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NEW SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS DEFINED BY A GENERALISED DIFFERENTIAL OPERATOR

(Subkelas Baharu Fungsi Univalen Harmonik yang Ditakrif oleh Pengoperasi Pembeza Teritlak)

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ABSTRACT

Let $S_{\mathcal{H}}$ denote the class of functions $f = h + \overline{g}$ which are harmonic univalent and sense preserving in the unit disk *U*. Earlier we have introduced a class of harmonic functions defined by a generalised differential operator. In this paper we introduce a new subclass of this class and obtain results on coefficient bounds, distortion and extreme points.

Keywords: Univalent functions; harmonic functions; generalised differential operator

ABSTRAK

Andaikan $S_{\mathcal{H}}$ melambangkan kelas fungsi $f = h + \overline{g}$ harmonik uivalen dan kekal orientasi dalam cakera unit *U*. Terlebih dahulu telah diperkenalkan suatu kelas fungsi harmonik yang ditakrif oleh suatu pengoperasi pembeza teritlak. Dalam makalah ini diperkenalkan pula suatu subkelas bagi kelas tersebut, dan mendapatkan keputusan mengenai batas-batas pekali, erotan dan titik-titik ekstrem.

Kata kunci: Fungsi univalen; fungsi harmonik; pengoperasi pembeza teritlak

1. Introduction

A continuous functions $f = u + iv$ is a complex harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $D \subseteq \mathbb{C}$ we can write $f(z) = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in *D* (See Clunie & Shell-Small 1984). Denote by $S_{\mathcal{H}}$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $U = \{z : |z| < 1\}$ for which $f(0) = h(0) = f_z(0) - 1 = 0.$ For $f = h + \overline{g} \in S_{\mathrm{H}}$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \quad |b_1| < 1. \tag{1.1}$$

The class T is defined as the subclass of $S_{\mathcal{H}}$ consisting of all functions $f(z) = h + \overline{g}$, where h and g are given by

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |b_n| z^n. \tag{1.2}$$

Clunie and Shell-Small (1984) investigated the class $S_{\mathcal{H}}$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there has been several related papers on $S_{\mathcal{H}}$ and its subclasses such that Silverman (1998), Silverman and Silvia (1999), and Jahangiri (1999) studied the harmonic univalent functions.

We denote by $\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$ the class of all function of the form (1.1) that satisfy the condition

$$\mathrm{Re}\left(D^k_{\alpha,\beta,\lambda,\delta}f(z)\right)' > 1-|\gamma|,\ z\in U \tag{1.3}$$

where $\gamma\in\mathrm{C}, k\in\mathrm{N}_0$, $D^k_{\alpha,\beta,\lambda,\delta}f(z)=D^k_{\alpha,\beta,\lambda,\delta}h(z)+\overline{D^k_{\alpha,\beta,\lambda,\delta}g(z)}$ and $D^k_{\alpha,\beta,\lambda,\delta}f(z)$ denote the operator introduced by Ramadan and Darus (2011) and given by

$$D^0 f(z)=f(z)$$

$$D^1_{\alpha,\beta,\lambda,\delta}f(z)=\left[1-(\lambda-\delta)(\beta-\alpha)\right]f(z)+(\lambda-\delta)(\beta-\alpha)zf'(z)$$

$$=z+\sum_{n=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]a_n z^n$$

that is

$$D^k_{\alpha,\beta,\lambda,\delta}f(z)=D^1_{\alpha,\beta,\lambda,\delta}\left(D^{k-1}_{\alpha,\beta,\lambda,\delta}f(z)\right)$$

$$D^k_{\alpha,\beta,\lambda,\delta}f(z)=z+\sum_{n=2}^{\infty}\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k a_n z^n, \tag{1.4}$$

for $\alpha\ge 0,\beta\ge 0,\lambda\ge 0,\delta\ge 0,\lambda>\delta,\beta>\alpha$ and $k\in\{0,1,2,...\}$.

Remark 1.1. *(i) When* $\alpha=0,\delta=0,\lambda=1,\beta=1$ we get Salagean differential operator (see Salagean 1983).

(ii) When $\alpha=0$ we get Darus & Ibrahim differential operator (Darus & Ibrahim 2009).

(iii) And when $\alpha=0,\delta=0,\beta=1$ we get Al- Oboudi differential operator (Al-Oboudi 2004).

Note that:

$\mathrm{H}^0(0,1,\lambda,0,1)\equiv\mathrm{H}(\lambda)$ studied by Yalcin and Ozturk (2004).

