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SOME SUBCLASSES OF GENERALISED PASCU CLASSES OF FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

(Beberapa Subkelas bagi Fungsi Kelas Pascu Teritlak terhadap Titik Simetri)

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ABSTRACT

In this paper, some subclasses of generalised Pascu classes of functions with respect to symmetric points are introduced. Here, integral representation formulae are established and sharp coefficient estimates are determined. Further, Fekete Szegö problem is solved and the second Hankel determinant is considered for these classes.

Keywords: **c**oefficient problems; Fekete Szegö problem; second Hankel determinant

ABSTRAK

Dalam makalah ini diperkenalkan beberapa subkelas untuk fungsi kelas Pascu teritlak terhadap titik simetri. Di sini rumus perwakilan kamiran dibina dan anggaran pekali terbaik ditentukan. Seterusnya permasalahan Fekete Szegö diselesaikan dan penentu kedua Hankel dipertimbangkan bagi kelas tersebut.

Kata kunci: masalah pekali; masalah Fekete Szegö; penentu kedua Hankel

1. Introduction

Let $f(z)$ and $F(z)$ be two analytic functions in the unit disc $E=\{z:|z|<1\}$. Then $f(z)$ is called a subordinate to $F(z)$ if there exists a function $w(z)$ analytic in *E* and satisfy the condition $w(0)=0,\ |w(z)|<1$ such that $f(z)=F(w(z))$ and we write $f(z)\prec F(z)$. If $f(z)$ is univalent in *E*, the above definition is equivalent to $f(0)=F(0)$ and $f(E)\subset F(E)$. The concept of s*ubordination* is due to Littlewood (1925) and Rogosinski (1932).

By *U*, we denote the class of analytic functions of the form $w(z)=\sum_{k=1}^{\infty} d_k z^k,\ z\in E$ and satisfying the conditions $w(0)=0$, and $|w(z)|<1$.

Let $\wp$ be the class of analytic functions *P*(*z*) of the form $P(z)=1+\sum_{k=1}^{\infty} p_k z^k$ which satisfies the condition $Re\{P(z)\}>0$, $z\in E$. The functions of the class $\wp$ are known as *Carathéodory Functions* (Carathéodory 1911).

Let A denote the class of functions $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ which are analytic in the unit disc $E=\{z:|z|<1\}$. Denote by *S*, the subclass of functions of A which are also univalent in *E*.

Fekete and Szegö (1933) made an early study for the estimate of $|a_3-\mu a_2^2|$ when $f(z)$ is analytic and univalent in *E*. The well-known result due to them states that if $f(z)$ is analytic univalent in *E*, then

$$|a_3-\mu a_2^2|\le\begin{cases}4\mu-3, & \mu\ge 1,\\ 1+2\exp\left(\dfrac{-2\mu}{1-\mu}\right), & 0\le\mu\le 1,\\ 3-4\mu, & \mu\le 0.\end{cases}$$

For any subclass of A the determination of upper bound for the functional $|a_3-\mu a_2^2|$ is known as Fekete Szegö problem.

Let $f(z)=\sum_{n=0}^{\infty}a_n z^n$ be analytic in *E*. For $q\geq 1$, the q^{th} Hankel determinant of f is defined by

$$H_q(n)=\begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

Firstly the Hankel determinant was studied by Noonan and Thomas (1972; 1976) and subsequently various authors including Hayman (1958), Pommerenke (1966; 1967), Janteng *et al.* (2007; 2008), and Oqlah and Darus (2009) considered and discussed the Hankel determinant. We consider the Hankel determinant $H_2(2)$ and obtain the sharp upper bounds for the functional $\left|a_2a_4-a_3^2\right|$.

Sakaguchi (1959) introduced the concept of univalent starlike functions with respect to symmetric points. A function $f\in \mathrm{A}$ is called univalent starlike functions with respect to symmetric points if and only if

$$Re\left\{\frac{zf'(z)}{f(z)-f(-z)}\right\}>0, \quad z\in E$$

and the corresponding class of functions may be denoted by S_s^*.

Das and Singh (1977) extended the concept of symmetric points to convex and close-to-convex functions. A function $f\in \mathrm{A}$ is said to be univalent starlike functions with respect to symmetric points if and only if

$$Re\left\{\frac{(zf'(z))'}{(f(z)-f(-z))'}\right\}>0, \quad z\in E$$

and class of such functions is denoted by K_s.

C_s is the class of close-to-convex functions *f* in A with respect to symmetric points if there exists a function $g\in S_s^*$ such that

$$Re\left\{\frac{zf'(z)}{g(z)-g(-z)}\right\}>0, \quad z\in E \tag{1}$$

Let $h\in K_s$. Then $C_{1(s)}$ is the class of functions in A with respect to symmetric points satisfies the condition obtained by replacing g by h in (1). Obviously $C_{1(s)}\subset C_s$.

Goel and Mehrok (1982) defined the subclass $S_s^*(A,B)$ of S_s^*. A function *f* in A belongs to $S_s^*(A,B)$ if

$$\left\{\frac{2zf'(z)}{f(z)-f(-z)}\right\}\prec\frac{1+Az}{1+Bz}, \quad -1\leq B<A\leq 1, \quad z\in E.$$

The class $K_s(A,B)$ consists of functions f in A which satisfies the condition

$$\left\{\frac{2(zf'(z))'}{\left(f(z)-f(-z)\right)'}\right\} \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1, \quad z \in E.$$

It is obvious that $z \in S_s^*(A,B) \Rightarrow zf'(z) \in K_s(A,B)$.

Let $\alpha \ge 0$ and $\frac{f(z)f'(z)}{z} \ne 0$. Then $C_s^*(\alpha)$ is the class of functions f in A with respect to symmetric points if there exists a function $g \in S_s^*$ such that

$$Re\left\{\frac{(1-\alpha)f(z)}{g(z)-g(-z)} + \frac{\alpha zf'(z)}{g(z)-g(-z)}\right\} > 0, \quad z \in E \tag{2}$$

For $h \in K_s$, $C_{1(s)}^*(\alpha)$ is the class of functions f in A which satisfies the condition obtained by replacing g by h in (2).

The classes $T_s^*(\alpha)$ and $T_{1(s)}^*(\alpha)$ are respectively defined as

$$\left\{f \in \mathrm{A};\quad Re\left(\frac{zf'(z)}{g(z)-g(-z)} + \frac{\alpha z^2 f''(z)}{g(z)-g(-z)}\right) > 0, \quad g \in \mathrm{S}_s^*\right\},$$

$$\left\{f \in \mathrm{A};\quad Re\left(\frac{zf'(z)}{h(z)-h(-z)} + \frac{\alpha z^2 f''(z)}{h(z)-h(-z)}\right) > 0, \quad h \in K_s\right\}.$$

We have the following observations:

(i) $\quad f(z) \in C_s^*(\alpha) \Rightarrow zf'(z) \in T_s^*(\alpha)$,

(ii) $\quad f(z) \in C_{1(s)}^*(\alpha) \Rightarrow zf'(z) \in T_{1(s)}^*(\alpha)$.

