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HYBRID CONDITIONAL PLOT OF GOODNESS-OF-FIT FOR GUMBEL DISTRIBUTION

(Plot Bersyarat Hibrid bagi Ujian Kebagusan Penyuaian untuk Taburan Gumbel)

NAHDIYA ZAINAL ABIDIN, MOHD BAKRI ADAM & HABSHAH MIDI

ABSTRACT

A Gumbel model is an extreme value model that describes the event of extreme behaviour. The Gumbel model has an exponential tail. Generally, the goodness-of-fit for the Gumbel model is evaluated by the graphical form of probability plot (PP) and quantiles plot (QQ). The model fits the observed values if the probability and the quantiles of the hypothetical distribution are linearly plotted against that of the observed values. However, the QQ plot is quite sensitive to the deviation at the tail of the plot, as opposed to the PP plot which is somewhat robust. Thus, distribution of extreme values is likely to deviate from the linear line at the tail of the QQ plot. An alternative approach of plotting the Gumbel model is given, in which the approach is expected to produce the linear plot. The conditional plot and stabilised plot are employed and the performances of both are compared. The plots are transformed into the hybrid plot so that the departures of the hypothetical quantiles values from the observed quantiles values are illustrated. The result shows that the hybrid conditional QQ plot is a better plot of goodness-of-fit for Gumbel model.

Keywords: Gumbel; QQ plot; conditional plot; stabilised plot; hybrid plot

ABSTRAK

Model Gumbel ialah sebuah model nilai ekstrem yang menerangkan sifat kejadian ekstrem. Model Gumbel mempunyai ekor eksponen. Secara umumnya, ujian kebagusan penyuaian bagi model Gumbel dinilai melalui bentuk gambar rajah plot kebarangkalian (PP) dan kuantil (QQ). Model ini adalah sesuai dengan nilai cerapan sekiranya kebarangkalian dan kuantil bagi taburan hipotesis diplotkan secara linear terhadap kebarangkalian dan kuantil daripada nilai cerapan. Walau bagaimanapun, plot QQ adalah sensitif terhadap sisihan pada ekor plot berbanding dengan plot PP yang agak teguh. Oleh itu, sebarang taburan yang mempunyai nilai ekstrem adalah cenderung untuk menyimpang dari garisan linear di bahagian ekor plot QQ. Pendekatan alternatif bagi model Gumbel diberikan, yang dengan pendekatan ini dijangkakan akan menghasilkan plot linear. Plot bersyarat dan plot stabil digunakan dan prestasi kedua-dua plot dibandingkan. Plot-plot ini diubah kepada plot hibrid supaya sisihan kuantil hipotesis daripada kuantil cerapan dapat digambarkan. Keputusan menunjukkan plot hibrid bersyarat adalah plot yang lebih bagus bagi ujian kebagusan penyuaian untuk taburan Gumbel.

Kata kunci: Gumbel; plot QQ; plot bersyarat; plot stabil; plot hibrid

1. Introduction

A common plot used to assess the goodness-of-fit between the statistical model and the observed values is called QQ plot. For the set of probability values $p\in [0,1]$, the QQ plot compares the quantiles values of two respective distributions, such as hypothetical values $F^{-1}(p)$ and observed values $G^{-1}(p)$ (Wilk & Gnanadesikan 1968). However, there are several weaknesses in the QQ plot (Fisher 1983; Kafadar & Spiegelman 1986). This plot is quite sensitive to the deviation at the tail of the distribution, whereby instead of a linear line, a wavy look at the end of the line usually appears. As a result, distribution with extreme values leads to higher tendency of departure from linearity. Moreover, since QQ plot is assessed by eyes, it does not have concrete measure on the degree of fit other than the straight line with 45^o slope. Even so, the QQ plot does not necessarily rely on the linear line with 45^o. This is because a plot that exhibits a straight line away from 45^o implies two set of data in an agreement but differ only in terms of shift in location (El-Sadek 2010).

