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OPEN PROBLEM FOR GENERAL CLASS OF SUPERORDINATION-PRESERVING CONVEX INTEGRAL OPERATOR

(Masalah Terbuka untuk Kelas Umum Pengoperasi Kamiran Cembung yang Mengawet Superordinasi)

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ABSTRACT

Let $H(U)$ be the space of analytic functions in the unit disc U. For the integral operator $A_{\alpha,\beta,\gamma}^{\phi,\varphi}[f](z): K \to H(U)$ with $K \subset H(U)$, we will determine sufficient conditions on certain parameters $\alpha, \beta, \gamma, \delta$ for *sandwich-type theorem*. We will also give some particular cases of the main result obtained for appropriate choices of function h, which also generalises classical results of the theory of differential subordination and superordination.

Keywords: Analytic function; starlike and convex function; differential operator; differential subordination

ABSTRAK

Misalkan $H(U)$ suatu ruang fungsi analisis dalam cakera unit U. Diberi pengoperasi kamiran $A_{\alpha,\beta,\gamma}^{\phi,\varphi}[f](z): K \to H(U)$, dengan $K \subset H(U)$, akan ditentukan syarat cukup bagi beberapa parameter $\alpha, \beta, \gamma, \delta$ untuk *teorem jenis-sandwic*. Seterusnya, akan diberi kes-kes penting bagi hasil utama yang diperoleh untuk menentukan pilihan bagi fungsi h, dan juga hasil teori klasik subordinasi dan superordinasi pembeza yang mengitlak.

Kata kunci: Fungsi analisis; fungsi bak bintang dan cembung; pengoperasi pembeza; subordinasi pembeza

1. Introduction

Let $H(U)$ be the space of all analytic functions in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. If $f, F \in H(U)$ and F is univalent in U we say that the function f is subordinate to F, or F is superordinate to f, written $f(z) \prec F(z)$, if $f(0) = F(0)$ and $f(U) \subseteq F(U)$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we denote $H[a,n] = \{f \in H(U) : f(z) = a + a_n z^n + \ldots\}$. Letting $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, $h \in H(U)$ and $q \in H[a,n]$ in Miller and Mocanu (2003), the authors determine conditions on φ such that $h(z) \prec \varphi(p(z), zp'(z), z^2 p''(z); z)$ implies $q(z) \prec p(z)$, for all p functions that satisfy the above superordination. Moreover, they find sufficient conditions so that the q function is the largest function with this property, called the *best* subordinant of this superordination.

For the integral operator $A_{\beta,\gamma} : K_{\beta,\gamma} \to H(U)$, $K_{\beta,\gamma} \subset H(U)$, defined by

$$A_{\beta,\gamma}[f](z) = \left[\frac{\beta+\gamma}{z^\gamma}\int_0^z f^\beta(t) t^{\gamma-1} dt\right]^{1/\beta}, \quad \beta, \gamma \in \mathbb{C}, \tag{1}$$

the author determined in Bulboacă (1997) the conditions on g, β and γ so that

$$z\left[\frac{f(z)}{z}\right]^{\beta}\prec z\left[\frac{g(z)}{z}\right]^{\beta}\quad\text{implies}\quad z\left[\frac{A_{\beta,\gamma}[f](z)}{z}\right]^{\beta}\prec z\left[\frac{A_{\beta,\gamma}[g](z)}{z}\right]^{\beta},$$

and this result was further improved in Bulboacă (2002a). We also studied in Bulboacă (2002b) the reverse problem, to find sufficient conditions on g, β and γ such that the next superordination holds:

$$z\left[\frac{g(z)}{z}\right]^{\beta}\prec z\left[\frac{f(z)}{z}\right]^{\beta}\quad\text{implies}\quad z\left[\frac{A_{\beta,\gamma}[g](z)}{z}\right]^{\beta}\prec z\left[\frac{A_{\beta,\gamma}[f](z)}{z}\right]^{\beta}.$$