Let $0 < \delta \le 1, -1 \le D \le B < A \le C \le 1,$ $g \in S_s^*(A,B)$ and $h \in K_s(A,B)$. Then we shall also deal with the following classes:

$$C_s^*(\alpha;\delta;A,B;C,D) = \left\{f \in \mathrm{A};\ \frac{2(1-\alpha)f(z)}{g(z)-g(-z)} + \frac{2\alpha zf'(z)}{g(z)-g(-z)} \prec \left(\frac{1+Cz}{1+Dz}\right)^\delta\right\}$$

$$C_{1(s)}^*(\alpha;\delta;A,B;C,D) = \left\{f \in \mathrm{A};\ \frac{2(1-\alpha)f(z)}{h(z)-h(-z)} + \frac{2\alpha zf'(z)}{h(z)-h(-z)} \prec \left(\frac{1+Cz}{1+Dz}\right)^\delta\right\}$$

$$T_s^*(\alpha;\delta;A,B;C,D) = \left\{f \in \mathrm{A};\ \frac{2zf'(z)}{g(z)-g(-z)} + \frac{2\alpha z^2 f''(z)}{g(z)-g(-z)} \prec \left(\frac{1+Cz}{1+Dz}\right)^\delta\right\}$$

$$T_{1(s)}^*(\alpha;\delta;A,B;C,D) = \left\{f \in \mathrm{A};\ \frac{2zf'(z)}{h(z)-h(-z)} + \frac{2\alpha z^2 f''(z)}{h(z)-h(-z)} \prec \left(\frac{1+Cz}{1+Dz}\right)^\delta\right\}.$$

For $\delta = 1$, we write

$$C_s^*(\alpha;1;A,B;C,D) \equiv C_s^*(\alpha;A,B;C,D),$$

$$C_{1(s)}^*(\alpha;1;A,B;C,D) \equiv C_{1(s)}^*(\alpha;A,B;C,D),$$

$$T_s^*\left(\alpha;1;A,B;C,D\right)\equiv T_s^*\left(\alpha;A,B;C,D\right),$$

$$T_{1(s)}^*\left(\alpha;1;A,B;C,D\right)\equiv T_{1(s)}^*\left(\alpha;A,B;C,D\right).$$

Throughout this paper we assume (unless mentioned otherwise) that

$$\left\{\begin{array}{l} z\in E, 0\le\alpha, 0<\delta\le 1, -1\le D\le B<A\le C\le 1, w(z)=1+\sum_{k=1}^{\infty}d_k z^k\in U,\\ g(z)=z+\sum_{n=2}^{\infty}b_n z^n\in S_s^*\left(A,B\right),\ \ h(z)=z+\sum_{n=2}^{\infty}c_n z^n\in K_s\left(A,B\right)\\ G(z)=\dfrac{g(z)-g(-z)}{2}, H(z)=\dfrac{h(z)-h(-z)}{2}, \mathrm{P}(z)=1+\sum_{k=1}^{\infty}p_k z^k, Q(z)=\dfrac{1+Cw(z)}{1+Dw(z)}. \end{array}\right\}$$

2. Preliminaries

The following lemmas are required to establish our results.

Lemma 2.1. (Goel & Mehrok 1982). *If* $Q(z)=\dfrac{1+Cw(z)}{1+Dw(z)}=1+\sum_{k=1}^{\infty}q_k z^k$, *then*

$$\left|q_k\right|\le(C-D).$$

The results are sharp for the functions $Q_n(z)=\dfrac{1+C\delta z^n}{1+D\delta z^n}$, $n\ge 1$ and $\left|\delta\right|=1$.

Lemma 2.2. (Goel & Mehrok 1982). *Let* $g\in S_s^*(A,B)$. *Then*

$$\left|b_{2n}\right|\le\frac{\left(A-B\right)}{2^n n!}\prod_{j=1}^{n-1}\left(A-B+2j\right),$$

$$\left|b_{2n+1}\right|\le\frac{\left(A-B\right)}{2^n n!}\prod_{j=1}^{n-1}\left(A-B+2j\right).$$

The bounds are sharp being attained for the functions $g_0(z)$ defined by

$$g_0(z)=\left\{\begin{array}{ll} log\left(\dfrac{(1+z)^{(1-A)/2(1-B)}(1+Bz)^{(A-B)/(1-B^2)}}{(1-z)^{(1+A)/2(1+B)}}\right), & B\ne -1;\\ log\left(\dfrac{1+z}{1-z}\right)^{(1-A)/4}+\left(\dfrac{1+A}{2}\right)\left(\dfrac{z}{1-z}\right), & B=-1;\\ log\left(\dfrac{(1+z)^{(1-A)/2}}{(1-z)^{(1+A)/2}}\right), & B=0. \end{array}\right\} \qquad (3)$$

Since $g\in S_s^*\left(A,B\right)$ implies that $zg'(z)\in K_s(A,B)$, we have the following:

Lemma 2.3. *Let* $h\in K_s(A,B)$. *Then*

$$\left|c_{2n}\right|\le\frac{1}{2n}\left[\frac{\left(A-B\right)}{2^n n!}\prod_{j=1}^{n-1}\left(A-B+2j\right)\right],$$

$$\left|c_{2n+1}\right| \le \frac{1}{2n+1}\left[\frac{\left(A-B\right)}{2^n n!}\prod_{j=1}^{n-1}\left(A-B+2j\right)\right].$$

The results are sharp for the function $h_0(z)$ defined by

$$h_0\left(z\right)=\int_0^z \frac{g_0\left(t\right)}{t}dt \tag{4}$$

where $g_0\left(z\right)$ is defined by (3).

Lemma 2.4. (Nehari 1952). *Let* $w\left(z\right)\in U$. *Then*

$$\left|d_1\right| \le 1 \text{ and } \left|d_2\right| \le \left(1-\left|d_1\right|^2\right).$$

Lemma 2.5. (Duren 1983). *If* $P\left(z\right)\in \wp$, *then* $\left|p_k\right| \le 2\left(k=1,2,...\right)$.