Wilk and Gnanadesikan (1968) have worked on the hybrid plot. The hybrid plot provides a view of the fit that is different from the QQ plot. The y-axis is represented by the deviation values between quantiles of the hypothetical data and that of observed data. The probability values, $F(x)$ are plotted on the x-axis. A good fit indicates small deviation between the hypothetical and the observed quantiles values, which is approximately zero. The fit is evaluated based on the line of deviation. Hence, small deviation points should fall along the horizontal line at the zero scale of the y-axis.

The conditional QQ plot is introduced by Kafadar and Spiegelman (1986). The purpose of the conditional QQ plot is to give a better plot than the QQ plot does for plotting the normal distribution. The conditional QQ plot is the modification of the QQ plot by conditioning the adjacent random variables or quantiles values and the order of the conditioned points are plotted. Furthermore, John (1983) has proposed the stabilised plot as the alternative plot. This plot is better than the existing plot in such a way that the transformed arcsine function is able to stabilise the variance of uniform distribution. In addition, Kimber (1985) has employed the stabilised plot for Exponential, Gumbel and Weibull models.

The aim of this study is to compare the linearity of classical QQ plot, conditional QQ plot, stabilised QQ plot and stabilised conditional QQ plot. In addition, as there are several weaknesses in QQ plot, this study suggested the hybrid plot as the alternative illustration to the existing linear plot. The QQ plot, conditional QQ plot, stabilised QQ plot and stabilised conditional QQ plot are transformed into hybrid plot. Hence, the comparisons are made in the form of hybrid QQ plot, hybrid conditional QQ plot, hybrid stabilised QQ plot and hybrid stabilised conditional QQ plot.

1.1 *Extreme value theory*

The Extreme value theory (EVT) is a statistical area which specifies on the extreme behaviour. EVT provides techniques for describing the distribution at maximum and minimum levels (Coles 2001). EVT is represented by the statistical extreme models. A Gumbel model is one of the extreme models. The Gumbel and normal models look almost alike, but they are not because Gumbel has a tail that is exponentially decayed (Coles 2001; Kotz & Nadarajah 2000).

The Gumbel model concentrates on the statistical behaviour of $M_n = \max(X_1,.., X_n)$, where M_n is the maximum value of the observation under distribution function over n time. The limiting distribution for M_n is highlighted by Theorem 1.1.

Theorem 1.1: *Let* $X_1,...,X_n$ *be an independent random variables with the distribution F, and let asymptotic argument be* $M_n = max\left(X_1,...,X_n\right)$. *As the sequences of constants* $a_n > 0$ *and* b_n *exist, denote*

$$\lim_{n\to\infty}\Pr\left(\frac{M_n - b_n}{a_n} \le x\right) \to F(x),$$

If the non-degenerate function, F exists, then F of Gumbel is:

$$F(x) = \exp\left\{-\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right\},$$

where $-\infty \le \mu \le \infty$ *and* $\sigma > 0$ *are the location and scale respectively.*

The quantiles of the Gumbel distribution is

$$x = \mu - \sigma\log\left[-\log(F(x)\right], \tag{1}$$

where U is the hypothetical distribution function. The hypothetical distribution is a uniform type which is defined as $\frac{i-0.5}{n}$.

2. Methodology

The statistical simulation, plot development and statistical analysis were done by R programming language 2.12.0. The simulation of Gumbel random variables with μ=100 and σ=10 was generated based on the quantiles function of Gumbel. This quantiles function was obtained by taking the inverse of the Gumbel distribution function. The simulation was replicated 10,000 times. The simulations were generated for sample of size n=10, 50 and 100. These simulated random variables are the quantiles values which were used to plot the QQ plot.

The Maximum Likelihood Estimation (MLE) approach was employed to approximate the values of μ and σ. The loglikelihood of Gumbel model is

$$l(\mu,\sigma) = -\sum_{i=1}^{n}\exp\left[-\left(\frac{x_i-\mu}{\sigma}\right)\right] - \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right) - n\log(\sigma), \tag{2}$$

The MLE maximise the loglikelihood function with respect to the parameters μ and σ. The maximum values denote the estimated values of the respective parameters.

2.1 *Conditional plot*

The conditional quantiles values are the points scattered on the conditional QQ plot. The conditional QQ plot is expected to be able to meet the linearity assumption.