$\mathrm{H}^0(0,1,\lambda,0,\gamma) \equiv \mathrm{H}(\lambda,\gamma)$ studied by Janteng *et al.* (2007). Also we note that for the analytic part of the class $\mathrm{H}^0(0,1,\lambda,0,\gamma)$ was introduced and studied by Altintas and Ertekin (1992). We further denote by $T\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$ the subclass of $\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$, where $T\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma) = T \cap \mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$.

2. Coefficients Bounds

Theorem 2.1 *Let* $f(z) = h + \overline{g}$, *with* h *and* g *be given by* (1.1). *Let*

$$\sum_{n=2}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)\right]^k \left(|a_n|+|b_n|\right) \le |\gamma| - |b_1|, \tag{2.1}$$

where $a_1 = 1, \gamma \in \mathrm{C}$, $\alpha \ge 0, \beta \ge 0, \lambda \ge 0, \delta \ge 0, \lambda > \delta, \beta > \alpha$ and $k \in \mathrm{N}_0$. Then f is harmonic univalent sense preserving in U and $f \in \mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$.

Proof: For $|z_1| \le |z_2|$, we have by (2.1),

$$|f(z_1) - f(z_2)| \ge |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)|$$

$$= \left|(z_1 - z_2) + \sum_{n=2}^{\infty} a_n \left(z_1^n - z_2^n\right)\right| - \left|\sum_{n=1}^{\infty} b_n \left(z_1^n - z_2^n\right)\right|$$

$$\ge |z_1 - z_2| \left(1 - |b_1| - \sum_{n=2}^{\infty} n\left(|a_n| + |b_n|\right)|z|^{n-1}\right)$$

$$\ge |z_1 - z_2| \left(1 - |b_1| - |z_2| \sum_{n=2}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k \left(|a_n| + |b_n|\right)\right)$$

$$\ge |z_1 - z_2| \left[1 - |b_1| - |z_2|\left(|\gamma| - |b_1|\right)\right] > 0.$$

Consequently, f is univalent in U. We note that f is sense preserving in U. This is because

$$|h'(z)| \ge 1 - \sum_{n=2}^{\infty} n|a_n||z|^{n-1} > 1 - \sum_{n=2}^{\infty} n|a_n|$$

$$\ge 1 - \sum_{n=2}^{\infty} \frac{n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k}{|\gamma|}|a_n|$$

$$\geq \sum_{n=1}^{\infty} \frac{n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k}}{|\gamma|}|b_n|$$

$$> \sum_{n=1}^{\infty} n|b_n||z|^{n-1} \geq |g'(z)|.$$

Now we show that $f \in \mathrm{H}^{k}(\alpha,\beta,\lambda,\delta,\gamma)$. Using the fact that $\mathrm{Re}\{w\} > 1-|\gamma|$ if and only if $||\gamma|+w| \geq |2-|\gamma|-w|$, it suffices to show that

$$\left|\gamma+\left(D^{k}_{\alpha,\beta,\lambda,\delta}f(z)\right)' - \overline{\left(D^{k}_{\alpha,\beta,\lambda,\delta}g(z)\right)'}\right| - \left|2-\gamma-\left(D^{k}_{\alpha,\beta,\lambda,\delta}f(z)\right)' + \overline{\left(D^{k}_{\alpha,\beta,\lambda,\delta}g(z)\right)'}\right|$$

$$= \Bigg|\gamma+1+\sum_{n=2}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k} a_n z^{n-1}$$

$$\overline{-\sum_{n=1}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k} b_n z^{n-1}}\Bigg|$$

$$-\Bigg|2-\gamma-1-\sum_{n=2}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k} a_n z^{n-1}$$

$$+\overline{\sum_{n=1}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k} b_n z^{n-1}}\Bigg|$$

$$\geq 2|\gamma| - \sum_{n=2}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k} |a_n||z|^{n-1}$$

$$-\sum_{n=1}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k} |b_n||z|^{n-1}$$

$$-\sum_{n=2}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k} |a_n||z|^{n-1}$$

$$-\sum_{n=1}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k} |b_n||z|^{n-1}$$

$$= 2|\gamma| - 2\sum_{n=2}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k}|a_n||z|^{n-1}$$

$$-2\sum_{n=1}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k}|b_n||z|^{n-1}$$

$$\geq 2\left\{|\gamma| - \left(\sum_{n=2}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k}|a_n|\right.\right.$$

$$\left.\left.+\sum_{n=1}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k}|b_n|\right)\right\} \geq 0,$$

by (2.1). The harmonic mappings

$$f(z) = z + \sum_{n=2}^{\infty} \frac{|\gamma| x_n}{n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k}} z^n$$

$$+\overline{\sum_{n=1}^{\infty} \frac{|\gamma| y_n}{n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k}} z^n}, \tag{2.2}$$

where $\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1,$ show that the coefficient bound given by (2.1) is sharp.