Lemma 2.6. (Libera & Zlotkiewicz 1982). If $P\left(z\right)\in\wp$, then

$$2p_2 = p_1^2+\left(4-p_1^2\right)x$$

$$4p_3 = p_1^3+2p_1\left(4-p_1^2\right)x-p_1\left(4-p_1^2\right)x^2+2\left(4-p_1^2\right)\left(1-\left|x\right|^2\right)z$$

for some *x* and *z* satisfying $\left|x\right|\le 1$ and $\left|z\right|\le 1$.

3. Integral Representation Formulae

Theorem 3.1. *Let* $f\in C_s^*\left(\alpha;A,B;C,D\right)$, *then*

$$f\left(z\right)=\begin{cases} Q\left(z\right)G\left(z\right) & \text{if } \alpha=0, \\ \dfrac{1}{\alpha z^{\frac{1}{\alpha}-1}}\left\{\displaystyle\int_0^z t^{\frac{1}{\alpha}-1}Q\left(t\right)G\left(t\right)dt\right\} & \text{if } \alpha>0. \end{cases}$$

Proof. For $f\in C_s^*\left(\alpha;A,B;C,D\right)$, we have

$$\left(1-\alpha\right)f\left(z\right)+\alpha zf'\left(z\right)=Q\left(z\right)G\left(z\right) \tag{5}$$

For $\alpha=0$, there is nothing to prove.
Consider the case when $\alpha>0$.

Dividing (5) by α and putting $c=\frac{1}{\alpha}-1$,

$$f\left(z\right)+zf'\left(z\right)=\left(1+c\right)Q\left(z\right)G\left(z\right) \tag{6}$$

Multiplying (6) by z^{c-1}, we get

$$cz^{c-1}f\left(z\right)+z^c f'\left(z\right)=(1+c)\left[z^{c-1}Q(z)G\left(z\right)\right]$$

which reduces to

$$\frac{d}{dz}\left[z^c f\left(z\right)\right]=\left(1+c\right)\left[z^{c-1}Q\left(z\right)G\left(z\right)\right].$$

On integrating, we get the desired result. □

On the same lines we can prove

Theorem 3.2. *Let* $f \in C^*_{1(s)}(\alpha; A, B; C, D)$, *then*

$$f(z) = \begin{cases} Q(z)H(z) & \text{if } \alpha = 0, \\ \dfrac{1}{\alpha z^{\frac{1}{\alpha}-1}} \left\{ \displaystyle\int_0^z t^{\frac{1}{\alpha}-1} Q(t)H(t)\,dt \right\} & \text{if } \alpha > 0. \end{cases}$$

Theorem 3.3. *Let* $f \in T^*_s(\alpha; A, B; C, D)$, *then*

$$f(z) = \begin{cases} \displaystyle\int_0^z \frac{Q(t)G(t)}{t}\,dt & \text{if } \alpha = 0, \\ \dfrac{1}{\alpha}\left[\displaystyle\int_0^z \frac{1}{u^{\frac{1}{\alpha}}} \left\{ \int_0^u t^{\frac{1}{\alpha}-2} Q(t)G(t)\,dt \right\} du \right] & \text{if } \alpha > 0. \end{cases}$$

Proof. For $f \in T^*_s(\alpha; A, B; C, D)$, by definition

$$zf'(z) + \alpha z^2 f''(z) = Q(z)G(z) \tag{7}$$

The result is obvious for $\alpha = 0$.

Let $\alpha > 0$. Dividing (7) by α and putting $c = \frac{1}{\alpha} - 1$, (7) becomes

$$(1+c)zf'(z) + z^2 f''(z) = (1+c)Q(z)G(z) \tag{8}$$

Multiplying (8) by z^{c-1}, we get

$$(1+c)z^c f'(z) + z^{c+1} f''(z) = (1+c)\left[z^{c-1} Q(z)G(z) \right]$$

which reduces to

$$\frac{d}{dz}\left[z^{c+1} f'(z) \right] = (1+c)\left[z^{c-1} Q(z)G(z) \right] \tag{9}$$

On integrating (9), we have

$$f'(z) = \frac{1}{\alpha z^{\frac{1}{\alpha}}} \left\{ \int_0^z t^{\frac{1}{\alpha}-2} Q(t)G(t)\,dt \right\}$$

Integration of the above equation gives the required result. □

Similarly we obtain the following

Theorem 3.4. *Let* $f \in T^*_{1(s)}(\alpha; A, B; C, D)$, *then*

$$f(z) = \begin{cases} \displaystyle\int_0^z \frac{Q(t)H(t)}{t}\,dt & \text{if } \alpha = 0, \\ \dfrac{1}{\alpha}\left[\displaystyle\int_0^z \frac{1}{u^{\frac{1}{\alpha}}} \left\{ \int_0^u t^{\frac{1}{\alpha}-2} Q(t)H(t)\,dt \right\} du \right] & \text{if } \alpha > 0. \end{cases}$$

4. Coefficient Estimates

Theorem 4.1. *If* $f \in C^*_s(\alpha; A, B; C, D)$, *then*

$$|a_{2n}| \le \frac{(C-D)}{(1+(2n-1)\alpha)}\left\{1+\sum_{k=2}^{n}\frac{(A-B)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right\}$$

$$|a_{2n+1}| \le \frac{1}{(1+2n\alpha)}\left[\begin{array}{l}\dfrac{(A-B)}{2^n n!}\prod_{j=1}^{n-1}(A-B+2j)\\ +(C-D)\left\{1+\sum_{k=2}^{n}\dfrac{(A-B)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right\}\end{array}\right].$$

Proof. Since $f \in C_S^*(\alpha;\ A,B;C,D)$, it implies that

$$\frac{(1-\alpha)f(z)}{G(z)}+\frac{\alpha z f'(z)}{G(z)}=\frac{1+Cw(z)}{1+Dw(z)}=Q(z) \tag{10}$$

Simplifying (10), we have

$$\begin{aligned}&1+(1+\alpha)a_2 z+(1+2\alpha)a_3 z^2+\ldots+(1+(2n-1)\alpha)a_{2n}z^{2n-1}+(1+2n\alpha)a_{2n+1}z^{2n}+\ldots\\&=(1+q_1 z+q_2 z^2+\ldots+q_{2n}z^{2n}+\ldots)(1+b_3 z^2+b_5 z^4+\ldots+b_{2n+1}z^{2n}+\ldots).\end{aligned} \tag{11}$$

Equating coefficients of z^{2n-1}, we get

$$\{1+(2n-1)\alpha\}a_{2n}=q_1 b_{2n-1}+q_3 b_{2n-3}+\ldots+q_{2n-3}b_3+q_{2n-1}.$$