Theorem 2.1: *Denote the ordered values of uniform distribution function,* U_i *by* $U_1, U_2, \ldots, U_n$*. The lower and upper end points are U=0 and U=1 respectively. The conditional* U_i *given by the order statistics* $U_1, \ldots, U_{i-1}, U_{i+1}, \ldots, U_n$ *is uniform in the interval* $[U_{i-1}, U_{i+1}]$*. Hence,*

$$E\left[U_i \mid U_{i-1}, U_{i+1}\right] = \frac{1}{2}\left[U_{i-1} + U_{i+1}\right]$$

2.2 *Stabilised Plot*

The stabilised plot is based on the arcsine function. The ordered statistics of the sine distribution have approximately equal variance. A stable plot implies equal variance. The stabilised plot is plotted as:

$$\left[\frac{2}{\pi}\text{arcsine}\left(F^{\frac{1}{2}}\right)\right] \tag{3}$$

The quantiles and conditional quantiles values were substituted into the stabilised plot function to generate the stabilised quantiles and stabilised conditional quantiles values respectively. The stabilised quantiles and stabilised conditional quantiles values are the plotted points on the stabilised QQ plot and stabilised conditional QQ plot respectively.

2.3 *Hybrid Plot*

The hybrid plot depicts the distance between the hypothetical and the observed quantiles values. The degree of deviation of the hypothetical quantiles from the observed quantiles indicates the degree of agreement between the model and the observed data. The closer the deviation values towards the zero value, the more likely the hypothetical and the observed quantiles values are comparable. The hybrid plot is different from the QQ plot in the sense that the horizontal axis of hybrid plot is plotted by the probability values, $F(x)$ while the plot in the vertical axis is built by the deviation values which are obtained from the differences between the hypothetical and the observed quantiles values. In other words, hybrid plot is developed by plotting the deviation values of quantiles against $F(x)$. The model fits the observed values well if the deviation values for $F(x)$ fall along the zero scale on the vertical axis. The horizontal scale is bounded from 0 to 1 because $F(x)$ has values ranged from 0 to 1.

Another drawback of QQ plot is the difficulty to compare the quantiles values from several models. The scale on the horizontal axis that varies with different type of statistical models is the main reason why comparison is difficult to be made by the QQ plot (Das & Resnick 2008). Unlike the QQ plot, the hybrid plot allows for comparing the differences in quantiles values from several models. This comparison can be done because the hybrid plot standardises the horizontal scale by plotting the $F(x)$ values which is bounded from 0 to 1.

3. Results

Figure 1(a) illustrates the hybrid QQ and hybrid conditional QQ plots of Gumbel distribution for n=50. Figures 1(b) show the hybrid stabilised QQ and hybrid stabilised conditional QQ plots of Gumbel distribution for n=50. For Figures 1(a) and (b), the symbol (Δ) implies the quantiles and stabilised quantiles points respectively while symbol (+) exhibits the conditional quantiles and stabilised conditional quantiles points respectively. Based on Figure 1(a), the high end point of quantiles values has large deviation value which is beyond -4. On the other hand, the end point of conditional quantiles values is smaller and closer to zero which is below 2. The hybrid stabilised QQ and hybrid stabilised conditional QQ plots shown by Figure 1(b) do not have outlying points because the plots produce points approximately along the zero scale.

