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A WEIGHTED FUZZY TIME SERIES MODEL FOR FORECASTING SEASONAL DATA

(Suatu Model Siri Masa Kabur Berpemberat untuk Meramal Data Bermusim)

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ABSTRACT

This study proposes a weighted model and graphical order selection in fuzzy seasonal time series forecasting. Initially, the fuzzy relationships were treated as if they were equally important, which might not properly reflected the importance of each individual fuzzy relationship in forecasting. Then, a linear chronological weight is introduced to handle the importance of each chronological individual fuzzy relationship. This paper proposes a naïve, uniform, and exponential chronological weight which is developed based on the concept of naïve, moving average, and exponential smoothing methods. In addition, graphical order fuzzy relationship is proposed to identify the best Fuzzy Logical Relationship order of fuzzy time series model. A quarterly data set is selected to illustrate the proposed method and to compare the forecasting accuracy with three other fuzzy time series models and two classical time series models. The results of the comparison using the test data show that the proposed method produces more precise forecast values than the other methods.

Keywords: Naïve, uniform, exponential weight; fuzzy time series; graphical order; seasonality

ABSTRAK

Kajian ini mencadangkan model berpemberat dan pemilihan tertib bergraf dalam peramalan siri masa kabur bermusim. Pada mulanya, hubungan kabur diandaikan sama penting, yang kemungkinannya tidak mencerminkan kepentingan sebenar setiap hubungan kabur dalam peramalan. Kemudian, satu pemberat linear kronologi diperkenalkan untuk menangani kepentingan setiap hubungan kabur individu kronologi. Makalah ini mencadangkan pemberat naif, seragam dan eksponen berkronologi yang dibangunkan berdasarkan konsep naif, purata bergerak dan kaedah pelicinan eksponen. Selanjutnya, hubungan kabur tertib bergraf diperkenalkan untuk mengenal pasti tertib hubungan logik kabur terbaik bagi model siri masa kabur. Data sukuan dipilih untuk menjelaskan kaedah yang dicadangkan dan untuk membuat perbandingan ketepatan peramalan dengan tiga model siri masa kabur dan dua model siri masa klasik. Hasil dapatan menggunakan data ujian menunjukkan kaedah yang dicadangkan menghasilkan nilai ramalan yang lebih tepat berbanding dengan kaedah lain.

Kata kunci: Pemberat naif, seragam, bereksponen; siri masa kabur; peringkat bergraf; kebermusiman

1. Introduction

Fuzzy set theory was introduced by Zadeh (1965). Based on this paper, fuzzy set theory has found many application areas in science. Fuzzy time series approach based on fuzzy set theory was introduced as an alternative method for conventional time series models. Recently, fuzzy time series has received much attention.

Song and Chissom first introduced the definitions of fuzzy time series, and developed their model by using fuzzy relation equations and approximate reasoning (see Song & Chissom 1993a; 1993b). Furthermore, Song and Chissom (1993b) divided the fuzzy time series into two types, namely time-variant and time-invariant, whose difference relies on whether there exists the same relation between time t and its prior time $t-k$ (where $k = 1,2,\ldots,m$). If the relations are all the same, it is a time-invariant fuzzy time series; likewise, if the relations are not the same, then it is time-variant.

Recently, Liu (2009) proposed an integrated fuzzy time series forecasting system in which the forecasted value will be a trapezoidal fuzzy number instead of a single-point value, and effectively deal with stationary, trend, and seasonal time series. Additionally, Egrioglu *et al.* (2009) proposed a new hybrid approach based on SARIMA and partial high order bivariate fuzzy time series for forecasting seasonal data.

In this paper, a new weighted fuzzy time series model based on the Exponential Smoothing method and graphical order selection are proposed to improve the forecast accuracy in seasonal data. In this model, a linear chronological weight from Yu (2005) is expanded to an exponential chronological weight. Moreover, most of the previous studies did not show how to choose easily and effectively the appropriate order of fuzzy time series. Egrioglu *et al.* (2009) stated that it is possible to model seasonal time series when high order models are used. These high-order models generally include chronological lagged variables which are corresponding to the period number. This paper shows that the graphical order fuzzy relationship could be used easily to select an appropriate order of fuzzy time series. Additionally, this study also shows that by using a quarterly electricity usage data (Hanke & Wichern 2009) the only seasonal-order fuzzy time series or subset high-order fuzzy time series model outperforms the fuzzy time series proposed by Chen (1996), Yu (2005), and Cheng *et al.* (2008).

