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THE INTERACTION OF PREDATOR PREY WITH UNCERTAIN INITIAL POPULATION SIZES

(Interaksi Pemangsa-Mangsa dengan Saiz Awal Populasi Tak Pasti**)**

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ABSTRACT

Differential model of dynamical predator-prey system contains some factors that constitute a formal description of features of the interaction between the predator and its prey. The initial population sizes are the factors which affect the behaviour of the predator-prey interaction. These factors may not be uniquely defined. In this paper we study the effect of uncertain initial population sizes of predator and prey on the behaviour of predator-prey interaction. Results based on numerical simulations are discussed.

Keywords: Predator-prey; initial conditions; uncertainty; normal distribution

ABSTRAK

Model pembezaan sistem dinamik pemangsa-mangsa mengandungi beberapa faktor yang merupakan perihalan formal ciri interaksi antara pemangsa dengan mangsanya. Saiz awal populasi merupakan antara faktor yang mempengaruhi tingkah laku interaksi pemangsa-mangsa. Faktor ini tidak semestinya tertakrif secara bitara. Dalam makalah ini dikaji kesan saiz populasi awal tak pasti terhadap tingkah laku interaksi pemangsa-mangsa. Hasil kajian berdasarkan simulasi berangka dibincangkan.

Kata kunci: Pemangsa-mangsa; keadaan awal; ketidakpastian; taburan normal

1. Introduction

When real phenomenon is described by a deterministic model, we usually cannot be sure the modelling process is perfect. In the predator-prey model, the initial state of population processes may be available but they cannot define with certainty the model of the system. Low accuracy of measurement process and imprecise human knowledge as well as difficulties in assessing the actual population sizes causes this uncertainty. According to Diniz *et al.* (2001), the uncertainty can arise in the experimental part, the data collection, the measurement process, as well as when determining the initial conditions. Soong (1973) considered the uncertainty in differential equation (the predator-prey model) in different forms; the first is characterised by the presence of a random input term or source term, the second when the uncertainty enters through the parameters, and finally when the initial conditions are random. Randomness is a basic type of objective uncertainty, and in system theory uncertainty is classically treated in probabilistic form by the theory of stochastic processes. However, when the underlying structure is not probabilistic, then it may be appropriate to use fuzzy numbers instead of randomness. Kegan and West (2005) investigated the effect of random initial conditions on the deterministic model for the susceptible-infectious epidemic by assuming a Beta distribution on the initial proportion of susceptible. A distribution for the proportion of susceptible at time t was developed and explored and they obtained the distribution of time until a given proportion of population remains susceptible. Gonzaléz *et al.* (2008) studied the effect of uncertainty in the dynamics behaviour of the overweight and obesity childhood population by assuming the initial condition of normal

weight and latent classes are random value taken from a uniform and normal distributions. The stability conditions for a fairly general predator-prey model which exhibits self-limiting density effects and minimum viable population levels for both the predator and the prey is investigated by Tu and Wilman (1992). da Silva Peixoto *et al.* (2008) studied the fuzzy predator-prey population model. The classical deterministic model by means of a fuzzy rule-based system is elaborated, and the stability of the critical points of the Holling–Tanner model is also studied. Xu and Gertner (2009) introduced an uncertainty analysis technique, the general Fourier Amplitude Sensitivity Test (FAST), to study uncertainties in transient population dynamics. They found that the general FAST is able to identify the amount of uncertainty in transient dynamics and contributions by different demographic parameters. The general FAST is applied to a mountain goat (Oreamnos americanus) matrix population model to give a clear illustration of how uncertainty analysis can be conducted for transient dynamics arising from matrix population models. Pollett *et al.* (2010) presented a general method for incorporating random initial conditions in population models where a deterministic model is sufficient to describe the dynamics of the population. They obtained for a large class of stochastic models the overall variation is the sum of variation due to random initial conditions and variation due to random dynamics.

The main concern of this paper is to focus on dynamical systems of differential equation of predator-prey model where the initial conditions are described by uncertainty distributions. We suggest that the initial conditions of prey and predator distribute as normal (Gaussian) distributions and investigate the effect of random initial conditions on the interaction between a prey and its predator.

2. Predator-Prey Models

In the natural world, we find that species compete, evolve and disperse for the purpose of finding resources to carry on the struggle for existence. These behaviours can be modelled mathematically by the predator-prey models. The generic model was first proposed by Lotka and Volterra (Murray 2002). Any form of interactions, be it win-win or loss-win, within and sometimes outside of ecology can be modelled by these models. Depending on their specific settings of applications, they can take the forms of resource-consumer, plant-herbivore, parasite-host, tumour cells (virus)-immune system, susceptible-infectious interactions, and others.