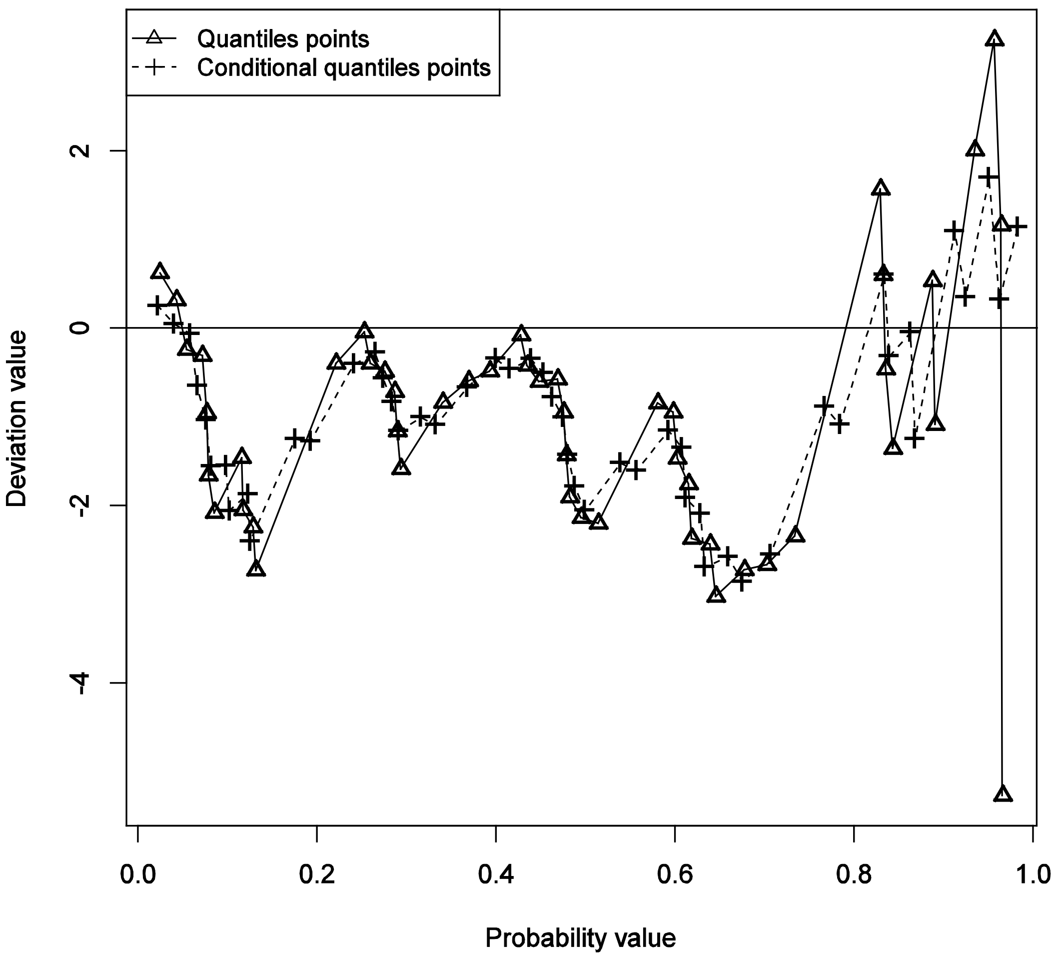

Figure 1(a): Hybrid QQ plot and hybrid conditional QQ plot of Gumbel distribution for n=50. Symbol (Δ) implies the quantiles points while symbol (+) exhibits the conditional quantiles points

