Journal of Quality Measurement and Analysis
Jurnal Pengukuran Kualiti dan Analisis

JQMA 9(1) 2013, 57-64

SOLVING A BACKPACK VIBRATION SYSTEM USING FOURTH-ORDER RUNGE-KUTTA METHODS

(Penyelesaian Sistem Regangan Beg Galas Menggunakan Kaedah Runge-Kutta Peringkat Keempat)

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ABSTRACT

A backpack vibration system involving ordinary differential equations has been developed in this study. The purpose of this research is to solve a biomechanical model of a backpack vibration system numerically. The model is a backpack-human trunk system, represented by a mass-spring system that uses a Fourier series as an external force to describe a motion of the backpack. The displacement and velocity of the backpack vibration system are determined using two numerical methods; the classical fourth-order Runge-Kutta method (RK4) and a modified fourth-order Runge-Kutta method, namely Arithmetic Mean fourth-order Runge-Kutta methods (AM-RK4). The numerical results obtained from these two methods are compared using the absolute errors. The result shows that RK4 gives better agreement and accuracy with the exact solution when compared to AM-RK4.

Keywords: biomechanical model; suspension system; numerical method

ABSTRAK

Suatu sistem regangan beg galas melibatkan persamaan pembezaan biasa telah dibangunkan dalam kajian ini. Kajian ini dijalankan dengan tujuan untuk menyelesaikan satu model biomekanik sistem regangan beg galas secara berangka. Model ini terdiri dari sistem badan dan beg galas yang diwakili oleh sistem spring-peredam menggunakan siri Fourier sebagai daya luaran untuk menerangkan pergerakan beg galas. Kedudukan dan halaju sistem regangan beg galas diperoleh menggunakan dua kaedah berangka, iaitu kaedah Runge-Kutta Peringkat-4 klasik (RK4) dan kaedah Runge-Kutta Peringkat-4 Min Aritmetik (AM-RK4). Penyelesaian berangka bagi kedua-dua kaedah ini dibandingkan menggunakan ralat mutlak. Keputusan menunjukkan bahawa kaedah RK4 memberikan nilai yang lebih baik dan jitu, dengan penyelesaian sebenar berbanding dengan kaedah AM-RK4.

Kata kunci: model biomekanik; sistem regangan; kaedah berangka

1. Introduction

Ordinary differential equations arise frequently in almost every discipline of science and engineering. Among the models using ordinary differential equations are various physical problems including biomechanical system. Numerical methods are techniques for solving these ordinary differential equations to give approximate solutions and one of the widely used numerical methods is the Runge-Kutta method which comprise the second-order, third-order and fourth-order Runge-Kutta methods. However this study focuses on the classical fourth-order Runge-Kutta method (RK4) and a modified fourth-order Runge-Kutta method known as Arithmetic Mean fourth order Runge-Kutta method (AM-RK4) formed by Noorhelyna and Rokiah Rozita (2008). The methods are applied to a biomechanical problem, which is a linear differential equation, in order to obtain the numerical solutions for the displacement and velocity of a backpack suspension system. AM-RK4 and the classical RK4 are chosen to solve this model in order to identify the suitable method which can be utilised on other simulation in future. A biomechanical model of vibration system with different conditions may turn out to be a system of nonlinear differential equation which does not has an exact solution.

The classical RK4 (Steven & Raymond 1998) is represented by

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

with
$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + h/2, y_n + k_1 h/2)$$
$$k_3 = f(x_n + h/2, y_n + k_2 h/2)$$
$$k_4 = f(x_n + h, y_n + k_3 h) .$$

AM-RK4 (Noorhelyna & Rokiah Rozita 2008) which utilised the arithmetic mean in the stages is given by

$$y_{n+1} = y_n + h(b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4)$$

with
$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + ha_2, y_n + ha_2 k_1)$$
$$k_3 = f\left(x_n + ha_3, y_n + ha_3 \frac{k_1 + k_2}{2}\right)$$
$$k_4 = f\left(x_n + ha_4, y_n + ha_4 \frac{k_1 + k_2 + k_3}{3}\right).$$

The values of the parameters $b_1, b_2, b_3, b_4, a_2, a_3, a_4$ were determined by comparing the expansion of the above equations to the expansion of Taylor series, y_{n+1}. Let, $a_4 = 0.4$, the values of the parameters are set to be

$$b_1 = -1.06539415143497 65$$
$$b_2 = 0.34680394371268713$$
$$b_3 = 0.02552440724502229$$
$$b_4 = 1.69306580047726720$$
$$a_2 = 0.82120765335178770$$
$$a_3 = 1.79809727693180180 .$$

These two methods can be used to obtain approximate solutions in biomechanics model such as walking, running and jumping. The objective of this study is to achieve numerical values of displacement and velocity of the backpack vibration system using the classical RK4 and AM-RK4 methods and to compare the accuracy between both methods. Absolute errors are calculated by comparing the numerical solutions with the exact solutions. This is to ensure that the better numerical results will justify the method that can be utilised in future for any other related system, especially the nonlinear backpack vibration system.