2. Proposed Model and Algorithm

Yu (2005) explained that there are two reasons why weighted fuzzy time series models could be proposed. The first was to resolve recurrent fuzzy relationships, and the other was to assign proper weights to various fuzzy relationships to reflect the differences in their importance.

2.1 Weights

Initially, the repeated fuzzy logical relationships (FLRs) were simply ignored when fuzzy relationships were established. In many previous studies, each FLR was treated as if it was of equal importance, which may not have reflected the real world situation (see Chen 1996; Huarng 2001; Huarng & Yu 2003; Song & Chissom 1993a; 1993b; 1994). In this scenario, the occurrences of the same FLRs are regarded as if there were only one occurrence. In other words, the recent identical FLRs are simply ignored. To explain this, suppose there are FLRs in chronological order that have the same left han3d side (LHS), A_1 as follows:

$$(t=1)\ A_1 \to A_2, (t=2)\ A_1 \to A_1, (t=3)\ A_1 \to A_1, (t=4)\ A_1 \to A_3, (t=5)\ A_1 \to A_1 \quad (1)$$

Following Chen (1996), these FLRs in Eq. (1) are used to establish an FLRG as

$$\mathrm{A}_1 \to \mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3 \quad (2)$$

The ignoring of recurrence, however, is questionable. Yu (2005) argued that the occurrence of a particular FLR represents the number of its appearances in the past. For instance, in Eq.,

$A_1 \to A_1$ appears three times, both $A_1 \to A_2$ and $A_1 \to A_3$ only once. The recurrence can be used to indicate how the FLR may appear in the future.

Yu (2005) proposed the chronological weights to deal with recurrent fuzzy relationships and their importance. To illustrate it, suppose there are FLRs in chronological order as in Eq. , and then the weights are as follows:

$$\begin{aligned}&(t=1)\ A_1 \to A_2 \text{ with weight } 1, (t=2)\ A_1 \to A_1 \text{ with weight } 2, (t=3)\ A_1 \to A_1\\ &\text{with weight } 3, (t=4)\ A_1 \to A_3 \text{with weight } 4, (t=5)\ A_1 \to A_1 \text{ with weight } 5\end{aligned} \tag{3}$$

As a result, the most recent FLR $(t = 5)$ is assigned the highest weight of 5, which means that the probability of its appearance in the near future is higher than in the case of the others. On the other hand, the most aged FLR $(t = 1)$ is assigned the lowest weight of 1, which means that the probability of its appearance in the near future is lower than in the case of the others.

Cheng *et al.* (2008) proposed the weights focused on the probability of its appearance and ignored the importance of chronological FLR. To explain it, suppose there are FLRs in chronological order as in Eq. , and then the weights are as follows:

$$\begin{aligned}&(t=1)\ A_1 \to A_2 \text{ with weight } 1, (t=2)\ A_1 \to A_1 \text{ with weight } 1, (t=3)\ A_1 \to A_1\\ &\text{with weight } 2, (t=4)\ A_1 \to A_3 \text{ with weight } 1, (t=5)\ A_1 \to A_1 \text{ with weight } 3\end{aligned} \tag{4}$$

2.2 Naïve, Uniform, and Exponential Chronological Weights

In classical time series analysis, there are two forecasting methods that use the concept of weights for calculating the forecasts, namely moving average and exponential smoothing methods. The method that was proposed by Yu (2005) could be seen as a linear smoothing method, particularly about the chronological weights. This indicates that the importance of chronological FLRs increase linearly.

In this section, we propose a new different weight schemes based on Yu's article and combine to average and exponential smoothing concept. We propose that the importance of FLRs chronological incrementally following uniform or exponential pattern, not linear ones as compared to Yu's. In this case, suppose the forecast of $F(t)$ is $A_{j1}, A_{j2}, \ldots, A_{jk}$. The corresponding weights for $A_{j1}, A_{j2}, \ldots, A_{jk}$, say $w'_1, w'_2, \ldots, w'_k$ are specified as:

$$w'_i = \frac{w_i}{\sum_{h=1}^{k} w_h}$$

where $w_1 = 1$, and $w_i = c^{i-1}$, for $c \geq 1$ and $2 \leq i \leq k$. Then the weight matrix can be obtained and written as

$$W(t) = \left[w'_1, w'_2, \ldots, w'_k\right] = \left[\frac{w_1}{\sum_{h=1}^{k} w_h}, \frac{w_2}{\sum_{h=1}^{k} w_h}, \ldots, \frac{w_k}{\sum_{h=1}^{k} w_h}\right]$$
$$= \left[\frac{1}{\sum_{h=1}^{k} w_h}, \frac{c}{\sum_{h=1}^{k} w_h}, \frac{c^2}{\sum_{h=1}^{k} w_h}, \ldots, \frac{c^{k-1}}{\sum_{h=1}^{k} w_h}\right]$$

where w_h is the corresponding weight for A_{jh}.