There are various models to describe the predator prey interactions, and these models are normally written as a system of ordinary differential equations. In this paper, we consider the following system:

$$\frac{dx}{dt} = a\,x(t)\left(1-\frac{x(t)}{k}\right)-\frac{b\,x(t)\,y(t)}{y(t)+b\,h_1 x(t)}, \qquad x(t_0)=x_0, \tag{1}$$

$$\frac{dy}{dt} = \frac{c\,b\,x(t)\,y(t)}{y(t)+b\,h_1 x(t)} - d\,y(t), \qquad y(t_0)=y_0, \tag{2}$$

where $x(t)$ represents the prey density and $y(t)$ the predator density. $\left(1-x(t)/k\right)$ is the usual logistic form of the prey in the absence of the predator, and $\left(b\,x(t)/y(t)+b\,h_1 x(t)\right)$ is the ratio-dependent functional response which is approximated by a function of prey. This functional response is known as the Michaelis-Menten-Holling type functional response

(Bandyopadhyay 2008). The initial conditions x_0 and y_0 are assumed to be positive. The parameters a, b, c, d, h_1 and k are positive constants given by:

a : the intrinsic growth rate of prey,

b : total attack rate for predator,

c : interpreted as conversion efficiency ($0 < c < 1$),

d : death rate of predator in the absence of their prey,

h_1 : represent the handling time, and

k : the environmental carrying capacity.

3. Distributions and Randomness

The normal probability distribution or Gaussian probability distribution is perhaps the most used distribution in science. It is a continuous probability distribution and often used to describe at least approximately any variable that tends to cluster around the mean. It is given by the following probability distribution function (PDF) (Krishnamoorthy 2006):

$$f(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(z-\mu)^2}{2\sigma^2}}. \tag{3}$$

Its distribution function is the integral

$$F(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{\frac{-(z-\mu)^2}{2\sigma^2}} dz, \tag{4}$$

where the normal distribution is completely characterised by the parameters, μ (mean of distribution, at the same place as mode and median), σ^2 (variance of the distribution) and z (continuous variable, $-\infty \le z \le \infty$).

Our concern in this paper is to see whether a normally distributed data will stay normal after the data undergo changes. Normality tests assess the likelihood that the given data set $\{x_1, x_2, \ldots x_n\}$ comes from a normal distribution. There are three main tests for normality and these are the Kolmogorov-Smirnov, Anderson-Darling and Shapiro-Wilk tests.

The Shapiro-Walk test is our numerical means of assessing normality. The reason is that the Shapiro-Wilk test for normality is one of three general normality tests designed to detect all departures from normality. It is comparable in power to the other two tests, and is more appropriate for small sample sizes (less than 50) but can also handle sample sizes as large as

2000. The null hypothesis (H_0) of the test is that the population is normally distributed, and the alternative hypothesis (H_1) is that it is not normally distributed (de Sá 2003).

In Eqs. (1) and (2), the initial population sizes $x(t_0)$ and $y(t_0)$ are deterministic. Here we want to see the behaviour of the solutions when the initial conditions are in the neighbourhood of the deterministic values. We assume that the initial population sizes are non-deterministic and uncertain. Thus, to ensure uncertainty in the model, the initial population sizes are considered random variables having a particular probability distribution which is the normal distribution. Under these assumptions, Eqs. (1) and (2) take the following form

$$\frac{dx}{dt} = a\,x(t)\left(1-\frac{x(t)}{k}\right)-\frac{b\,x(t)y(t)}{y(t)+b\,h_1x(t)}, \qquad x(t_0)\sim N(\mu_1,\sigma_1), \tag{5}$$

$$\frac{dy}{dt} = \frac{c\,b\,x(t)y(t)}{y(t)+b\,h_1x(t)}-d\,y(t), \qquad y(t_0)\sim N(\mu_2,\sigma_2). \tag{6}$$