The functions of the form (2.2) are in $\mathrm{H}^{k}(\alpha,\beta,\lambda,\delta,\gamma)$ because

$$\sum_{n=1}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k}\left(|a_n|+|b_n|\right)$$

$$= 1 + |\gamma|\left\{\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n|\right\} = 1 + |\gamma|.$$

The restriction placed in Theorem 2.1 on the moduli of coefficients of $f(z) = h + \overline{g}$, enables us to conclude for arbitrary rotation of the coefficients of f that the resulting functions would still be harmonic univalent and $f \in H^{k}(\alpha,\beta,\lambda,\delta,\gamma)$. We next show that the condition (2.1) is also necessary for functions f in $T\,\mathrm{H}^{k}(\alpha,\beta,\lambda,\delta,\gamma)$.

Theorem 2.2 *Let* $f(z) = h + \overline{g}$, *with* h *and* g *be given by* (1.1). *Then* $f \in T\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$ *if and only if*

$$\sum_{n=2}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)\right]^k\left(|a_n|+|b_n|\right) \le |\gamma| - |b_1|, \tag{2.3}$$

where $a_1 = 1, \gamma \in \mathrm{C}$ $\alpha \ge 0, \beta \ge 0, \lambda \ge 0, \delta \ge 0, \lambda > \delta, \beta > \alpha$ and $k \in \mathrm{N}_0$.

Proof: We first suppose that $f \in T\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$, then by (1.3) we have

$$\mathrm{Re}\left\{\left(D^k_{\alpha,\beta,\lambda,\delta}h(z)\right)' - \overline{\left(D^k_{\alpha,\beta,\lambda,\delta}g(z)\right)'}\right\}$$

$$= \mathrm{Re}\left\{1 - \sum_{n=2}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k |a_n| z^{n-1}\right.$$

$$\left.\overline{-\sum_{n=1}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k |b_n| z^{n-1}}\right\}$$

$$> 1 - |\gamma|.$$

If we choose z to be real and let $z \to 1^-$, we get

$$1 - \sum_{n=2}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k |a_n| -$$

$$\sum_{n=1}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k |b_n| \ge 1 - |\gamma|,$$

which is precisely the assertion (2.3).

Conversely, suppose that the inequality (2.3) holds true. Then we find from the equation (1.3) that

$$\mathrm{Re}\left\{\left(D^k_{\alpha,\beta,\lambda,\delta}h(z)\right)' - \overline{\left(D^k_{\alpha,\beta,\lambda,\delta}g(z)\right)'}\right\}$$

$$= \mathrm{Re}\left\{1 - \sum_{n=2}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k |a_n| z^{n-1}\right.$$

$$\overline{-\sum_{n=1}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k |b_n| z^{n-1}}$$

$$\geq 2-\sum_{n=1}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k}\left(|a_n|+|b_n|\right) z^{n-1}$$

$$>2-\sum_{n=1}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^{k}\left(|a_n|+|b_n|\right) \geq 1-|\gamma|,$$

provided that the inequality (2.3) is satisfied.

Corollary 2.3 *If* $f \in T\mathrm{H}^{k}(\alpha,\beta,\lambda,\delta,\gamma)$, *then*

$$\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right) \leq \frac{|\gamma|-|b_1|}{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{k}}. \tag{2.4}$$

Corollary 2.4 *Suppose that* $\gamma,\gamma^{*} \in \mathrm{C}$ *such that* $|\gamma|<|\gamma^{*}|$. *Then for* $\alpha \geq 0, \beta \geq 0, \lambda \geq 0, \delta \geq 0, \lambda>\delta, \beta>\alpha$ *and* $k \in \mathrm{N}_0$ *we have* $T\mathrm{H}^{k}(\alpha,\beta,\lambda,\delta,\gamma) \subset T\mathrm{H}^{k}(\alpha,\beta,\lambda,\delta,\gamma^{*})$.

3. Distortion Bounds and Extreme Points

In this section, we shall obtain distortion bounds for functions in $T\mathrm{H}^{k}(\alpha,\beta,\lambda,\delta,\gamma)$ and also provide extreme points for this class.