To illustrate these, suppose there are FLRs in chronological order as in Eq. , and $c = 2$. The assign weights are shown in Table 1.

Table 1: Assigned weights of Yu's, Cheng's, and the proposed methods

Chronological	FLR	Assigned weight		
		Yu's	Cheng's	Proposed method
$(t=1)$	$A_1 \rightarrow A_2$	1	1	1
$(t=2)$	$A_1 \rightarrow A_1$	2	1	2
$(t=3)$	$A_1 \rightarrow A_1$	3	2	4
$(t=4)$	$A_1 \rightarrow A_3$	4	1	8
$(t=5)$	$A_1 \rightarrow A_1$	5	3	16
Total		15	8	31

These weights are standardised to obtain the weight matrix as shown in Table 2. Graphically, the comparison of the FLRs importance between the proposed weights and others could be seen in Figure 1.

Table 2: The standardised weights of Yu's, Cheng's, and the proposed methods

Chronological	FLR	Assigned weight		
		Yu's	Cheng's	Proposed method
$(t=1)$	$A_1 \rightarrow A_2$	1/15	1/8	1/31
$(t=2)$	$A_1 \rightarrow A_1$	2/15	1/8	2/31
$(t=3)$	$A_1 \rightarrow A_1$	3/15	2/8	4/31
$(t=4)$	$A_1 \rightarrow A_3$	4/15	1/8	8/31
$(t=5)$	$A_1 \rightarrow A_1$	5/15	3/8	16/31
Total		1	1	1

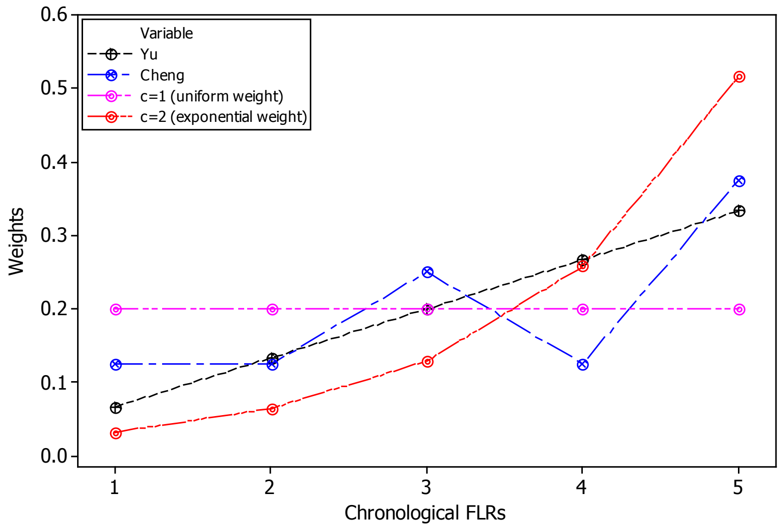

Figure 1: Comparison between Yu's, Cheng's, and the proposed weights

The results from Figure 1 show that the proposed weights have uniformly or exponentially increasing weights over the FLRs chronological. The basic idea is the same with Yu's weights and classical exponential smoothing method (Hanke & Wichern 2009) that the recent FLRs are more important than the older ones; hence, higher weights are assigned to the recent ones. Obviously, the main difference is on the weights increments which imply the different pattern of the chronological weights. It means that the proposed weights tend to give the recent FLRs, as more important than the older ones and generally higher values than Yu's weights. In addition, these proposed weights also show that when $c = 1$ then the weights will have uniformly chronological pattern which imply the same important of chronological time relationship. Moreover, when $c \to \infty$ then the weights only involved with the last chronological time relationship and this concept have similarity with Naïve method (Hanke&Wichern 2009). Hence, we shall call this weight as Naïve weight.

2.1 Graphical Order of FLRs

One of the issues that is still of interest by researchers in fuzzy time series is how to select easily and effectively the order of fuzzy time series for certain real data. In this section, we propose graphical order of FLRs to identify the best order of fuzzy time series. To illustrate it, we use a set of data on electricity usage for Washington Water Power Company (Hanke & Wichern 2009). The data consist of 66 quarterly records of the electricity usage from the first quarter (Q1) in 1980 until the second quarter (Q2) in 1996.