4. Numerical Method

In this section, the fourth order Runge-Kutta method is applied to numerically solve the prey-predator model (5) and (6) with the normally distributed initial population sizes. For $r=1,...,m$, let x_0^r and y_0^r be random variables, where $x_0^r \sim N(\mu_1,\sigma_1^2)$ is the initial population size of prey, and $y_0^r \sim N(\mu_2,\sigma_2^2)$ is the initial population size of predator. Let $0=t_0<t_1<...<t_N=T$ be a discrete set of points in $[t_0,T]$, and $h=T-t_0/N=t_{i+1}-t_i, i=0,...,N$ be the step size. When solving our model numerically, we try to approximate every $x^r(t_i)$ and $y^r(t_i)$, $i=0,...,N$, $r=1,...,m$. In this case, the fourth order Runge-Kutta method takes the following formula:

$$x^r(t_{i+1}) = x^r(t_i)+\frac{h}{6}[K_1^r+2K_2^r+2K_3^r+K_4^r] \tag{7}$$

$$y^r(t_{i+1}) = y^r(t_i)+\frac{h}{6}[L_1^r+2L_2^r+2L_3^r+L_4^r] \tag{8}$$

where

$$K_1^r = f(t_i, x^r(t_i), y^r(t_i)); \qquad L_1^r = g(t_i, x^r(t_i), y^r(t_i));$$

$$K_2^r = f(t_i+\frac{h}{2}, x^r(t_i)+\frac{h}{2}K_1^r, y^r(t_i)+\frac{h}{2}L_1^r);$$

$$L_2^r = g(t_i+\frac{h}{2}, x^r(t_i)+\frac{h}{2}K_1^r, y^r(t_i)+\frac{h}{2}L_1^r);$$

$$K_3^r = f\,(t_i + \frac{h}{2}, x^r(t_i) + \frac{h}{2}K_2^r, y^r(t_i) + \frac{h}{2}L_2^r)\,;$$
$$L_3^r = g\,(t_i + \frac{h}{2}, x^r(t_i) + \frac{h}{2}K_2^r, y^r(t_i) + \frac{h}{2}L_2^r)\,;$$
$$K_4^r = f\,(t_i + h, x^r(t_i) + h\,K_3^r, y^r(t_i) + h\,L_3^r)\,;$$
$$L_4^r = g\,(t_i + h, x^r(t_i) + h\,K_3^r, y^r(t_i) + h\,L_3^r)\,,$$

and

$$f\,(t_i, x(t_i), y(t_i)) = a\,x(t_i)\left(1 - \frac{x(t_i)}{k}\right) - \frac{b\,x(t_i)y(t_i)}{y(t_i) + b\,h_1 x(t_i)}\,,$$
$$g\,(t_i, x(t_i), y(t_i)) = \frac{c\,b\,x(t_i)y(t_i)}{y(t_i) + b\,h_1 x(t_i)} - d\;y(t_i)\,.$$

5. Simulation

The goal here is to investigate the effect of assuming the normal distribution as the initial population sizes of our model (5) and (6) on the behaviour of the solution. By using the numerical method of the previous section, the solution of the model seems a family of pairs of curves $x^r(t)$, $y^r(t)$; $r = 1,...,m$. Every pair of curves $x^{r^*}(t)$, $y^{r^*}(t)$ describe the behaviour of the prey and its predator for special initial population sizes $x_0^{r^*}$, $y_0^{r^*}$. For $r = 1,...,m$, equations (7) and (8) basically define an algebraic transformation of random variables $x^r(t_i)$ and $y^r(t_i)$ into $x^r(t_{i+1})$ and $y^r(t_{i+1})$ respectively for every r starting from the random initial population sizes $x^r(t_0)$ and $y^r(t_0)$. Mean of the distribution determines the location, and variance measures of the width of the distribution. For that, choosing the mean and the variance affect on the distribution of the initial population sizes and then affect on the distribution of prey and predator populations in any time t. Since our model is nonlinear and using the fact that the family of normal distribution is closed under linear transformations, the distribution functions of the solution $x^r(t_i)$ and $y^r(t_i)$; $i > 0$ is unknown. Before solving our model, we first non-dimensionalise the model (5), (6). Hence our model that we solve does not have any units. The idea behind that is to reduce the number of undetermined constants. For the sake of simplicity it is convenient to scale the variables as $P = x/k$, $N = y/kbh_1$ and consider the dimensionless time $\tau = at$. The non dimensional form of the governing equations of the model are then given by

$$\frac{dP}{d\tau} = P(\tau)(1 - P(\tau)) - \frac{\alpha P(\tau)N(\tau)}{P(\tau) + N(\tau)}, \qquad P(\tau_0) \sim N(\mu_1, \sigma_1), \tag{9}$$

$$\frac{dN}{d\tau} = \frac{\beta P(\tau)N(\tau)}{P(\tau) + N(\tau)} - \delta N(\tau), \qquad N(\tau_0) \sim N(\mu_2, \sigma_2). \tag{10}$$

Where in the equations the number of constants is a minimum (Bandyopadhyay 2008). The new constants are given by

$$\alpha = \frac{b}{a}, \qquad \beta = \frac{c}{ah_1}, \qquad \delta = \frac{d}{a}. \tag{11}$$

In the following, prey $P(\tau)$, predator $N(\tau)$ and time τ are replaced by $x(t)$, $y(\tau)$ and t as the dimensionless prey, predator and time respectively.