Furthermore, for the integral operator $A^{h}_{\beta,\gamma}:K\to H(U),$ with $K\subset H(U),$ defined by

$$A^{h}_{\beta,\gamma}[f](z)=\left[\frac{\beta+\gamma}{z^{\gamma}}\int_0^z f^{\alpha}(t)h(t)t^{\delta-1}dt\right]^{1/\beta},$$

where $\alpha,\beta,\gamma,\delta\in\mathbb{C},$ and $h\in H(U),$ we also determine in Siregar *et al.* (2009) the sufficient conditions on g_1, g_2, α, β and γ such that

$$zh(z)\left[\frac{g_1(z)}{z}\right]^{\alpha}\prec zh(z)\left[\frac{f(z)}{z}\right]^{\alpha}\prec zh(z)\left[\frac{g_2(z)}{z}\right]^{\alpha}$$

implies

$$z\left[\frac{A^{h}_{\beta,\gamma}[g_1](z)}{z}\right]^{\beta}\prec z\left[\frac{A^{h}_{\beta,\gamma}[f](z)}{z}\right]^{\beta}\prec z\left[\frac{A^{h}_{\beta,\gamma}[g_2](z)}{z}\right]^{\beta}.$$

Then we proved that, under our assumptions, this result is sharp, and combining the above two implications we obtained *a sandwich-type theorem*.

Now, let consider the integral operator $A^{\phi,\varphi}_{\beta,\gamma}:K\to H(U),$ with $K\subset H(U),$ defined by

$$A^{\phi,\varphi}_{\beta,\gamma}[f](z)=\left[\frac{\beta+\gamma}{z^{\gamma}\phi(z)}\int_0^z f^{\alpha}(t)\varphi(t)t^{\delta-1}dt\right]^{1/\beta},\qquad(2)$$

where $\alpha,\beta,\gamma,\delta\in\mathbb{C},$ and $h\in H(U),$ (all powers are principal ones).

In the present paper we generalise these previous results, in the sense to give sufficient conditions on g_1 and g_2 functions and on α, β, γ and δ parameters, such that the next *sandwich-type result* holds:

$$z\varphi(z)\left[\frac{g_1(z)}{z}\right]^{\alpha}\prec z\varphi(z)\left[\frac{f(z)}{z}\right]^{\alpha}\prec z\varphi(z)\left[\frac{g_2(z)}{z}\right]^{\alpha}$$

implies

$$z\left[\frac{A^{h}_{\alpha,\beta,\gamma}[g_1](z)}{z}\right]^{\beta}\prec z\left[\frac{A^{h}_{\alpha,\beta,\gamma}[f](z)}{z}\right]^{\beta}\prec z\left[\frac{A^{h}_{\alpha,\beta,\gamma}[g_2](z)}{z}\right]^{\beta}.$$

Moreover, the functions $z\left[\frac{A^{h}_{\alpha,\beta,\gamma}[g_1](z)}{z}\right]^{\beta}$ and $z\left[\frac{A^{h}_{\alpha,\beta,\gamma}[g_2](z)}{z}\right]^{\beta}$ are respectively the best subordinant and the best dominant.

2. Preliminaries

To prove our main results, we will need the following definitions and lemmas presented in this section. Let $c\in\mathbb{C}$ with Re $c>0,$ let $n\in\mathbb{N}^{*}$ and let

$$C_n = C_n(c) = \frac{n}{\operatorname{Re} c}\left[|c|\sqrt{1+2\operatorname{Re}\left(\frac{c}{n}\right)} + \operatorname{Im} c\right].$$

If R is the univalent function $R(z) = \frac{2C_n z}{1-z^2}$, then the *open door function* $R_{c,n}$ is defined by $R_{c,n}(z) = R\left(\frac{z+b}{1+\bar{b}z}\right)$, $z \in U$, where $b = R^{-1}(c)$.

Remark that $R_{c,n}$ is univalent in U, $R_{c,n}(0) = c$ and $R_{c,n}(U) = R(U)$ is the complex plane slit along the half-lines $\operatorname{Re}(w) = 0$, $\operatorname{Im} w \geq C_n$ and $\operatorname{Re}(w) = 0$, $\operatorname{Im} w \leq -C_n$.