The graphical order of FLRs to identify the best order of fuzzy time series is used in step 5 of the proposed algorithm. To demonstrate the proposed algorithm, these electrical usage data is used as a numerical example to detail the proposed algorithm, particularly the concept of graphical order of FLRs, as follows.

Step 1. Define the universe of discourse and partition it into intervals.
We use histogram to identify the partition for the universe of discourse $U = [D_{min} - D_1, D_{max} + D_2]$ into even lengthy of equal length intervals $u_1, u_2, \ldots, u_m$. The results show that $D_{min} = 480$ and $D_{max} = 1086$. We partition $U = [475,1125]$ into 13 intervals $u_1, u_2, \ldots, u_{13}$, where $u_1 = [475,525]$, $u_2 = [525,575]$, $u_3 = [575,625]$, $u_4 = [625,675]$, $u_5 = [675,725]$, $u_6 = [725,775]$, $u_7 = [775,825]$, $u_8 = [825,875]$, $u_9 = [875,925]$, $u_{10} = [925,975]$, $u_{11} = [975,1025]$, $u_{12} = [1025,1075]$, and $u_{13} = [1075,1125]$. Thus, in this paper we let $D_1 = 5$ and $D_2 = 39$.

Step 2. Establish a related fuzzy set (linguistic value) for each observation in the training dataset.
In this step, the fuzzy sets, $A_1, A_2, \mathrm{K}, A_k$, for the universe of discourse are defined by Eq. (5), where the value of a_{ij} indicates the grade of membership of u_j in fuzzy set A_i, where $a_{ij} \in [0,1]$, $1 \le i \le k$ and $1 \le j \le m$. Ascertain the degree of each electricity usage belonging to each $A_i\,(i = 1, \ldots, m)$. If the maximum membership of the electricity usage is under A_k, then the fuzzified electricity usage is labelled as A_k [1].

$$\begin{aligned} A_1 &= a_{11}/u_1 + a_{12}/u_2 + \cdots + a_{1m}/u_m \\ A_2 &= a_{21}/u_1 + a_{22}/u_2 + \cdots + a_{2m}/u_m \\ &\vdots \\ A_k &= a_{k1}/u_1 + a_{k2}/u_2 + \cdots + a_{km}/u_m \end{aligned} \tag{5}$$

Step 3. Establish fuzzy relationships.
Two consecutive fuzzy sets $A_i(t-p)$ and $A_j(t)$ can be established into a single FLR as $A_i \to A_j$. In this step we present $p = 1, 2, \ldots, p^*$ and will be used in the next step to find the best p of two consecutive fuzzy sets as an appropriate order of fuzzy time series model.

Step 4. Establish fuzzy relationship groups for all FLRs.
The FLRs with the same LHSs can be grouped to form a FLR Group by applying chronological relationships as proposed by Yu (2005). For instance $A_i \to A_j$, $A_i \to A_k$, $A_i \to A_k$, $A_i \to A_m$ can be grouped as $A_i \to A_j, A_k, A_k, A_m$.

Step 5. Select the best order of FLRs.
The graphical orders for FLRs and fluctuation-type matrices are two useful tools to identify easily and effectively the best order of FLRs. The smallest fluctuation order graph is chosen as the best order of fuzzy time series. The graphical orders in Figure 2 show the fluctuation for FLRs at order 1 and 4.

These graphs illustrate that the smallest fluctuation for all FLRs is at order 4. This means that the best order for fuzzy time series is order 4 only or seasonal fuzzy time series with period 4. In addition, the graphical orders also show that the relationship patterns in fuzzy time series are nonlinear. Thus, fuzzy time series could be seen as nonlinear time series models. To validate the goodness of the order selection, we present and compare the results of the first order and only order four (seasonal) fuzzy time series in the calculation in the following steps.