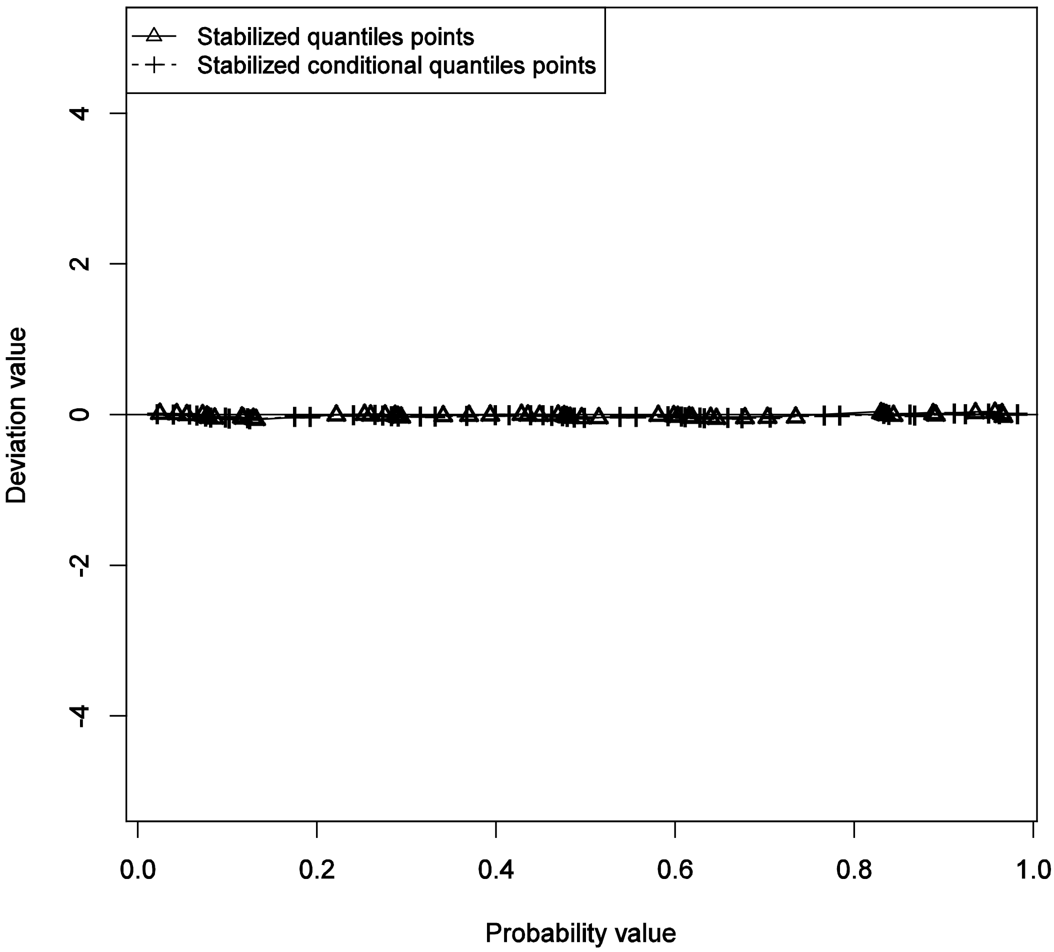

Figure 1(b): Hybrid stabilised QQ plot and hybrid stabilised conditional QQ plot of Gumbel distribution for n=50. Symbol (Δ) implies the stabilised quantiles points while symbol (+) exhibits the stabilised conditional quantiles points

We validated the goodness-of-fit of the plotting methods by plotting the alternative distribution against the hypothetical Gumbel distribution. The chosen alternative distribution was Fréchet distribution. Good plots will display the discrepancy of Fréchet data over Gumbel model. Figure 2(a) and Figure 2(b) show the fit for Fréchet random variables. The hybrid QQ and hybrid conditional QQ plots for n=50 is represented by Figure 2(a), while hybrid stabilised QQ and stabilised conditional QQ plots for n=50 is shown by Figure 2(b). The symbol (Δ) in Figure 2(a) and (b) represents the quantiles and stabilised quantiles points respectively. Likewise symbol (+) denotes the conditional quantiles and stabilised conditional quantiles points. Based on Figure 2(a), the highest deviation shown by the end point of quantiles values is approximately 60. In contrast, the deviation at end point of conditional quantiles values is greater than of quantiles values, which is close to 90. On contrary, Figure 2(b) illustrates the points along the zero scale, which implies stabilised and stabilised conditional values have low discrepancies.