Moreover, if $c > 0$, then $C_{n+1} > C_n$ and $\lim_{n\to\infty} C_n = \infty$, hence $R_{c,n} \prec R_{c,n+1}$ and $\lim_{n\to\infty} R_{c,n}(U) = C$. We will use the notation $R_c \equiv R_{c,1}$. Let denote the class of functions $A_n = \{f \in H(U): f(z) = z + a_{n+1}z^{n+1} + ...\}$ and let $A \equiv A_1$.

Lemma 2.1. [Integral Existence Theorem] (Miller & Mocanu 1989; 1991) *Let* $\phi, \Phi \in H[1,n]$ *with* $\phi \neq 0, \Phi \neq 0$, *for* $z \in U$. *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, *with* $\beta \neq 0$, $\alpha + \delta = \beta + \gamma$ *and* $\operatorname{Re}(\alpha+\delta) > 0$. *If the function* $f(z) = z + a_{n+1}z^{n+1} + ... \in A_n$ *and satisfies* $\alpha\frac{zf'(z)}{f(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta \prec R_{\alpha+\delta,n}(z)$ *then*

$$F(z) = \left[\frac{\beta+\gamma}{z^{\gamma}\Phi(z)}\int_0^z f^{\alpha}(t)\phi(t)t^{\gamma-1}dt\right]^{1/\beta} = z + b_{n+1}z^{n+1} + ... \in A_n,$$

$\frac{F(z)}{z} \neq 0$, $z \in U$. *and* $\operatorname{Re}\left(\beta\frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right) > 0$, $z \in U$. (*All powers are principal ones*).

A function $L(z;t): U \times [0,+\infty) \to \mathbb{C}$ is called a subordination (or a Loewner) chain if $L(\cdot;t)$ is analytic and univalent in U for all $t > 0$, $L(z;\cdot)$ is continuously differentiable on $[0,+\infty)$ for all $z \in U$ and $L(z;s) \prec L(z;t)$ when $0 \leq s \leq t$. The next well-known lemma gives us necessary and sufficient conditions so that the $L(z;t)$ function will be a subordination chain.

Lemma 2.2. (Pommerenke 1975, p. 159) *The function* $L(z;t) = a_1(t)z + a_2(t)z^2 + ...$, *with* $a_1 \neq 0$, *for all* $t \geq 0$, *and* $\lim_{t\to+\infty}|a_1(t)| = +\infty$, *is a subordination chain if and only if* $\operatorname{Re}\left(z\frac{\partial L/\partial z}{\partial L/\partial t}\right) > 0$, $z \in U$, $t \geq 0$.

We denote by $K(\alpha)$, $\alpha < 1$, the class of *convex functions of order* α in the unit disc U, that is

$$K(\alpha) = \left\{f \in A: \operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha, z \in U\right\}.$$

In particular, the class $K \equiv K(0)$ represents the class of *convex (and univalent) functions* in the unit disc. The class of starlike function of order α in U, $\alpha < 1$, denoted by $S^*(\alpha)$ is

$$S^*(\alpha)=\left\{f\in A:\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right]>\alpha, z\in U\right\}.$$

In particular, the class $S^*=S^*(0)$ represents the class of starlike (and univalent) functions in the unit disc.

As in Mocanu *et al.* (1981), if $\beta>0,$ and $\beta+\gamma>0,$ for a given we define the order of starlikeness of the class $A_{\beta,\gamma}$ by the largest number $\delta=\delta(\alpha;\beta,\gamma)$ such that $A_{\alpha,\beta}(S^*(\alpha))\subset S^*(\delta)$, where $A_{\beta,\gamma}$ is given by (1).

Lemma 2.3. (Mocanu *et al.* 1981) *Let* $\beta>0,$ $\beta+\gamma>0.$ *If* $\alpha\in[\alpha_0,1),$ *where* $\alpha_0=\max\left\{\frac{\beta-\gamma-1}{2\beta};-\frac{\gamma}{\beta}\right\},$ *then the order of starlikeness of the class* $A^{\phi,\varphi}_{\alpha,\beta,\gamma}(S^*(\alpha))$ *is given by*

$$\delta(\alpha;\beta,\gamma)=\frac{1}{\beta}\left[\frac{\beta+\gamma}{{}_2F_1(1,2\beta(1-\alpha),\beta+\gamma+1;1/2)}-\gamma\right].$$

Here ${}_2F_1$ *represents the (Gaussian) hypergeometric function, that is*

$${}_2F_1(a,b,c;z)=\sum_{k=0}^{+\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!},$$

where $(d)_k=d(d+1)...(d+k-1)$ *and* $(d)_0=1.$

The next result deals with the solutions of the Briot-Bouquet differential equation (3). The more general forms of the following lemma may be found in Miller and Mocanu (2003,Theorem 1).