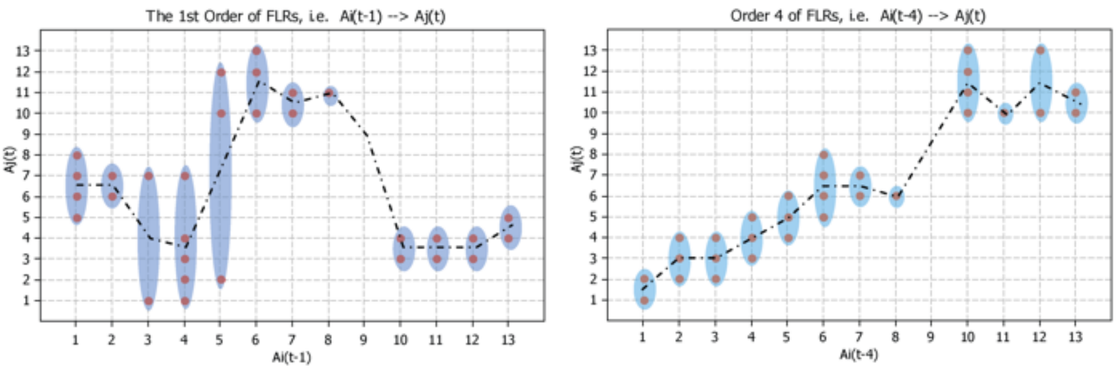

Figure 2: Graphical order of FLR Groups for the electricity usage data

Step 6. Forecast.
The forecast uses the same rule as Yu (2005).

Step 7. Defuzzify.

Suppose the forecast of $F(t)$ is $A_{j1}, A_{j2}, \ldots, A_{jk}$. The defuzzified matrix is equal to a matrix of the midpoints of $A_{j1}, A_{j2}, \ldots, A_{jk}$:

$$M(t) = \left[m_{j1}, m_{j2}, \ldots, m_{jk}\right],$$

where $M(t)$ represents the defuzzified forecast of $F(t)$.

Step 8. Assigning weights.

As discussed in Section 2.2, suppose that the forecast of $F(t)$ is $A_{j1}, A_{j2}, \ldots, A_{jk}$. The corresponding weight matrix for $A_{j1}, A_{j2}, \ldots, A_{jk}$, say $w'_1, w'_2, \ldots, w'_k$ are

$$W(t) = \left[\frac{1}{\sum_{h=1}^{k} w_h}, \frac{c}{\sum_{h=1}^{k} w_h}, \frac{c^2}{\sum_{h=1}^{k} w_h}, \ldots, \frac{c^{k-1}}{\sum_{h=1}^{k} w_h}\right] \tag{6}$$

where $w_1 = 1, w_i = c^{i-1}$, for $c > 1$, $2 \le i \le k$, and w_h is the corresponding weight for A_{jh}.

Step 9. Calculate forecast value.
In the weighted model, the final forecast is equal to the product of the defuzzified matrix and the transpose of the weight matrix:

$$\hat{F}(t) = M(t) \times W(t)^T = \left[m_{j1}, m_{j2}, \ldots, m_{jk}\right] \times \left[w'_1, w'_2, \ldots, w'_k\right]^T$$
$$= \left[m_{j1}, m_{j2}, \ldots, m_{jk}\right] \times \left[\frac{1}{\sum_{h=1}^{k} w_h}, \frac{c}{\sum_{h=1}^{k} w_h}, \ldots, \frac{c^{k-1}}{\sum_{h=1}^{k} w_h}\right]^T \quad (7)$$

where is the matrix product operator, and $M(t)$ is a $1 \times k$ matrix and $W(t)^T$ is a $k \times 1$ matrix, respectively. For example, consider Group 1 in the first order FLRs where $F(t-1) = A_1$ and the forecast of $F(t)$ is $A_6, A_6, A_7, A_6, A_6, A_8, A_6, A_5, A_5, A_5$. From Eq. (7), for $c = 2$ the final forecast value is determined as

$$\hat{F}(t) = \left[m_6, m_6, m_7, m_6, m_6, m_8, m_6, m_5, m_5, m_5\right] \times \left[\frac{1}{1023}, \frac{2^1}{1023}, \ldots, \frac{2^9}{1023}\right]^T$$
$$= 709.53$$

By optimising the RMSE for the test data, the optimal value of c for the 4th order FLRs is 2.6 and the resulting RMSE of various values of c starting 1 to 5 with grid 0.1 can be seen in Figure 3.