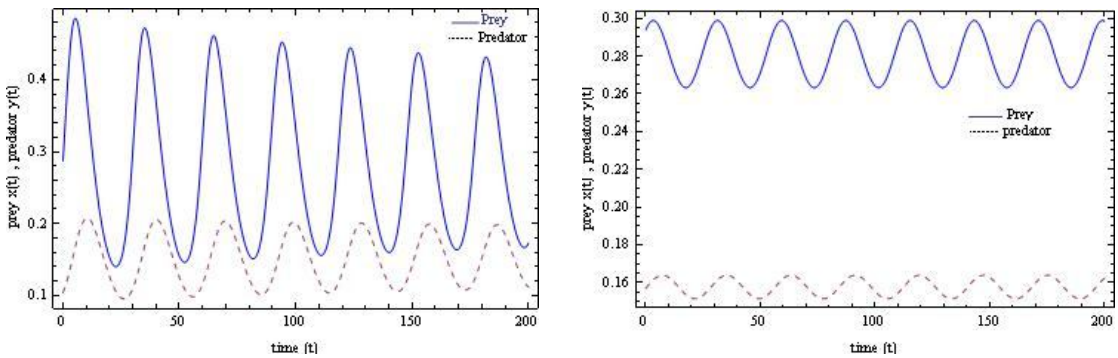

Figure 1: The evolution of prey and predator with different initial conditions which distribute normally as $x_0 \sim N(0.3, 0.02)$ and $y_0 \sim N(0.15, 0.02)$, where $\alpha = 2$, $\beta = 0.7808$ and $\delta = 0.5$

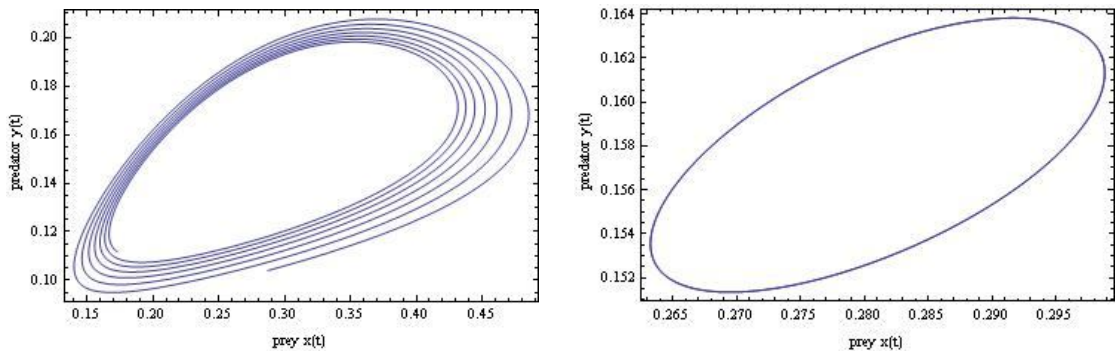

Figure 2: Phase-plane of prey and predator with different initial conditions which distribute normally as $x_0 \sim N(0.3, 0.02)$ and $y_0 \sim N(0.15, 0.02)$, where $\alpha = 2$, $\beta = 0.7808$ and $\delta = 0.5$

We are interested in the statistical properties of the random solution of Eqs. (9) and (10). Since we assume the initial population sizes are distribute normally, first of all we check at every time whether the solution still distributes normally or is significantly different from a normal distribution. One good way for testing the solution is to use the Shapiro-Wilk test. Using this test, Figure 3. and Table 1. show us that the *p*-values decrease and increase. From this, the random solution sometimes distributes normally or closes to the normal, and sometimes does not. This means keeping the solution to distribute normally does not depend on the sample size m .