Lemma 2.4. (Miller & Mocanu 1985) *Let* $\beta,\gamma\in\mathbb{C}$ *with* $\beta\neq0$ *and let* $h\in H(U)$ *with* $h(0)=c.$ *If* $\operatorname{Re}[\beta h(z)+\gamma]>0,$ $z\in U,$ *then the solution of the differential equation*

$$q(z)+\frac{zq'(z)}{\beta q(z)+\beta}=h(z), \tag{3}$$

with $q(0)=c$ *is analytic in* U *and satisfies* $\operatorname{Re}[\beta q(z)+\gamma]>0,$ $z\in U.$

For the next lemma that we will use to prove our first result, we need to introduce the following definition.

As in Miller and Mocanu (2003), let the set of functions f that are analytic and injective on $\overline{U}\setminus E(f),$ denote by Q where $E(f)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}f(z)=\infty\right\},$ and such that $f'(\zeta)\neq0$ for $\zeta\in\partial U\setminus E(f).$

Lemma 2.5. (Miller & Mocanu 2003, Theorem 7) *Let* $q\in H[a,1],$ *let* $\chi:\mathbb{C}^2\to\mathbb{C}$ *and set* $\chi(q(z),zq'(z))\equiv h(z).$ *If* $L(z;t)=\chi(q(z),tzq'(z))$ *is a subordination chain and*

$$p\in H[a,1]\cap Q,\quad \text{then } h(z)\prec\chi(p(z),zp'(z)) \text{ implies } q(z)\prec p(z).$$

Furthermore, if $\chi(q(z),zq'(z))=h(z)$ *has a univalent solution* $q\in Q,$ *then* q *is the best subordinant.*

Like in Miller and Mocanu (1981; 1985), let $\Omega \in \mathbb{C}, q \in Q,$ and n be a positive integer. Then, the class of admissible functions $\Psi_n[\Omega, q]$ is the class of those functions $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ that satisfy the admissibility condition $\psi(r,s,t;z) \notin \Omega,$ whenever $r = q(\zeta), s = m\zeta q'(\zeta),$

$$\mathrm{Re}\frac{t}{s} + 1 \geq m\,\mathrm{Re}\left[\frac{\zeta q''(\zeta)}{q'(\zeta)} + 1\right], \; z \in U, \; \zeta \in \partial U \setminus E(q) \text{ and } m \geq n.$$

This class will be denoted by $\Psi_n[\Omega, q]$. We write $\Psi[\Omega, q] \equiv \Psi_1[\Omega, q]$. For the special case when $\Omega \neq \mathbb{C}$ is a simply connected domain and h is a conformal mapping of U onto $\Omega,$ we use the notation $\Psi_n[h,q] \equiv \Psi_n[\Omega, q].$

Remark 2.6. If $\psi : \mathbb{C}^2 \times U \to \mathbb{C}$, then the above defined admissibility condition reduces to $\psi(q(\zeta), m\zeta q'(\zeta); z) \notin \Omega,$ when $z \in U, \; \zeta \in \partial U \setminus E(q)$ and $m \geq n.$

Lemma 2.7. (Miller & Mocanu 1981; 1999) *Let h be univalent in U and $\psi : \mathbb{C}^3 \times U \to \mathbb{C}.$ Suppose that the differential equation $\psi(q(z), zq'(z), z^2 q''(z); z) = h(z)$ has a solution $q,$ with $q(0) = a,$ and one of the following conditions is satisfied:*