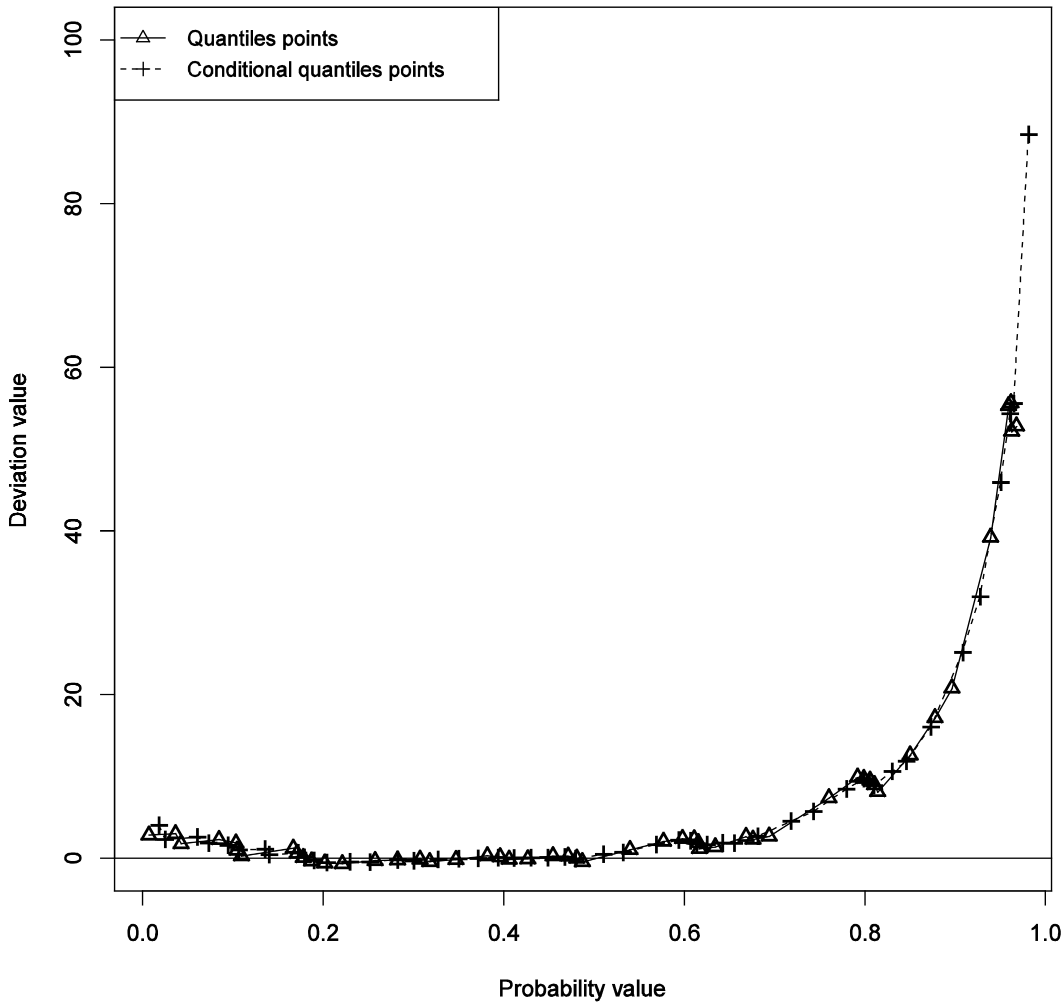

Figure 2(a): Hybrid QQ plot and hybrid conditional QQ plot of Fréchet distribution for n=50. Symbol (Δ) represents the quantiles points while symbol (+) denotes the conditional quantiles points