Applying lemma 2.1, we obtain

$$|a_{2n}| \le \frac{(C-D)}{(1+(2n-1)\alpha)}\left(1+\sum_{k=2}^{n}|b_{2k-1}|\right) \tag{12}$$

Using lemma 2.2 in (12), we get the first required inequality.
Equating coefficients of z^{2n} in (11), we get

$$(1+2n\alpha)a_{2n+1}=b_{2n+1}+q_2 b_{2n-1}+\ldots+q_{2n-2}b_3+q_{2n}.$$

Applying lemma 2.1, we have

$$|a_{2n+1}| \le \frac{1}{(1+2n\alpha)}\left[|b_{2n+1}|+(C-D)\left(1+\sum_{k=2}^{n}|b_{2k-1}|\right)\right]. \tag{13}$$

Using lemma 2.2, we obtain the second required inequality from (13).
External function is obtained by choosing $G(z)=\frac{g_0(z)-g_0(-z)}{2}$, $g_0(z)$ is defined by (3) and $Q(z)=\frac{1+Cz}{1+Dz}$ in the integral representation formula proved in the theorem 3.1. □

On putting $A=C=1$ and $B=D=-1$, we have the following:

Corollary 4.1. *If* $f \in C_s^*(\alpha)$, *then* $|a_{2n}| \le \dfrac{2n}{1+(2n-1)\alpha}$ and $|a_{2n+1}| \le \dfrac{2n+1}{1+2n\alpha}$.

On the same pattern, we can easily prove the following results:

Theorem 4.2. *If* $f \in C_{1(S)}^*(\alpha;\ A,B;C,D)$ *, then*

$$|a_{2n}| \le \frac{(C-D)}{1+(2n-1)\alpha}\left\{1+\sum_{k=2}^{n}\frac{1}{2k-1}\left(\frac{(A-B)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right)\right\}$$

$$|a_{2n+1}| \le \frac{1}{(1+2n\alpha)}\left[\frac{1}{(2n+1)}\left(\frac{(A-B)}{2^{n}n!}\prod_{j=1}^{n-1}(A-B+2j)\right)\right.$$

$$\left.+(C-D)\left\{1+\sum_{k=2}^{n}\frac{1}{2k-1}\left(\frac{(A-B)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right)\right\}\right].$$

The results are sharp for the function obtained by taking $H(z)=\frac{h_0(z)-h_0(-z)}{2}, h_0(z)$ is defined by (9) and $Q(z)=\frac{1+Cz}{1+Dz}$ in the integral representation formula proved in the theorem 3.2.

Theorem 4.3. *If* $f \in T_s^*(\alpha; A,B; C,D)$, *then*

$$|a_{2n}| \le \frac{(C-D)}{2n(1+(2n-1)\alpha)}\left\{1+\sum_{k=2}^{n}\frac{(A-B)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right\}$$

$$|a_{2n+1}| \le \frac{1}{(2n+1)(1+2n\alpha)}\left[\frac{(A-B)}{2^{n}n!}\prod_{j=1}^{n-1}(A-B+2j)\right.$$

$$\left.+(C-D)\left\{1+\sum_{k=2}^{n}\frac{(A-B)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right\}\right]$$

Proof. For $f \in T_s^*(\alpha; A,B; C,D)$, it implies from definitions that

$$\frac{zf'(z)}{G(z)}+\frac{az^2f''(z)}{G(z)}=\frac{1+Cw(z)}{1+Dw(z)}=Q(z).$$

On simplification, the above equation becomes

$$1+2(1+\alpha)a_2z+\ldots+2n(1+(2n-1)\alpha)a_{2n}z^{2n-1}+(2n+1)(1+2n\alpha)a_{2n+1}z^{2n}+\ldots$$

$$=(1+q_1z+q_2z^2+\ldots+q_{2n}z^{2n}+\ldots)(1+b_3z^2+b_5z^4+\ldots+b_{2n+1}z^{2n}+\ldots) \tag{14}$$

Equating coefficients of z^{2n-1} in (14), we get

$$2n(1+(2n-1)\alpha)a_{2n}=q_1b_{2n-1}+q_3b_{2n-3}+\ldots+q_{2n-3}b_3+q_{2n-1}.$$

By applying lemma 2.1, we have

$$|a_{2n}| \le \frac{(C-D)}{2n(1+(2n-1)\alpha)}\left(1+\sum_{k=2}^{n}|b_{2k-1}|\right)$$

Using lemma 2.2, the above inequality implies the first result of our theorem.

On equating coefficients of z^{2n} in (14), we obtain

$$(2n+1)(1+2n\alpha)a_{2n+1}=b_{2n+1}+q_2b_{2n-1}+\ldots+q_{2n-2}b_3+q_{2n}.$$

Applying lemma 2.1, we have

$$|a_{2n+1}| \le \frac{1}{(2n+1)(1+2n\alpha)}\left[|b_{2n+1}| + (C-D)\left(1+\sum_{k=2}^{n}|b_{2k-1}|\right)\right]$$

Using lemma 2.2, we get the second required result. The external function is the same as in the theorem 4.1. □

On putting $A=C=1$ and $B=D=-1$, we have the following:

Corollary 4.2. *If* $f \in T_s^*(\alpha)$, *then* $|a_{2n}| \le \frac{1}{1+(2n-1)\alpha}$ *and* $|a_{2n+1}| \le \frac{1}{1+2n\alpha}$.

Similarly we can prove the following:

Theorem 4.4. *If* $f \in T_{1(s)}^*(\alpha; A,B;C,D)$, *then*

$$|a_{2n}| \le \frac{(C\text{-}D)}{2n(1+(2n-1)\alpha)}\left\{1+\sum_{k=2}^{n}\frac{1}{2k-1}\left(\frac{(A-B)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right)\right\},$$

$$|a_{2n+1}| \le \frac{1}{(2n+1)(1+2n\alpha)}\left[\frac{1}{2n+1}\left(\frac{(A-B)}{2^n n!}\prod_{j=1}^{n-1}(A-B+2j)\right)\right.$$
$$\left. +(C-D)\left\{1+\sum_{k=2}^{n}\frac{1}{2k-1}\left(\frac{(A-B)}{2^{k-1}(k-1)!}\prod_{j=1}^{k-2}(A-B+2j)\right)\right\}\right].$$

These estimates are sharp for the function obtained by taking $H(z)=\frac{h_0(z)-h_0(-z)}{2}, h_0(z)$ is defined by (4) and $Q(z)=\frac{1+Cz}{1+Dz}$ in the integral representation formula proved in theorem 3.2.