(i) $q \in Q$ *and* $\psi \in \Psi[h,q]$
(ii) q *is univalent in* U *and* $\psi \in \Psi[h, q_\rho]$ *for some* $\rho \in (0,1),$ *where* $q_\rho(z) = q(\rho z),$ *or*
(iii) q *is univalent in* U *and there exists* $\rho_o \in (0,1),$ *such that* $\psi \in \Psi[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_o, 1),$ *where* $h_\rho(z) = h(\rho z)$ *and* $q_\rho(z) = q(\rho z).$

If $p(z) = a + a_1 z + ... \in H(U)$ *and* $\psi(p(z), zp'(z), z^2 p''(z); z) \in H(U),$ *then* $\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z)$ *implies* $p(z) \prec q(z)$ *and* q *is the best dominant.*

3. Main Result

First, we can see the result in Siregar *et al.* (2009) as follows:

Theorem 3.1 (Siregar *et al.* 2009) *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{C},$ *with* $\beta \neq 0, \; 1 < \beta + \gamma) \leq 2,$ $\alpha + \delta = \beta + \gamma.$ *Let* $g \in K_{\alpha,\delta}^{\phi,\varphi}$ *, and for* $\alpha \neq 1$ *suppose in addition that* $\frac{g(z)}{z} \neq 0,$ *for* $z \in U.$ Suppose that

$$\mathrm{Re}\left[1 + \frac{z\varphi''(z)}{\varphi'(z)}\right] > \frac{1 - (\beta + \gamma)}{2}, \; z \in U,$$

where $\varphi(z) = zh(z)\left[\frac{g(z)}{z}\right]^\alpha.$ Let $f \in Q \cap K_{\alpha,\delta}^{\phi,\varphi}$ such that $zh(z)\left[\frac{f(z)}{z}\right]^\alpha$ and $z\left[\frac{A_{\alpha,\beta,\gamma}^h[f](z)}{z}\right]^\beta$ are univalent functions in U, and for $\alpha \neq 1$ suppose in addition that

$\frac{g(z)}{z}\neq 0,$ for $z\in U$. Then $zh(z)\left[\frac{g(z)}{z}\right]^{\alpha}\prec zh(z)\left[\frac{f(z)}{z}\right]^{\alpha}$ implies $z\left[\frac{A^{h}_{\alpha,\beta,\gamma}[g](z)}{z}\right]^{\beta}\prec z\left[\frac{A^{h}_{\alpha,\beta,\gamma}[f](z)}{z}\right]^{\beta}$. The function $z\left[\frac{A^{h}_{\alpha,\beta,\gamma}[g](z)}{z}\right]^{\beta}$ is the best subordinant.

In the above theorem we need the following Lemma:

Lemma 3.1. (Siregar *et al.* 2009) *Let* $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, *with* $\beta\neq 0$, $\alpha+\delta=\beta+\gamma$ *and* $\operatorname{Re}(\beta+\gamma)>0$. *For the functions* $h\in H[1,1]$, *with* $h(z)\neq 0$ *for all* $z\in U$, *we define the set* $K\subset H(U)$ *by*

$$K=K^{h}_{\alpha,\delta}=\left\{f\in A:\alpha\frac{zf'(z)}{f(z)}+\frac{z\varphi'(z)}{\varphi(z)}+\delta\prec R_{\alpha+\delta}(z)\right\}.$$

Then $f\in K^{h}_{\alpha,\delta}$ *implies* $F\in A$, $\frac{F(z)}{z}\neq 0$, *and* $\operatorname{Re}\left(\beta\frac{zF'(z)}{F(z)}+\gamma\right)>0$, $z\in U$, *where* $F(z)=A^{h}_{\alpha,\beta,\gamma}[f](z).$

According to Lemma 3.1, we need to determine the subset $K\subset H(U)$ such that the integral operator $A^{\phi,\varphi}_{\alpha,\beta,\gamma}$ given by (1.2) will be well-defined. If we choose in Lemma 2.1 the correspondent functions $\Phi\equiv\phi\in H[1,1]$ and $\phi\equiv\varphi\in H[1,1]$, with $\phi(z)\varphi(z)\neq 0$ for all $z\in U$, then we get the set K where the integral operator $A^{\phi,\varphi}_{\alpha,\beta,\gamma}$ is well-defined.