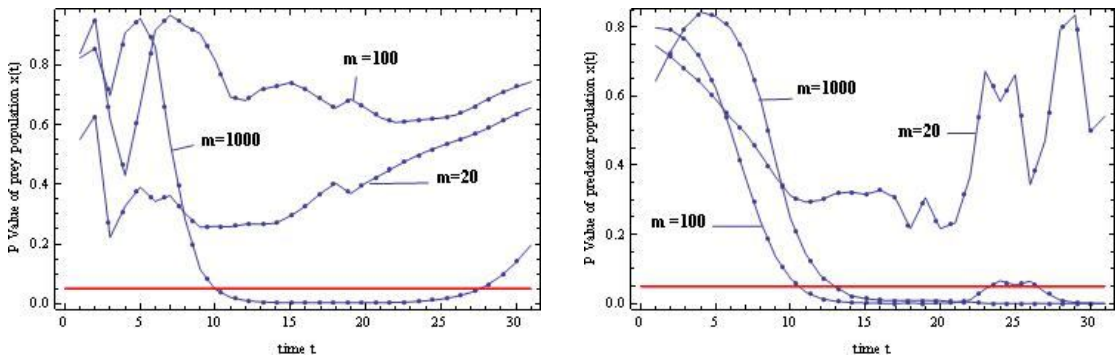

Figure 3: The p-values of prey and predator populations over time with different sample sizes

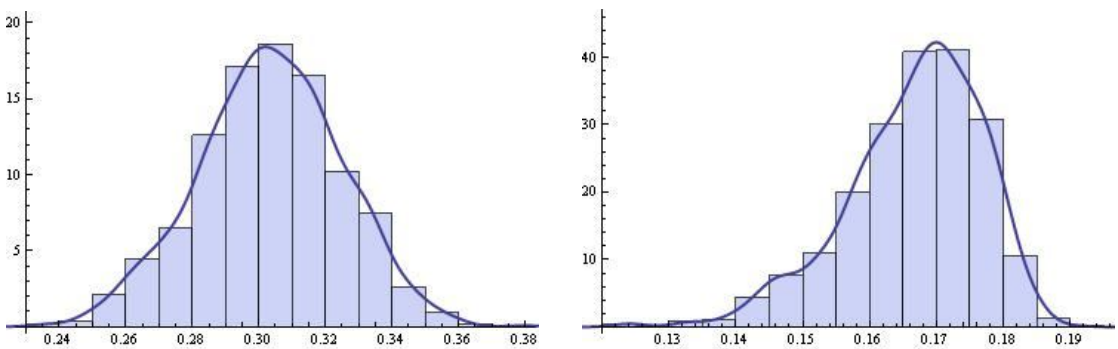

Figure 4: The left one is the prey population when the sample size is 1000 and $t = 2$, it is very close to normal and the right one is the predator population when the sample size is 1000 and $t = 20$, it is clearly not a symmetrical distribution, and the solution is left skewed

Table 1: *P*-values of the distribution of the solution with different initial population sizes

Prey x_i Predator y_i	*p*-value with size 1000	*p*-value with size 100	*p*-value with size 20
x_0	0.824545	0.838885	0..551217
x_1	0.856342	0.956159	0.630693
x_5	0.863039	0.916659	0.343374
x_{10}	0.021168	0.693405	0.257885
x_{15}	0.002321	0.719180	0.327328
x_{20}	0.003096	0.623557	0.431388
x_{25}	0.023634	0.632895	0.536213
x_{30}	0.196281	0.744091	0.657008
y_0	0.644231	0.798561	0.746110
y_1	0.730623	0.791798	0.714491
y_5	0.797178	0.501154	0.537396
y_{10}	0.144655	0.033317	0.293632
y_{15}	0.009851	0.000959	0.327565
y_{20}	0.004552	0.002854	0.233896
y_{25}	0.7×10^{-10}	0.064741	0.343866
y_{30}	0.5×10^{-15}	0.000238	0.542380

If the random solution is not distributed normally at specific time t_i , then we can use another statistical idea that allows us to use the principles of normal distribution with non-normal solution. Transformation of the solution is one of these statistical ideas. Transforming a non-normal distribution solution into a normal distribution solution is performed in a number of different ways, such as a logarithmic, square root, or arcsine square root which can be useful for transformation of the solution to normality. The Central Limit Theorem is another statistical idea. It states that if we add together large number independent identically distributed random variables which are drawn from any distribution, then the resulting sum will have a normal distribution (Roe 2001).

6. Conclusion

In this paper, we have studied the prey predator model with the ratio-dependent functional response which has uncertainty in the initial population sizes of prey and predator by assuming the normal distribution as the initial condition of prey and predator. Numerical simulation is provided and the affect of assuming uncertain initial conditions is investigated. The distribution of the solution has tested for normality and developed in period time with a variety of sample sizes. We have seen that the distribution of the random solution is sometimes unknown and we suggest two statistical ideas to use principles of the normal distribution with non-normal solution.

Acknowledgments

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