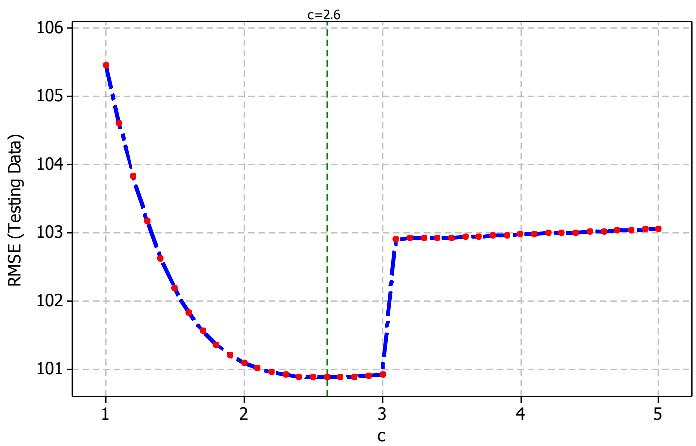

Figure 3: The optimal value of c at order 4 of FLRs for the test data

3. Results

The forecasting performance of the proposed model with $c = 2.6$ is verified by using the electricity usage data and three fuzzy time series models, which are Chen's (1996), Yu's (2005), and Cheng's *et al*. (2008), with the optimal weighted parameter (the 1st order, and the 4th order, $\alpha = 0.34$), are employed as comparison models.

Table 3 shows the complete results of the final forecast values for each fuzzy time series model for the 1st order and order 4 (seasonal) models respectively. To evaluate the performance of uniform and exponential weighted fuzzy time series, the root mean squared error (RMSE) is selected as an evaluation index, both in training and testing data.

The results of the RMSEs obtained using the uniform $(c = 1)$, exponential $(c = 2.6)$ and other three weighted fuzzy time series and two classical time series models, both in training and testing data, are listed in Table 4. These results show that the proposed uniform seasonal fuzzy time series yields the best forecast in training data, whereas the proposed exponential weighted seasonal fuzzy time series yields the best forecast in testing data. Based on the performance evaluation in testing data, the results show that the proposed seasonal method generates more accurate forecasted values than the three other fuzzy time series models and the two classical time series models.

Table 3: The final forecast values for the test data

		The 1st order fuzzy time series				The seasonal fuzzy time series			
t	*Y(t)*	Chen's (1996)	Yu's (2005)	Cheng's *et al.* (2008)	Proposed method	Chen's (1996)	Yu's (2005)	Cheng's *et al.* (2008)	Proposed method
61	1002	975	965.00	914.61	962.10	1025	1033.33	988.14	1027.49
62	887	625	633.93	653.58	647.31	650	647.22	646.23	652.54
63	615	620	604.55	546.06	728.86	650	647.22	635.01	652.54
64	828	650	585.71	754.46	1061.17	775	795.00	804.83	798.21
65	1003	620	585.71	935.09	618.54	1025	995.00	965.21	986.30
66	706	650	585.71	673.62	685.22	650	647.22	646.23	652.54
	RMSE	204.494	228.337	113.976	213.937	103.563	103.378	103.361	100.874

Table 4: Comparison of RMSEs found for the training and the test data

The order of fuzzy time series	Method	Training		Testing	
		RMSE	Rank	RMSE	Rank
The 1st	Chen's (1996)	89.874	11	204.494	10
	Yu's (2005)	84.147	10	228.337	12
	Cheng's *et al*. (2008)	80.014	9	113.976	8
	Proposed	78.785	8	115.814	9
	Proposed	110.877	12	213.937	11

To be continued...

...Continuation

The 4th	Chen's (1996)	35.482	3	103.563	4
(Seasonal)	Yu's (2005)	33.387	2	103.378	3
	Cheng's *et al*. (2008)	43.514	5	103.360	2
	Proposed,	33.026	1	105.353	6
	Proposed,	35.902	4	100.874	1
Time Series Regression		43.576	6	103.643	5
ARIMA([2,4],0,0)		54.479	7	111.71	7

4. Conclusion

In this paper, we proposed uniform and exponential weighted fuzzy time series models and graphical order of FLRs to identify an appropriate order of fuzzy time series. This new weight can provide decision analyst various degrees of importance to the chronological relationships. Meanwhile, the graphical order of FLRs can easily and effectively deal with order identification of fuzzy time series. An electricity usage data has been employed to compare the forecasting accuracy between uniform and exponential weighted fuzzy time series and three fuzzy time series methods (Chen 1996; Yu 2005; Cheng *et al*. 2008). The results using the test data show that an exponential weighted seasonal model provides more accurate forecasting values than those of three fuzzy time series.

This study proposes a weight scheme where the more recent ones have higher weights than the older ones as in Yu (2005). In addition, other issue that could be further investigated is the hybrid methods for forecasting data with trend and seasonality as proposed by Liu (2009) and Egrioglu *et al*. (2009). These hybrid methods could be expanded for resolving other time series patterns such as calendar variations and interventions.

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