Next we consider another lemma which we thought will give a helpful solution to the next result.

Lemma 3.2. *Let* $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, *with* $\beta\neq 0$, $\alpha+\delta=\beta+\gamma$ *and* $\operatorname{Re}(\alpha+\delta)>0$. *For the functions* $\phi,\varphi\in H[1,1]$, *with* $\phi(z)\varphi(z)\neq 0$ *for all* $z\in U$, *we define the set* $K\subset H(U)$ *by*

$$K=K^{\phi,\varphi}_{\alpha,\delta}=\left\{f\in A:\alpha\frac{zf'(z)}{f(z)}+\frac{z\varphi'(z)}{\varphi(z)}+\delta\prec R_{\alpha+\delta}(z)\right\}.$$

Then $f\in K^{\phi,\varphi}_{\alpha,\delta}$ *implies* $F\in A$, $\frac{F(z)}{z}\neq 0$, *and* $\operatorname{Re}\left(\beta\frac{zF'(z)}{F(z)}+\frac{z\phi'(z)}{\phi(z)}+\gamma\right)>0$, $z\in U$, *where* $F(z)=A^{\phi,\varphi}_{\alpha,\beta,\gamma}[f](z).$

Now, we conject a new result which is yet to be proven.

Theorem 3.2. *Let* $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, *with* $\beta\neq 0$, $\operatorname{Re}(\beta+\gamma)>1$ *and* $\alpha+\delta=\beta+\gamma$. *Let* $f,g\in K^{\phi,\varphi}_{\alpha,\delta}$ *and for* $\alpha\neq 1$ *suppose in addition that* $\frac{f(z)}{z}\neq 0$, $\frac{g(z)}{z}\neq 0$, *for* $z\in U$. *Suppose that*

$$\operatorname{Re}\left[1+\frac{zu''(z)}{u'(z)}+\frac{z\phi'(z)}{\phi(z)}\right]>\frac{1-(\beta+\gamma)}{2},\quad z\in U, \tag{4}$$

where $u(z) = z\varphi(z)\left[\frac{g(z)}{z}\right]^{\alpha}$.

Let $f \in Q \cap K_{\alpha,\delta}^{\phi,\varphi}$ *such that* $z\varphi(z)\left[\frac{f(z)}{z}\right]^{\alpha}$ *and* $z\left[\frac{A_{\alpha,\beta,\gamma}^{\phi,\varphi}[f](z)}{z}\right]^{\beta}$ *are univalent functions in* U.

Then

$$z\varphi(z)\left[\frac{g(z)}{z}\right]^{\alpha} \prec z\varphi(z)\left[\frac{f(z)}{z}\right]^{\alpha} \text{ implies } z\left[\frac{A_{\alpha,\beta,\gamma}^{\phi,\varphi}[g](z)}{z}\right]^{\beta} \prec z\left[\frac{A_{\alpha,\beta,\gamma}^{\phi,\varphi}[f](z)}{z}\right]^{\beta},$$

and the function $z\left[\frac{A_{\alpha,\beta,\gamma}^{\phi,\varphi}[g](z)}{z}\right]^{\beta}$ *is the best subordinant.*

Here, we gave a part for the conjecture above,

Proof. Denoting $G = A_{\alpha,\beta,\gamma}^{\phi,\varphi}[g]$, $F = A_{\alpha,\beta,\gamma}^{\phi,\varphi}[f]$, $u(z) = z\varphi(z)\left[\frac{g(z)}{z}\right]^{\alpha}$,

$$v(z) = z\varphi(z)\left[\frac{f(z)}{z}\right]^{\alpha}, \ \mathrm{U}(z) = z\left[\frac{G(z)}{z}\right]^{\beta} \text{ and } \mathrm{V}(z) = z\left[\frac{F(z)}{z}\right]^{\beta},$$

we need to prove that $u(z) \prec v(z)$ implies $\mathrm{U}(z) \prec \mathrm{V}(z)$.