5. Fekete-Szegö Problem

Theorem 5.1. *Let* $f \in C_s^*(\alpha,\delta; A,B;C,D)$, *then*

(i) *for* μ *complex*

$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{(A-B)+2\delta(C-D)}{2(1+2\alpha)} & \text{if } |\lambda+\mu| \le v, & (15)\\[2ex] \dfrac{(A-B)}{2(1+2\alpha)} + \dfrac{\delta^2(C-D)^2}{(1+\alpha)^2}|\lambda+\mu| & \text{if } |\lambda+\mu| \ge v; & (16)\end{cases}$$

(ii) *for* μ *real*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{(A-B)}{2(1+2\alpha)}-\dfrac{\delta^2(C-D)^2}{(1+\alpha)^2}(\lambda+\mu) & \text{if } \mu\le-(v+\lambda), \quad (17)\\[2mm] \dfrac{(A-B)+2\delta(C-D)}{2(1+2\alpha)} & \text{if } -(v+\lambda)\le\mu\le(v-\lambda), \quad (18)\\[2mm] \dfrac{(A-B)}{2(1+2\alpha)}+\dfrac{\delta^2(C-D)^2}{(1+\alpha)^2}(\lambda+\mu) & \text{if } \mu\ge(v-\lambda); \quad (19)\end{cases}$$

where

$$\lambda=\frac{(1+\alpha)^2\left[D+\frac{1}{2}(1-\delta)(C-D)\right]}{\delta(1+2\alpha)(C-D)},\qquad v=\frac{(1+\alpha)^2}{\delta(1+2\alpha)(C-D)}\ . \tag{20}$$

Proof. Since $f\in C_s^*(\alpha,\delta;A,B;C,D)$, it follows that

$$(1-\alpha)\frac{f(z)}{G(z)}+\alpha\frac{zf'(z)}{G(z)}=\left(\frac{1+Cw(z)}{1+Dw(z)}\right)^{\delta}$$

By expanding the above equation, we obtain

$$1+(1+\alpha)a_2z+\{(1+2\alpha)a_3-b_3\}z^2+\ldots$$
$$=1+\delta(C-D)d_1z+\left(\delta(C-D)d_2-\delta(C-D)\left[D+\frac{1}{2}(1-\delta)(C-D)\right]d_1^2\right)z^2+\ldots$$

Identifying the terms, we have

$$a_2=\frac{\delta(C-D)}{(1+\alpha)}d_1$$

$$a_3=\frac{\delta(C-D)}{(1+2\alpha)}d_2-\frac{\delta(C-D)\left[D+\frac{1}{2}(1-\delta)(C-D)\right]}{(1+2\alpha)}d_1^2+\frac{|b_3|}{(1+2\alpha)}\ .$$

and therefore

$$a_3-\mu a_2^2=\frac{\delta(C-D)}{(1+2\alpha)}d_2-\frac{\delta^2(C-D)^2}{(1+\alpha)^2}(\lambda+\mu)d_1^2+\frac{b_3}{(1+2\alpha)}$$

where λ is defined by (20). Applying triangular inequality,

$$\left|a_3-\mu a_2^2\right|\le\frac{\delta(C-D)}{(1+2\alpha)}|d_2|+\frac{\delta^2(C-D)^2}{(1+\alpha)^2}|\lambda+\mu||d_1|^2+\frac{|b_3|}{(1+2\alpha)}$$

Using lemma 2.2 and lemma 2.4, we obtain

$$\left|a_3-\mu a_2^2\right|\le\frac{\delta(C-D)}{(1+2\alpha)}+\frac{\delta^2(C-D)^2}{4(1+\alpha)^2}(|\lambda+\mu|-v)|d_1|^2+\frac{(A-B)}{2(1+2\alpha)} \tag{21}$$

where ν is defined by (20).

If $|\lambda+\mu|\le v$, then (21) implies (15).

If $|\lambda+\mu|\ge v$, then applying lemma 2.4, (16) follows from (21).

Consider the case when μ is real.

Case I. $\mu \le -\lambda$.

From (37), we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{\delta(C-D)}{(1+2\alpha)} - \frac{\delta^2 (C-D)^2}{4(1+\alpha)^2}(\mu+\lambda+v)|d_1|^2 + \frac{(A-B)}{2(1+2\alpha)} . \tag{22}$$

If $\mu \le -(v+\lambda)$, then from lemma 2.4 and (22), we get (17).

If $\mu \ge -(v+\lambda)$, then from (22) we have

$$\left|a_3 - \mu a_2^2\right| \le \frac{(A-B)+2\delta(C-D)}{2(1+2\alpha)} . \tag{23}$$

Case II. $\mu \ge -\lambda$.

Then (21) takes the form

$$\left|a_3 - \mu a_2^2\right| \le \frac{\delta(C-D)}{(1+2\alpha)} + \frac{\delta^2 (C-D)^2}{4(1+\alpha)^2}(\mu+\lambda-v)|d_1|^2 + \frac{(A-B)}{2(1+2\alpha)} . \tag{24}$$

If $\mu \le (v-\lambda)$, then from the above

$$\left|a_3 - \mu a_2^2\right| \le \frac{(A-B)+2\delta(C-D)}{2(1+2\alpha)} . \tag{25}$$

From (23) and (25), we obtain (18).

If $\mu \ge (v-\lambda)$, then from (24) and lemma 2.4, we have (19).

The estimates (15) and (18) are sharp for the function $f_0(z)$ defined by

$$(1-\alpha) f_0(z) + \alpha z f_0(z) = G(z)\left(\frac{1+Az^2}{1+Bz^2}\right)^{\delta} \tag{26}$$

and estimates (16), (17) and (19) are sharp for the function $f_1(z)$ defined by

$$(1-\alpha) f_1(z) + \alpha z f_1(z) = G(z)\left(\frac{1+Az}{1+Bz}\right)^{\delta} \tag{27}$$

where $G(z) = \frac{g_0(z) - g_0(-z)}{2}$ and $g_0(z)$ is defined by (3). □

On taking $\delta = 1,\ A = C = 1$ and $B = D = -1$ in the theorem, we get

Corollary 5.1. *Let* $f \in C_s^*(\alpha)$, *then*

(i) *for* μ *complex*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{3}{(1+2\alpha)} & \text{if } \ |\lambda+\mu| \le v, \\[2ex] \dfrac{1}{(1+2\alpha)} + \dfrac{4}{(1+\alpha)^2}|\lambda+\mu| & \text{if } \ |\lambda+\mu| \ge v \ ; \end{cases}$$

where $\lambda = -\frac{(1+\alpha)^2}{2(1+\alpha)}$ *and* $v = \frac{(1+\alpha)^2}{2(1+2\alpha)}$.