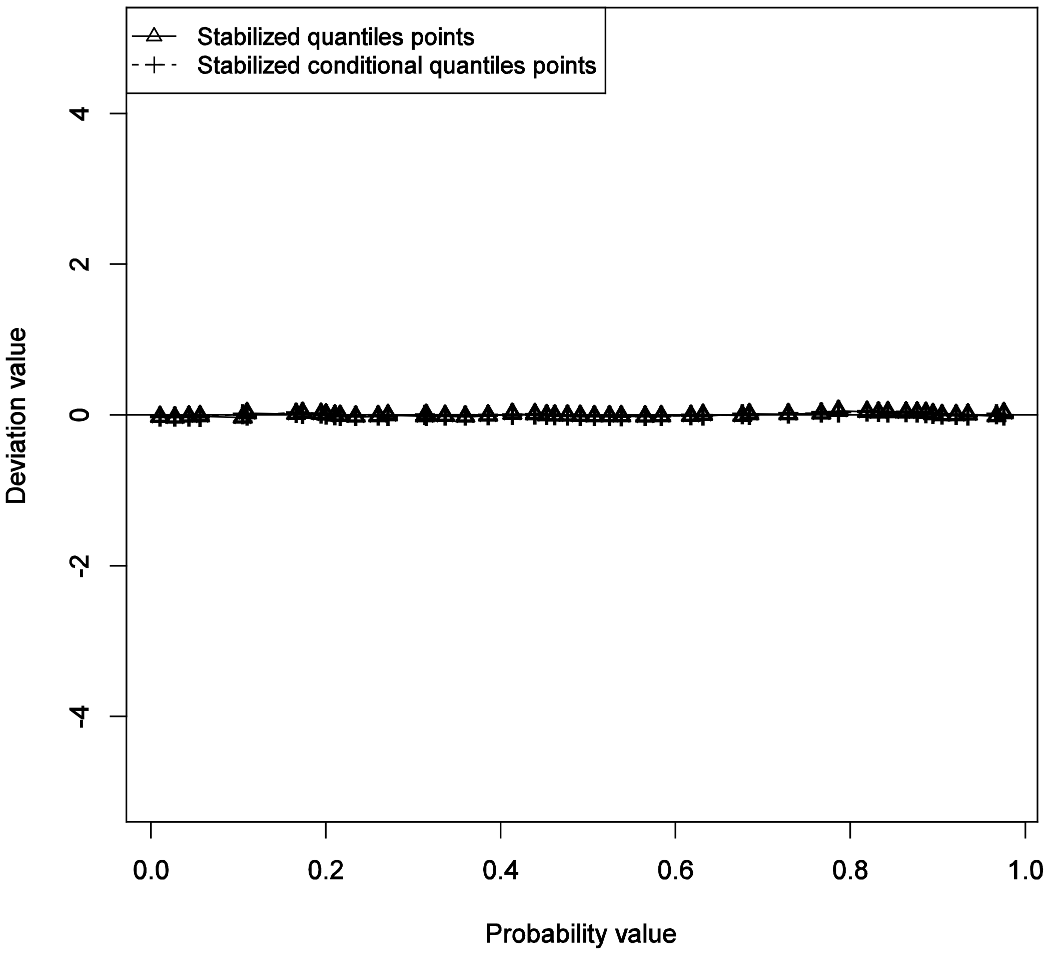

Figure 2(b): Hybrid stabilised QQ plot and hybrid stabilised conditional QQ plot of Fréchet distribution for n=50. Symbol (Δ) represents the stabilised quantiles points while symbol (+) denotes the stabilised conditional quantiles points

Table 1 shows the parameter values, the bias and root mean squares error (RMSE) of Gumbel for n = 10, 50 and 100. All the estimated parameters rest within their respective confidence intervals for all sample sizes. For the bias, the differences between the estimated parameters and the true parameter values (μ=10 and σ=100) were measured. As the sample size increases, the bias reduces as well. The same trend goes to the RMSE values. In addition, of the given types of the plots, the QQ plots has the biggest RMSE values, which is 3.145 for n=10. As we plotted the conditional QQ plot, stabilised QQ plot and stabilised conditional QQ plot, the RMSE values are getting smaller accordingly. This trend is similar for n=50 and 100.

Table 1: Parameter values, the bias and root mean squares error (RMSE) for Gumbel for sample of size n = 10, 50 and 100

n	Parameter		Bias		RMSE			
	μ	σ	μ	σ	QQ plot	Conditional QQ plot	Stabilised QQ plot	Stabilised conditional QQ plot
10	100.378 (95.06, 106.26)	9.277 (5.57,13.70)	0.378	0.723	3.145	1.940	0.073	0.046
50	100.045 (97.72, 102.63)	9.803 (8.13,11.72)	0.045	0.197	2.106	1.725	0.026	0.022
100	100.036 (98.31, 101.82)	9.942 (8.66,11.25)	0.036	0.058	1.332	1.187	0.018	0.017

Note: Values in the bracket are the confidence interval of the estimated parameters.

Table 2 displays the RMSE values for QQ plot, conditional QQ plot, stabilised QQ plot and stabilised conditional QQ plot of Gumbel and Fréchet random variables for n= 50. Based on Gumbel random variables, the QQ plot has RMSE value of 2.106. As the plot goes to the conditional QQ plot, the stabilised QQ plot and stabilised conditional QQ plot, the RMSE values become smaller respectively. In contrast, for Fréchet random variables, the RMSE value for QQ plot is 47.541 but the conditional QQ plot produces higher RMSE value which is 49.236. However, the stabilised plots have small RMSE values which are 0.028 and 0.024 respectively.