Theorem 3.1 *If* $f \in T\mathrm{H}^{k}(\alpha,\beta,\lambda,\delta,\gamma)$, *for* $\gamma \in \mathrm{C}$, $\alpha \geq 0, \beta \geq 0, \lambda \geq 0, \delta \geq 0, \lambda>\delta, \beta>\alpha$ *and* $k \in \mathrm{N}_0$ *and* $|z|=r>1$, *then*

$$|f(z)| \leq (1+b_1) r+\frac{|\gamma|-|b_1|}{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{k}} r^{2},$$

and

$$|f(z)| \geq (1-b_1) r-\frac{|\gamma|-|b_1|}{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^{k}} r^{2}.$$

Proof: We only proof the second inequality. The argument for first inequality is similar and will be omitted. Let $f \in T\mathrm{H}^{k}(\alpha,\beta,\lambda,\delta,\gamma)$. Taking the absolute value of f, we obtain

$$|f(z)| \geq (1-b_1) r-\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right) r^{n} \geq (1-b_1) r-\sum_{n=2}^{\infty}\left(|a_n|+|b_n|\right) r^{2}$$

$$= (1-b_1)r - \frac{1}{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k}\sum_{n=2}^{\infty} 2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k \left(|a_n|+|b_n|\right)r^2$$

$$\geq (1-b_1)r - \frac{1}{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k}\sum_{n=2}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k \times$$

$$\left(|a_n|+|b_n|\right)r^2$$

$$\geq (1-b_1)r - \frac{1}{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k}\left[|\gamma|-|b_1|\right]r^2.$$

The bounds given in Theorem 3.1 for functions $f(z) = h + \overline{g}$, of the form (1.2) also hold for functions of the form (1.1) if the coefficient condition (2.1) is satisfied. The functions

$$f(z) = z + \overline{|b_1|z} - \overline{\frac{|\gamma|-|b_1|}{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k} z^2}$$

and

$$f(z) = (1-|b_1|)z - \frac{|\gamma|-|b_1|}{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k} z^2,$$

for $|b_1| < 1$ show that the bounds given in Theorem 3.1 are sharp.

The following covering result follows from the second inequality in Theorem 3.1.

Corollary 3.2 *If* $f \in T\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$, *then*

$$\left\{w : |w| < 1 - \frac{|\gamma| - \left[1 - 2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k\right]|b_1|}{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k}\right\} \subset f(U).$$

Theorem 3.3 $f \in T\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$ *if and only if* f *can be expressed as*

$$f(z) = \sum_{n=1}^{\infty}\left(Y_n h_n + \Upsilon_n g_n\right), \tag{3.1}$$

where $z \in U$,

$$h_1(z) = z,\ h_n = z - \frac{|\gamma|}{n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k} z^n, (n = 2,3,....),$$

$$g_n = z - \frac{|\gamma|}{n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k}\overline{z}^n, (n=1,2,....),$$

$\sum_{n=1}^{\infty}(Y_n + \Upsilon_n) = 1, Y_n \geq 0$ and $\Upsilon_n \geq 0.$

In particular, the extreme points of $T\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$ are $\{h_n\}$ and $\{g_n\}$.
Proof: Note that for f we may write

$$f(z) = \sum_{n=1}^{\infty}(Y_n h_n + \Upsilon_n g_n)$$

$$= \sum_{n=1}^{\infty}(Y_n + \Upsilon_n)z - \sum_{n=2}^{\infty}\frac{|\gamma|}{n\left[(\lambda-\delta)(\beta-\alpha)(n-1)\right]^k}Y_n z^n$$

$$-\overline{\sum_{n=1}^{\infty}\frac{|\gamma|}{n\left[(\lambda-\delta)(\beta-\alpha)(n-1)\right]^k}\Upsilon_n z^n}.$$

Now the first part of the proof is complete, since by Theorem 2.2

$$\sum_{n=2}^{\infty}n\left[(\lambda-\delta)(\beta-\alpha)(n-1)\right]^k\frac{|\gamma|Y_n}{n\left[(\lambda-\delta)(\beta-\alpha)(n-1)\right]^k}$$

$$+\sum_{n=1}^{\infty}n\left[(\lambda-\delta)(\beta-\alpha)(n-1)\right]^k\frac{|\gamma|\Upsilon_n}{n\left[(\lambda-\delta)(\beta-\alpha)(n-1)\right]^k}$$

$$= |\gamma|\sum_{n=1}^{\infty}(Y_n + \Upsilon_n) - Y_1 = |\gamma| - Y_1 \leq |\gamma|.$$

Conversely, suppose that $f \in T\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$. Then

$$\sum_{n=1}^{\infty}n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k(|a_n|+|b_n|) \leq 1+|\gamma|.$$

Setting

$$Y_n = \frac{n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k}{|\gamma|}|a_n|, |\gamma| \neq 0, 0 \leq Y_n \leq 1, (n=2,3,....)$$

$$\Upsilon_n = \frac{n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k}{|\gamma|}|b_n|, |\gamma| \neq 0, 0 \le \Upsilon_n \le 1, (n = 1,2,....),$$

and $Y_1 = 1 - \sum_{n=2}^{\infty} Y_n + \sum_{n=2}^{\infty} \Upsilon_n$ we obtain

$$f(z) = \sum_{n=1}^{\infty} \left(Y_n h_n + \Upsilon_n g_n \right)$$

as required.