(ii) *for* μ *real*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{3}{(1+2\alpha)}-\dfrac{4\mu}{(1+\alpha)^2} & if \quad \mu\le 0,\\[2ex] \dfrac{3}{(1+2\alpha)} & if \quad 0\le\mu\le\dfrac{(1+\alpha)^2}{(1+2\alpha)};\\[2ex] \dfrac{4\mu}{(1+\alpha)^2}-\dfrac{1}{(1+2\alpha)} & if \quad \mu\ge\dfrac{(1+\alpha)^2}{(1+2\alpha)}.\end{cases}$$

On the same pattern as in the above theorem, we can have the following:

Theorem 5.2. *Let* $f\in C_{1(s)}^{*}(\alpha,\delta;A,B;C,D)$ *then*

(i) *for* μ *complex*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{(A-B)+6\delta(C-D)}{6(1+2\alpha)} & if \quad |\lambda+\mu|\le v,\\[2ex] \dfrac{(A-B)}{6(1+2\alpha)}+\dfrac{\delta^2(C-D)^2}{(1+\alpha)^2}|\lambda+\mu| & if \quad |\lambda+\mu|\ge v;\end{cases}$$

(ii) *for* μ *real*

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}\dfrac{(A-B)}{6(1+2\alpha)}-\dfrac{\delta^2(C-D)^2}{(1+\alpha)^2}(\lambda+\mu) & if \ \mu\le-(v+\lambda),\\[2ex] \dfrac{(A-B)+6\delta(C-D)}{6(1+2\alpha)} & if \ -(v+\lambda)\le\mu\le(v-\lambda),\\[2ex] \dfrac{(A-B)}{6(1+2\alpha)}+\dfrac{\delta^2(C-D)^2}{(1+\alpha)^2}(\lambda+\mu) & if \quad \mu\ge(v-\lambda);\end{cases}$$

where $\lambda=\dfrac{(1+\alpha)^2\left[D+\frac{1}{2}(1-\delta)(C-D)\right]}{\delta(1+2\alpha)(C-D)}$ *and* $v=\dfrac{(1+\alpha)^2}{\delta(1+2\alpha)(C-D)}$.

The first inequality of (i) and second inequality of (ii) are sharp for the function $f_2(z)$ defined by

$$(1-\alpha)f_2(z)+\alpha zf_2(z)=H(z)\left(\frac{1+Az^2}{1+Bz^2}\right)^{\delta} \tag{28}$$

And the second inequality of (i), first and third inequalities of (ii) are sharp for the function $f_3(z)$ defined by

$$(1-\alpha)f_3(z)+\alpha zf_3(z)=H(z)\left(\frac{1+Az}{1+Bz}\right)^{\delta} \tag{29}$$

where $H(z)=\dfrac{h_0(z)-h_0(-z)}{2}$ and $h_0(z)$ is defined by (4).

Theorem 5.3. *Let* $f \in T_s^*(\alpha,\delta;A,B;C,D)$, *then*

(i) *for* μ *complex*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{(A-B)+2\delta(C-D)}{6(1+2\alpha)} & \text{if } \ |\lambda+\mu| \le v, \quad (30) \\ \dfrac{(A-B)}{6(1+2\alpha)}+\dfrac{\delta^2(C-D)^2}{4(1+\alpha)^2}|\lambda+\mu| & \text{if } \ |\lambda+\mu| \ge v; \quad (31) \end{cases}$$

(ii) *for* μ *real*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{(A-B)}{6(1+2\alpha)}-\dfrac{\delta^2(C-D)^2}{4(1+\alpha)^2}(\lambda+\mu) & \text{if } \ \mu \le -(v+\lambda), \quad (32) \\ \dfrac{(A-B)+2\delta(C-D)}{6(1+2\alpha)} & \text{if } \ -(v+\lambda) \le \mu \le (v-\lambda), \quad (33) \\ \dfrac{(A-B)}{6(1+2\alpha)}+\dfrac{\delta^2(C-D)^2}{4(1+\alpha)^2}(\lambda+\mu) & \text{if } \ \mu \ge (v-\lambda); \quad (34) \end{cases}$$

where $\lambda = \dfrac{4(1+\alpha)^2\left[D+\frac{1}{2}(1-\delta)(C-D)\right]}{3\delta(1+2\alpha)(C-D)}$ *and* $v = \dfrac{4(1+\alpha)^2}{3\delta(1+2\alpha)(C-D)}$.

Proof. Since $f \in T_s^*(\alpha,\delta;A,B;C,D)$, we have

$$(1-\alpha)\frac{zf(z)}{G(z)}+\alpha\frac{z^2f'(z)}{G(z)} = \left(\frac{1+Cw(z)}{1+Dw(z)}\right)^{\delta}.$$

After a little computation, the above equation becomes

$$1+2(1+\alpha)a_2z+\left\{3(1+2\alpha)a_3-b_3\right\}z^2+\ldots$$

$$=1+\delta(C-D)d_1z+\left(\delta(C-D)d_2-\delta(C-D)\left\{D+\frac{1}{2}(1-\delta)(C-D)\right\}d_1^2\right)z^2+\ldots$$

On comparing coefficients, we get

$$a_2=\frac{\delta(C-D)}{2(1+\alpha)}d_1 \ ,$$

$$a_3=\frac{\delta(C-D)}{3(1+2\alpha)}d_2-\frac{\delta(C-D)\left[D+\frac{1}{2}(1-\delta)(C-D)\right]}{3(1+2\alpha)}d_1^2+\frac{b_3}{3(1+2\alpha)} \ .$$

Consequently, we have

$$\left|a_3-\mu a_2^2\right| \le \frac{\delta(C-D)}{3(1+2\alpha)}|d_2|+\frac{\delta^2(C-D)^2}{4(1+\alpha)^2}|\lambda+\mu||d_1|^2+\frac{|b_3|}{3(1+2\alpha)} \tag{35}$$

where λ is defined as in the theorem.

Using lemma 2.2 and lemma 2.4, (35) takes the form

$$\left|a_3-\mu a_2^2\right| \le \frac{\delta(C-D)}{3(1+2\alpha)}+\frac{\delta^2(C-D)^2}{4(1+\alpha)^2}\left(|\lambda+\mu|-v\right)|d_1|^2+\frac{(A-B)}{6(1+2\alpha)} \tag{36}$$

where ν is defined as in the theorem.

If $|\lambda+\mu| \le \nu$, then from (36), we get (30).

If $|\lambda+\mu| \ge \nu$, then using lemma 1.4, (31) follows from (36).

Consider the case when μ is real.

Case I. $\mu \le -\lambda$.