Table 2: Root mean squares errors (RMSE) values for Gumbel and Fréchet random variables for sample size n = 50

	RMSE			
Distribution	QQ plot	Conditional QQ plot	Stabilised QQ plot	Stabilised conditional QQ plot
Gumbel	2.106	1.725	0.026	0.022
Fréchet	47.541	49.236	0.028	0.024

4. Discussion

A good fit of statistical model to the observed values should have minimum deviation values which are close to zero. Therefore, the hybrid graph should be able to produce a horizontal line along the zero scale on the y-axis. Based on Figure 1(a), the Δ point at the end of the tail is plotted away from zero scale, of which the highest discrepancy is more than -4. However, the + point at the end of the tail has less discrepancy than the end point of Δ by having a plot closer to zero which is between 1 and 2. This indicates that conditional quantiles values have less deviation than the quantiles values. Hence, hybrid conditional QQ plot is a better plot than the hybrid QQ plot for Gumbel.

It has been signified that the hybrid conditional QQ plot produces smaller deviation than the hybrid QQ plot for Gumbel distribution. However, in order to check for the ability of the plots to illustrate the discrepancy between Gumbel model and non Gumbel observed values, an alternative random variables which is Fréchet is fitted against Gumbel model. For Fréchet random variables, the + points produce long tail of deviation in which the end point is over 80 as compared to Δ points with end point around 60. Therefore, the + points have larger deviation than the Δ points. Hence, the hybrid conditional QQ plot has better illustration for the discrepancy between Fréchet random variables and Gumbel model than the hybrid QQ plot. For hybrid stabilised QQ plot and hybrid stabilised conditional QQ plot, both plots do not illustrate the discrepancy between Fréchet random variables and Gumbel model.

The RMSE value is a value used to measure the degree of variation in error. Small RMSE value means small variation in error. Thus, a good fit should have small RMSE value. Of the given types of the plots, the smallest value of RMSE belongs to hybrid stabilised conditional QQ plot for both cases of Gumbel and Fréchet random variables. Unfortunately, these small values of RMSE do not indicate the hybrid stabilised conditional plot as the best plot. This is because hybrid stabilised conditional plot fail to verify that Fréchet distribution does not fit the Gumbel model. In contrast, hybrid QQ plot and hybrid conditional QQ plot have small RMSE values for Gumbel random variables and high values of RMSE for Fréchet random variables. Among them, hybrid conditional QQ plot has the smallest RMSE value for Gumbel and the greatest RMSE value for Fréchet. Therefore, hybrid conditional QQ plot is the best plot because it is able to identify the agreement between Gumbel distribution and Gumbel model as well to recognise the discrepancy between Fréchet distribution and Gumbel model.

Bias is the difference between the estimated values and true values of parameters. The bias that is close to zero indicates good estimate value. Based on the result, bias is getting smaller along with the increment of sample size. Hence, the greater the sample size, the more likely the estimated value of the parameter is precise.

For classical QQ plot, the fit of the model is assessed along the linear line at the angle of 45^o. The model loses the degree of fit if it departs from 45^o. Nevertheless, several models fit well even when the linear line does not perform the 45^o (Davidson & Mackinnon 2008). On contrary, hybrid plot evaluates the fit through horizontal line at zero scale of the vertical axis. The discrepancy can be observed whenever the line is not horizontal.

5. Conclusion

The hybrid conditional QQ plot is a better plot of goodness-of-fit for Gumbel model than the hybrid QQ plot, stabilised QQ plot and stabilised conditional QQ plot. This is because, given by Gumbel model, hybrid conditional QQ plot performs a good plot of goodness-of-fit for Gumbel random variables and a clearer plot of deviation for Fréchet random variables. Based on the RMSE values for Gumbel random variable, the validation is outstanding for large sample sizes, which are 50 and 100. Therefore, the larger the sample size, the higher the degree of fit of the distribution. The transformation of conditional QQ plot into the hybrid plot facilitates a clear observation on the degree of deviation between the hypothetical and observed values of conditional quantiles.

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Institute for Mathematical Research
Universiti Putra Malaysia
43400, UPM Serdang
Selangor DE, MALAYSIA
E-mails: aidan249@gmail.com, bakri@math.upm.edu.my, habshahmidi@gmail.com*

* Corresponding author