+\mu_2(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n |a_k| \leq |b|$$

if $\mu_2 \geq \mu_1$, implying that $1+\mu_2(k-1) \geq 1+\mu_1(k-1)$,

in such that

$$\sum_{k=2}^{\infty}(1+\mu_2(k-1))(\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha})^n \geq \sum_{k=2}^{\infty}(1+\mu_1(k-1))(\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha})^n.$$

This shows that

$$\sum_{k=2}^{\infty}[1+\mu_1(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n |a_k| \leq |b| \leq \sum_{k=2}^{\infty}[1+\mu_2(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n |a_k| \leq |b|$$

or

$$\sum_{k=2}^{\infty}[1+\mu_1(k-1)][\frac{\alpha+(\beta+\lambda)(k-1)}{\alpha}]^n\,|a_k|\leq|b|.$$

Hence $f\in G_n(\mu_1,\lambda,\alpha,\beta,b)$, which shows that $G_n(\mu_2,\lambda,\alpha,\beta,b)\subseteq G_n(\mu_1,\lambda,\alpha,\beta,b)\cdot$

Similarly, let $f\in G_n(\mu,\lambda,\alpha_1,\beta,b)$, then by using Theorem 1 we have $\alpha_2\geq\alpha_1$. This implies that

$$(1+\frac{(\beta+\lambda)(k-1)}{\alpha_1})^n\geq(1+\frac{(\beta+\lambda)(k-1)}{\alpha_2})^n,$$

$$\sum_{k=2}^{\infty}(1+\mu(k-1))(\frac{\alpha_1+(\beta+\lambda)(k-1)}{\alpha_1})^n\geq\sum_{k=2}^{\infty}(1+\mu(k-1))(\frac{\alpha_2+(\beta+\lambda)(k-1)}{\alpha_2})^n$$

$$\sum_{k=2}^{\infty}[1+\mu(k-1)][\frac{\alpha_1+(\beta+\lambda)(k-1)}{\alpha_1}]^k\left|a_k\right|\leq\left|b\right|$$

and hence

$$\sum_{k=2}^{\infty}(1+\mu(k-1))(\frac{\alpha_2+(\beta+\lambda)(k-1)}{\alpha_2})^n\leq\left|b\right|.$$

This proves that $f\in G_n(\mu,\lambda,\alpha_2,\beta,b)$, and finally implies that

$$G_n(\mu,\lambda,\alpha_1,\beta,b)\subseteq G_n(\mu,\lambda,\alpha_2,\beta,b).$$

Employing a similar procedure we can prove that

$$G_n(\mu,\lambda_2,\alpha,\beta,b)\subseteq G_n(\mu,\lambda_1,\alpha,\beta,b)$$

and

$$G_n(\mu,\lambda,\alpha,\beta_2,b)\subseteq G_n(\mu,\lambda,\alpha,\beta_1,b)\cdot$$

For more details about coefficient bounds we refer to Joshi (2007), Aouf (1987), Silverman (1975), Raina (1997), and Owa and Aouf (1989), respectively.

Acknowledgements

The work presented here is fully supported by UKM-ST-06-FRGS0107-2009.

References

Al-Oboudi F.M. 2004. On univalent functions defined by generalized Salagean operator. *Int. J. MatSci.* **25**:1429-1436.

Aouf M.K., El-Ashwah R.M. & El-Deeb S. M. 2009. Some inequalities for certain p-valent functions involving extended multiplier transformations. *Proc. Pakistan Acad. Sci.* **46**: 217-221.

Aouf M.K. 1987. On subclasses of univalent functions with negative coefficients III. *Bull. Soc. Roy. Sci. Liege* **56**: 465-473.

Chen M.P. 1974. On functions satisfying $\mathrm{Re}(\frac{f(z)}{z}) > \alpha$. *Tamking J. Math. Sci. Net.* **23**: 231-234.

Chen M.P. 1975. On the regular functions satisfying $\mathrm{Re}(\frac{f(z)}{z}) > \alpha$. *Bull. Ins. Math. Sci. Net. Acad. Sinica* **3**: 65-70.

Cho N.E. & Kim T.H. 2003. Multiplier transformation -s and strongly close to convex functions. *Bull. Korean Math. Soc.* **40**: 399-410.

Cho N.E. & Srivastava H.M. 2003. Argument estimates of certain analytic functions defined by a certain Multiplier transformations. *Math. Comput. Modeling* **37**: 39-49.

Ezrohi T.G. 1965. Certain estimates in special classes of univalent functions regular in the circle $|z|<1$. *Dopovidi Akademiji Nauk Koji RSR*: 984-988.

Halim S.A. 1999. On a class of functions of complex order. *Tamking J. Math.* **30**: 147-153.

Joshi S.B. 2007. A unified presentation of certain subclass of analytic function with negative coefficient. *Mathematika* **23**(1): 23-28.

Owa S. & Aouf M.K. 1989. On subclasses of univalent functions with the coefficients. *Pure App. Math. .Sci.* **29**:131-139.

Raina R.K. 1997. On certain classes of analytic functions and applications to fractional calculus operator. *Integral Transform and Special Functions* **5**:13-19.

Salagean G.S. 1983. Subclasses of univalent functions. *Lecture notes in mathematics* (*Springer-Verlag*) **1013**: 362-372.

Silverman H. 1975. Univalent functions with negative coefficients. *Proc. Amer. Math. Soc.* **51**:109-116.

Uralegaddi B.A. & Somanatha C. 1992. Certain classes of univalent function. In: Srivastava H.M. & Owa S. (Eds.). *Current Topics in Analytic Function Theory*. Singapore: World Scientific Publishing Company.

[1]Centre for Modelling and Data Analysis (DELTA)
[1,2]School of Mathematical Sciences
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600 UKM Bangi
Selangor DE, Malaysia
E-mail: maslina@ukm.my, imran_faisall@yahoo.com*

* Corresponding author