Because $f, g \in K_{\alpha,\delta}^{\phi,\varphi}$ then $\psi, \varphi \in A$ and by Lemma 2.1 we have $\frac{G(z)}{z} \neq 0$ and $\frac{F(z)}{z} \neq 0$, $z \in U$, $\mathrm{U},\mathrm{V} \in H(U)$ and moreover $\mathrm{U},\mathrm{V} \in A$.

If we differentiate the relation $G(z) = A_{\alpha,\beta,\gamma}^{\phi,\varphi}[g](z)$, we have

$$z^{\gamma}\left[\frac{G(z)}{z}\right]^{\beta}\left[\beta\frac{zG'(z)}{G(z)}\phi(z) + \gamma\phi(z) + z\phi'(z)\right] = (\beta+\gamma)g^{\alpha}(z)\varphi(z)z^{\delta-\beta}. \tag{5}$$

Since $\mathrm{U}(z) = z\left[\frac{G(z)}{z}\right]^{\beta}$ by differentiating this relation we obtain $\beta\frac{zG'(z)}{G(z)} = \beta - 1 + \frac{z\mathrm{U}'(z)}{\mathrm{U}(z)}$,

and replacing in (5) we get

$$u(z) = \left[\left(1 - \frac{1}{\beta+\gamma}\right)\phi(z) + \frac{1}{\beta+\gamma}z\phi'(z)\right]\mathrm{U}(z) + \frac{1}{\beta+\gamma}\phi(z)z\mathrm{U}'(z) = \chi(\mathrm{U}(z), z\mathrm{U}'(z)), \tag{6}$$

where we used the relation $\alpha + \delta = \beta + \gamma$.

Letting

$$L(z;t) = \left[\left(1 - \frac{1}{\beta+\gamma}\right)\phi(z) + \frac{1}{\beta+\gamma}z\phi'(z)\right]\mathrm{U}(z) + \frac{t}{\beta+\gamma}\phi(z)z\mathrm{U}'(z), \tag{7}$$

then $L(z;1) = u(z)$. If we denote $L(z;t) = a_1(t)z + \ldots$, then

$$a_1(t) = \frac{\partial L(0;t)}{\partial z} = \left(1 + \frac{t-1}{\beta+\gamma}\right)\phi(0)\mathrm{U}'(0) = 1 + \frac{t-1}{\beta+\gamma}.$$

Hence $\lim_{t\to+\infty}|a_1(t)| = +\infty$. Using the $\mathrm{Re}(\beta+\gamma) > 1$ we obtain $a_1 \neq 0$, $\forall t \geq 0$.

Now, by using Lemma 2.2 we will prove that $L(z;t)$ is a subordination chain. From (7), a simple computation shows the equality

$$\operatorname{Re}\left(z\frac{\partial L/\partial z}{\partial L/\partial t}\right)=t\operatorname{Re}\left[1+\frac{zU''(z)}{U'(z)}+\frac{z\phi'(z)}{\phi(z)}\right]+\operatorname{Re}(\beta+\gamma-1)$$
$$+\operatorname{Re}\left[\frac{1}{\phi(z)}[(\beta+\gamma)\phi'(z)+z\phi''(z)]\frac{zU'(z)}{U(z)}+\frac{z\phi'(z)}{\phi(z)}\right].$$

Using the above relation together with the assumption $\operatorname{Re}(\beta+\gamma)>1$ and according to Lemma 2.2, in order to prove that $\operatorname{Re}\left(z\frac{\partial L/\partial z}{\partial L/\partial t}\right)>0,\ \forall z\in U,\ \forall t\geq 0$, we need to prove that the next two inequalities hold:

$$\operatorname{Re}\left[1+\frac{zU''(z)}{U'(z)}+\frac{z\phi'(z)}{\phi(z)}\right]>0,\quad z\in U \tag{8}$$

and

$$\operatorname{Re}\left[\frac{1}{\phi(z)}[(\beta+\gamma)\phi'+z\phi'']\frac{U(z)}{U'(z)}+\frac{z\phi'(z)}{\phi(z)}\right]\geq 0,\quad z\in U. \tag{9}$$