4. Closure Theorem

Let the functions $f_j(z)$ be defined, for $j = 1,2,....,m$ by

$$f_j(z) = z - \sum_{n=2}^{\infty} |a_{n,j}| z^n + \overline{\sum_{n=1}^{\infty} |b_{n,j}| z^n}, \ z \in U. \tag{4.1}$$

Theorem 4.1 *Let the functions $f_j(z)$ defined by (4.1) be in the class $T\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$ for every $j = 1,2,....,m$. Then the functions $\Psi(z)$ defined by*

$$\Psi(z) = \sum_{j=1}^{\infty} t_j f_j(z) \left(t_j \ge 0 \right), \tag{4.2}$$

is also in the class $T\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$, where $\sum_{j=1}^{m} t_j = 1.$

Proof: According to the definition of Ψ, we can write

$$\Psi(z) = z - \sum_{n=2}^{\infty} \left(\sum_{j=1}^{m} t_j |a_{n,j}| \right) z^n - \overline{\sum_{n=1}^{\infty} \left(\sum_{j=1}^{m} t_j |b_{n,j}| \right) z^n}. \tag{4.3}$$

Further, since functions $f_j(z)$ are in $T\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$, for every $j = 1,2,....,m$ we get

$$\sum_{n=1}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)\right]^k \left(|a_{n,j}| + |b_{n,j}| \right) \le 1 + |\gamma|,$$

for every $j = 1,2,....,m$. We can see that

$$\sum_{n=1}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)(n-1)+1\right]^k \left(\sum_{j=1}^{m} t_j \left(|a_{n,j}| + |b_{n,j}| \right) \right)$$

$$= \sum_{j=1}^{m} t_j \left(\sum_{n=1}^{\infty} n \left[(\lambda-\delta)(\beta-\alpha)(n-1)+1 \right]^k \left(|a_{n,j}| + |b_{n,j}| \right) \right)$$

$$\leq (1+|\gamma|) \sum_{j=1}^{m} t_j = 1+|\gamma|,$$

by Theorem 2.2, we have $\Psi(z) \in T\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$.

5. An Applications of Neighbourhood

Following Yalcin and Ozturk (2004), we defined the *n*-neighbourhood of a function $f \in S_{\mathrm{H}}^*$ (S_{H}^* class of starlike harmonic functions in U) by

$$\mathrm{N}_\mu(f) = \left\{ F \in H : F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n} \text{ and} \right.$$

$$\left. \sum_{n=2}^{\infty} k \left(|a_n - A_n| + |b_n - B_n| \right) + |b_1 - B_1| \leq \mu \right\}.$$

In particular, for the identity function $I(z) = z$, we immediately have

$$\mathrm{N}_\mu(I) = \left\{ f : f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n + \overline{\sum_{n=1}^{\infty} |b_n| z^n} \text{ and} \right.$$

$$\left. \sum_{n=2}^{\infty} n \left(|a_n| + |b_n| \right) + |b_1| \leq \mu \right\}.$$

Theorem 5.1 *Let*

$$\mu = \frac{|\gamma|}{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k} + \frac{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k - 1}{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k} |b_1|.$$

Then, $T\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma) \subset \mathrm{N}_\mu(I)$.

Proof: Let $f \in T\mathrm{H}^k(\alpha,\beta,\lambda,\delta,\gamma)$. Then the proof follows since, by (2.1), we have

$$\sum_{n=2}^{\infty} n \left(|a_n| + |b_n| \right) + |b_1|$$

$$\leq |b_1| + \frac{1}{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k} \sum_{n=2}^{\infty} n\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k \left(|a_n|+|b_n|\right)$$

$$\leq |b_1| + \frac{1}{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k}\left[|\gamma|-|b_1|\right]$$

$$\leq \frac{|\gamma|}{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k} + \frac{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k-1}{2\left[(\lambda-\delta)(\beta-\alpha)+1\right]^k}|b_1| = \mu.$$

Hence $f \in \subset \mathrm{N}_{\mu}(I)$.

Some other work related to harmonic functions and differential operators can be found in Al-Shaqsi and Darus (2008; 2007; 2006), Al-Shaqsi *et al.* (2010), Darus and Al-Shaqsi (2006), and many elsewhere.

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