From (36), we have

$$\left|a_3-\mu a_2^2\right| \le \frac{\delta(C-D)}{3(1+2\alpha)}-\frac{\delta^2(C-D)^2}{4(1+\alpha)^2}(\lambda+\mu+v)|d_1|^2+\frac{(A-B)}{6(1+2\alpha)}\ . \tag{37}$$

If $\mu \le -(\nu+\lambda)$ then from (37) and lemma 2.4, we get (32).

If $\mu \ge -(\nu+\lambda)$ then from (37) we have

$$\left|a_3-\mu a_2^2\right| \le \frac{(A-B)+2\delta(C-D)}{6(1+2\alpha)}\ . \tag{38}$$

Case II. $\mu \ge -\lambda$.

Then (36) implies that

$$\left|a_3-\mu a_2^2\right| \le \frac{\delta(C-D)}{3(1+2\alpha)}+\frac{\delta^2(C-D)^2}{4(1+\alpha)^2}(\lambda+\mu-v)|c_1|^2+\frac{(A-B)}{6(1+2\alpha)}\ . \tag{39}$$

If $\mu \le (\nu-\lambda)$ then from (35) we obtain

$$\left|a_3-\mu a_2^2\right| \le \frac{(A-B)+2\delta(C-D)}{6(1+2\alpha)}\ . \tag{40}$$

From (38) and (40), we get (33).

If $\mu \ge (\nu-\lambda)$, then lemma 2.4 and (39) implies (34).

Eqs. (30) and (33) are sharp for the function $f_0(z)$ defined by (26) and estimates (31), (32) and (34) are sharp for the function $f_1(z)$ defined by (43). □

Corollary 5.2. *Let* $f \in T_s^*(\alpha)$, *then*

(i) *for* μ *complex*

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{1}{(1+2\alpha)} & \text{if } \ |\lambda+\mu| \le v, \\[2ex] \dfrac{1}{3(1+2\alpha)}+\dfrac{1}{(1+\alpha)^2}|\lambda+\mu| & \text{if } \ |\lambda+\mu| \ge v\ ; \end{cases}$$

where $\lambda = -\dfrac{2(1+\alpha)^2}{3(1+2\alpha)}$ *and* $v = \dfrac{2(1+\alpha)^2}{3(1+2\alpha)}$.

(ii) *for* μ *real*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{1}{(1+2\alpha)} - \dfrac{\mu}{(1+\alpha)^2} & \text{if } \mu \le 0, \\ \dfrac{1}{(1+2\alpha)} & \text{if } 0 \le \mu \le \dfrac{4(1+\alpha)^2}{3(1+2\alpha)}; \\ \dfrac{\mu}{(1+\alpha)^2} - \dfrac{1}{3(1+2\alpha)} & \text{if } \mu \ge \dfrac{4(1+\alpha)^2}{3(1+2\alpha)}. \end{cases}$$

Theorem 5.4. *Let* $f \in T^*_{1(s)}(\alpha,\beta;A,B;C,D)$, *then*

(i) *for* μ *complex*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{(A-B)+6\delta(C-D)}{18(1+2\alpha)} & \text{if } |\lambda+\mu| \le v, \\ \dfrac{(A-B)}{18(1+2\alpha)} + \dfrac{\delta^2(C-D)^2}{4(1+\alpha)^2}|\lambda+\mu| & \text{if } |\lambda+\mu| \ge v; \end{cases}$$

(ii) *for* μ *real*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \dfrac{(A-B)}{18(1+2\alpha)} - \dfrac{\delta^2(C-D)^2}{4(1+\alpha)^2}(\lambda+\mu) & \text{if } \mu \le -(v+\lambda), \\ \dfrac{(A-B)+6\delta(C-D)}{18(1+2\alpha)} & \text{if } -(v+\lambda) \le \mu \le (v-\lambda), \\ \dfrac{(A-B)}{18(1+2\alpha)} + \dfrac{\delta^2(C-D)^2}{4(1+\alpha)^2}(\lambda+\mu) & \text{if } \mu \ge (v-\lambda); \end{cases}$$

where λ *and* v *are defined as in the theorem 5.3. First result of* (i) *and second of* (ii) *are sharp for the function* $f_2(z)$ *defined by* (28) *and second estimate of* (i), *first and third of* (ii) *are sharp for the function* $f_3(z)$ *defined by* (29).

6. Second Hankel Functional

Theorem 6.1. *Let* $0 \le \alpha \le 1$ *and* $f \in C_s^*(\alpha)$. *Then* $\left|a_2 a_4 - a_3^2\right| \le \dfrac{9}{(1+2\alpha)^2}$.

Proof. Since $f \in \mathrm{C}_\mathrm{s}^*(\alpha)$, it follows that $(1-\alpha)f(z) + \alpha z f'(z) = P(z)G(z)$.
Identifying the terms, we get

$$\left.\begin{aligned} a_2 &= \frac{p_1}{(1+\alpha)} \\ a_3 &= \frac{p_2}{(1+2\alpha)} + \frac{b_3}{(1+2\alpha)} \\ a_4 &= \frac{p_3}{(1+3\alpha)} + \frac{p_1 b_3}{(1+3\alpha)} \end{aligned}\right\} \qquad (41)$$

As $g \in S^*$, by definition

$$zg'(z) = P(z)G(z).$$

Equating coefficients on both sides, we obtain

$$\left.\begin{aligned} b_2 &= \frac{p_1}{2} \\ b_3 &= \frac{p_2}{2} \\ b_4 &= \frac{p_3}{4} + \frac{p_1 b_3}{8} \end{aligned}\right\} \tag{42}$$

Eqs. (41) and (42) ensure that

$$a_2a_4 - a_3^2 = \frac{1}{C(\alpha)}\left\{4(1+2\alpha)^2 p_1(4p_3) + 4(1+2\alpha)^2 p_1^2(2p_2) - 9(1+\alpha)(1+3\alpha)(2p_2)^2\right\}$$

where $C(\alpha) = 16(1+\alpha)(1+3\alpha)(1+2\alpha)^2$.