From (6) $u(z)=\left[\left(1-\frac{1}{\beta+\gamma}\right)\phi(z)+\frac{1}{\beta+\gamma}z\phi'(z)\right]U(z)+\frac{1}{\beta+\gamma}\phi(z)zU'(z)$ then

$$u'(z)=\left[\left(1-\frac{1}{\beta+\gamma}\right)\phi'(z)+\frac{1}{\beta+\gamma}(\phi'(z)+z\phi''(z))\right]U(z)+\left[\left(1-\frac{1}{\beta+\gamma}\right)\phi(z)+\frac{1}{\beta+\gamma}z\phi'(z)\right]U'(z)$$
$$+\frac{1}{\beta+\gamma}\left(U'(z)\phi(z)+zU''(z)\phi(z)+zU'(z)\phi'(z)\right). \tag{10}$$

Next, from (6) and (10), we obtain that

$$\frac{zu'(z)}{u(z)}=\frac{\left[\phi'(z)+\frac{1}{\beta+\gamma}z\phi''(z))\right]zU(z)+\left[\phi(z)+\frac{1}{\beta+\gamma}z\phi'(z)\right]zU'(z)+\frac{1}{\beta+\gamma}\left(z^2U''(z)\phi(z)+z^2U'(z)\phi'(z)\right)}{\left[\left(1-\frac{1}{\beta+\gamma}\right)\phi(z)+\frac{1}{\beta+\gamma}z\phi'(z)\right]U(z)+\frac{1}{\beta+\gamma}zU'(z)\phi(z)}$$
$$=\frac{\left[(\beta+\gamma-1)\frac{z\phi'(z)}{\phi(z)}+\frac{z\phi'(z)}{\phi(z)}+\frac{z^2\phi''(z)}{\phi(z)}-\left(\frac{z\phi'(z)}{\phi(z)}\right)^2\right]+(\beta+\gamma-1)\frac{zU'(z)}{U(z)}}{\left[(\beta+\gamma-1)+\frac{z\phi'(z)}{\phi(z)}+\frac{zU'(z)}{U(z)}\right]}$$
$$+\frac{\left[\frac{zU'(z)}{U(z)}+\frac{z^2U''(z)}{U(z)}-\left(\frac{zU'(z)}{U(z)}\right)^2\right]+\left(\left(\frac{z\phi'(z)}{\phi(z)}\right)^2+\left(\frac{zU'(z)}{U(z)}\right)^2+\frac{zU'(z)}{U(z)}\cdot\frac{z\phi'(z)}{\phi(z)}\right)}{\left[(\beta+\gamma-1)+\frac{z\phi'(z)}{\phi(z)}+\frac{zU'(z)}{U(z)}\right]} \tag{11}$$

Let $p(z)=\frac{z\phi'(z)}{\phi(z)}$ and $q(z)=\frac{zU'(z)}{U(z)}$. Then

$$zp'(z)=\frac{z\phi'(z)}{\phi(z)}+\frac{z^2\phi''(z)}{\phi(z)}+\left(\frac{z\phi'(z)}{\phi(z)}\right)^2 \text{ and } zq'(z)=\frac{z\mathrm{U}'(z)}{\mathrm{U}(z)}+\frac{z^2\mathrm{U}''(z)}{\mathrm{U}(z)}+\left(\frac{z\mathrm{U}'(z)}{\mathrm{U}(z)}\right)^2.$$

Replacing the formula in eq. (11) , we get

$$\frac{zu'(z)}{u(z)}=\frac{(\beta+\gamma-1)p(z)+zp'(z)+(\beta+\gamma-1)q(z)+zq'(z)+p^2(z)+q^2(z)+p(z)+q(z)+p(z)q(z)}{(\beta+\gamma-1)+p(z)+q(z)}$$

$$=\frac{z(p'(z)+q'(z))}{(\beta+\gamma-1)+p(z)+q(z)}+\frac{(\beta+\gamma-1)(p(z)+q(z))+p^2(z)+q^2(z)+p(z)q(z)}{(\beta+\gamma-1)+p(z)+q(z)}. \tag{12}$$

The last equation cannot make to the form of Briot-Bouquet equations as required. So, to this end, we let the reader to figure out the full solution towards the conjecture.

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