Using lemma 2.6, we get

$$\begin{aligned} a_2a_4 - a_3^2 = \frac{1}{C(\alpha)}\Big\{ & -(1+4\alpha-5\alpha^2)p_1^4 - 6(1+4\alpha+\alpha^2)p_1^2(4-p_1^2)x \\ & -\left\{4(1+4\alpha+4\alpha^2)p_1^2 + 9(1+4\alpha+3\alpha^2)(4-p_1^2)\right\}(4-p_1^2)x^2 \\ & +8(1+4\alpha+4\alpha^2)p_1(4-p_1^2)(1-|x|^2)z\Big\} \end{aligned}$$

Assume that $p_1 = p$ and $p \in [0,2]$. Using triangular inequality and $|z| \le 1$, we have

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| \le \frac{1}{C(\alpha)}\Big\{ & (1+4\alpha-5\alpha^2)p^4 + 6(1+4\alpha+\alpha^2)p^2(4-p^2)\delta \\ & -\left\{4(1+4\alpha+4\alpha^2)p^2 + 9(1+4\alpha+3\alpha^2)(4-p^2)\right\}(4-p^2)\delta^2 \\ & +8(1+4\alpha+4\alpha^2)p(4-p^2)(1-\delta^2)\Big\} \end{aligned}$$

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| \le \frac{1}{C(\alpha)}\Big\{ & (1+4\alpha-5\alpha^2)p^4 + 6(1+4\alpha+\alpha^2)p^2(4-p^2)\delta \\ & +\left\{18(1+4\alpha+3\alpha^2) + (5+20\alpha+11\alpha^2)p\right\}(2-p)(4-p^2)\delta^2 \\ & +8(1+4\alpha+4\alpha^2)p(4-p^2)\Big\} \end{aligned}$$

$$\equiv \frac{1}{C(\alpha)}F(\delta) \text{ where } \delta = |x| \le 1.$$

Since $F(\delta)$ is an increasing function, therefore the maximum of $F(\delta) = F(1)$. Consequently

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{C(\alpha)}G(p) \tag{43}$$

where $G(p) = 144(1+4\alpha+3\alpha^2) - 32(1+4\alpha+4\alpha^2)p^2$.

Obviously $G(p) \le 144(1+4\alpha+3\alpha^2)$. The result follows from (43).

For $p_1 = 0,\ p_2 = -2$ and $p_3 = 0,$ we get the sharp results. □

Observe that $\left|a_2a_4 - a_3^2\right| \le \begin{cases} 9 & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha = 1. \end{cases}$

By the same pattern as in the above theorem, we can have the following theorems:

Theorem 6.2. *If* $0 \le \alpha \le 1$ *and* $f \in C^*_{1(s)}(\alpha),$ *then* $\left|a_2a_4 - a_3^2\right| \le \dfrac{49}{9(1+2\alpha)^2}.$

In particular , $\left|a_2a_4 - a_3^2\right| \le \begin{cases} \dfrac{49}{9} & \text{if } \alpha = 0, \\ \dfrac{49}{81} & \text{if } \alpha = 1. \end{cases}$

Theorem 6.3. *If* $0 \le \alpha \le 1$ *and* $f \in T^*_s(\alpha),$ *then* $\left|a_2a_4 - a_3^2\right| \le \dfrac{1}{(1+2\alpha)^2}.$

Proof. For $f \in T^*_s(\alpha)$, we have

$$zf'(z) + az^2 f''(z) = P(z)G(z).$$

On equating coefficients, we obtain

$$\left.\begin{aligned} a_2 &= \frac{p_1}{2(1+\alpha)} \\ a_3 &= \frac{p_2}{3(1+2\alpha)} + \frac{b_3}{3(1+2\alpha)} \\ a_4 &= \frac{p_3}{4(1+3\alpha)} + \frac{p_1b_3}{4(1+3\alpha)} \end{aligned}\right\} \tag{44}$$

$g \in S^*$ implies

$$zg'(z) = P(z)G(z).$$

From which, we have

$$\left.\begin{aligned} b_2 &= \frac{p_1}{2} \\ b_3 &= \frac{p_2}{2} \\ b_4 &= \frac{p_3}{4} + \frac{p_1b_3}{8} \end{aligned}\right\} \tag{45}$$

Eqs. (44) and (45) together ensure that

$$a_2a_4 - a_3^2 = \frac{1}{C(\alpha)}\left\{\begin{aligned} &(1+2\alpha)^2 p_1(4p_3) + (1+2\alpha)^2 p_1^2(2p_2) \\ &-2(1+\alpha)(1+3\alpha)(2p_2)^2 \end{aligned}\right\},$$

where $C(\alpha) = 32(1+\alpha)(1+3\alpha)(1+2\alpha)^2.$

Using lemma 2.6, we get

$$a_2a_4 - a_3^2 = \frac{1}{C(\alpha)}\begin{Bmatrix} 2\alpha^2 p_1^{\,4} - (1+4\alpha)p_1^{\,2}\left(4-p_1^{\,2}\right)x \\ -\left\{\left(1+4\alpha+4\alpha^2\right)p_1^{\,2} + 2\left(1+4\alpha+3\alpha^2\right)\left(4-p_1^{\,2}\right)\right\}\left(4-p_1^{\,2}\right)x^2 \\ +2\left(1+4\alpha+4\alpha^2\right)p_1\left(4-p_1^{\,2}\right)(1-|x|^2)z \end{Bmatrix}.$$

Setting $p_1 = p \in [0,2]$ and using $|z| \le 1$, we have

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{C(\alpha)}\begin{Bmatrix} 2\alpha^2 p^4 + (1+4\alpha)p^2\left(4-p^2\right)\delta + \left\{4\left(1+4\alpha+3\alpha^2\right) + \left(1+4\alpha+2\alpha^2\right)p\right\} \\ (2-p)\left(4-p^2\right)\delta^2 + 2\left(1+4\alpha+4\alpha^2\right)p\left(4-p^2\right) \end{Bmatrix}$$

$$\equiv \frac{1}{C(\alpha)}F(\delta), \text{ where } \delta = |x| \le 1.$$

One can easily see that maximum of $F(\delta) = F(1)$. Consequently

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{C(\alpha)}G(p) \tag{46}$$

where $G(p) = 32\left(1+4\alpha+3\alpha^2\right) - 8\left(1+4\alpha+4\alpha^2\right)p^2 + 4\alpha^2 p^4$.

$G(p)$ attains its maximum at $p = 0$. Required result follows from (46).

The sharp estimates are obtained for setting $p_1 = 0, p_2 = -2$ and $p_3 = 0$. □

Remark. $\left|a_2a_4 - a_3^2\right| \le \begin{cases} 1 \; if\; \alpha = 0, \\ \frac{1}{9} \; if\; \alpha = 1. \end{cases}$

On the footsteps of the above theorem, we can have

Theorem 6.4. *If* $f \in T^*_{1(s)}(\alpha)$, *then* $\left|a_2a_4 - a_3^2\right| \le \frac{49}{81(1+2\alpha)^2}$, $(0 \le \alpha \le 1)$.

In particular, $\left|a_2a_4 - a_3^2\right| \le \begin{cases} \frac{49}{81} \; if\; \alpha = 0, \\ \frac{49}{729} \; if\; \alpha = 1. \end{cases}$

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