2. The Biomechanical model of a backpack vibration system

Vibration can influence human body in many different ways. The response to the vibration exposure is primarily dependent on the frequency, amplitude and duration of the exposure and also may depend on the direction input, location and the presence of external forces that can produce various sensations to the human body. The dynamic responses of the body to the vibration of the backpack for example, exerted the forces on the bearer's torso which perceived discomfort, fatigue, tissue damage under straps, and the risk of injury such as rucksack palsy and back problem (Knapik *et al*. 1996). It is learned that while walking with a backpack, the interaction between a pack and a human trunk occurred in a dynamic way as a result of the cyclic motion of the trunk (Ren *et al*. 2005). The motion of the backpack relative to the body causes a vibration behaving like a spring that moves up and down following the movement of the body. As a result, a backpack suspension system which consists of a backpack and human trunk is developed. The model involves ordinary differential equations and is represented by a mass-spring system that uses a Fourier series as an external force to describe a motion of the backpack. A free body diagram of a backpack suspension system, is shown in Figure 1. T

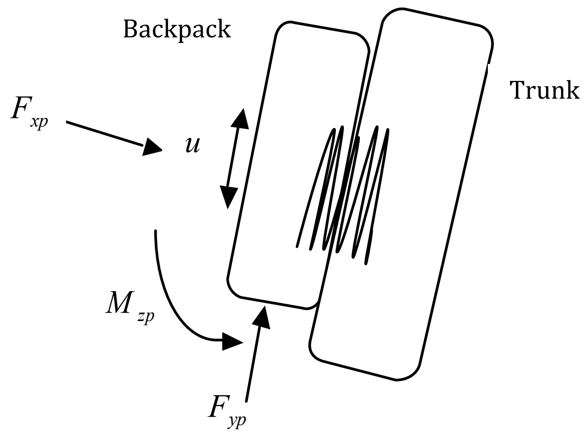


Figure1: A free body diagram of a backpack and a human trunk showing the forces exerted on the backpack centre of mass

The equation of motion of the back pack is derived as a linear differential equation of motion for free vibration of a damped spring-mass system and can be written as,

$$\ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = \sum_{n=1}^{\infty} \frac{4}{mn\pi}\sin(nt) \qquad \text{for } n = 1,3,5,\ldots \tag{1}$$

where u, $\dot{u}$ and $\ddot{u}$ are the displacement, velocity and acceleration of the backpack suspension system, respectively, t is time (s), m is a mass of the backpack (kg), c is a damping coefficient (Nsmm^{-1}), k is a spring stiffness (Nmm^{-1}), n is the number of selected period, and the external forces are represented by a Fourier series. Defining the parameters as the natural frequency ω_0 and $\xi = \dfrac{c}{2\sqrt{km}}$ as the damping ratio, equation (1) becomes

$$\ddot{u} + 2\xi\omega_o\dot{u} + \omega_o{}^2 u = \sum_{n=1}^{\infty} \frac{4}{mn\pi}\sin(nt) \quad \text{for } n = 1,3,5,... \tag{2}$$

Assuming that the oscillations will gradually decay to zero as the bearer stops walking, and taking 0 < ξ < 1, the system will vibrate at the natural damped frequency, $\omega_d = \omega_o\sqrt{1-\xi^2}$. Thus the analytical solution for the displacement of the backpack suspension system satisfies the differential equation,

$$u(t) = e^{-\zeta\omega_o t}\left(A\cos(\omega_d t) + B\sin(\omega_d t)\right) + \frac{1}{m}\left[\sum_{n=1}^{\infty}\frac{-8\zeta\omega_o n}{n\pi D_n}\cos(nt) + \sum_{n=1}^{\infty}\frac{4\left(\omega_o{}^2 - n^2\right)}{n\pi D_n}\sin(nt)\right] \quad n = 1,3,5,\ldots. \tag{3}$$

Taking the first derivative, the velocity of the backpack suspension system is represented by,

$$\dot{u}(t) = -\zeta\omega_o e^{-\zeta\omega_o t}\left[A\cos(\omega_d t) + B\sin(\omega_d t)\right] + e^{-\zeta\omega_o t}\left[-A\omega_d\sin(\omega_d t) + B\omega_d\cos(\omega_d t)\right] + \sum_{n=1}^{\infty}\frac{8\zeta\omega_o n^2}{mn\pi D_n}\sin(nt) + \sum_{n=1}^{\infty}\frac{4n\left(\omega_o{}^2 - n^2\right)}{mn\pi D_n}\cos(nt), \quad n = 1,3,5,\ldots \tag{4}$$

in which $D_n = (2\zeta\omega_o n)^2 + \left(\omega_o{}^2 - n\right)^2$.

The biomechanical model of a backpack vibration system is based on a linear differential equations. To approximate the solution for these differential equations, numerical methods are used. Thus in order to solve the equation of motion numerically using the Runge-Kutta method, the second order linear differential equation (2) has to be reduced into a system of first order linear differential equations. Letting $x_1 = u$ and $x_2 = \dot{u}$, the equations describing the state variables of the state-space system (Aplevich 1999) are given as

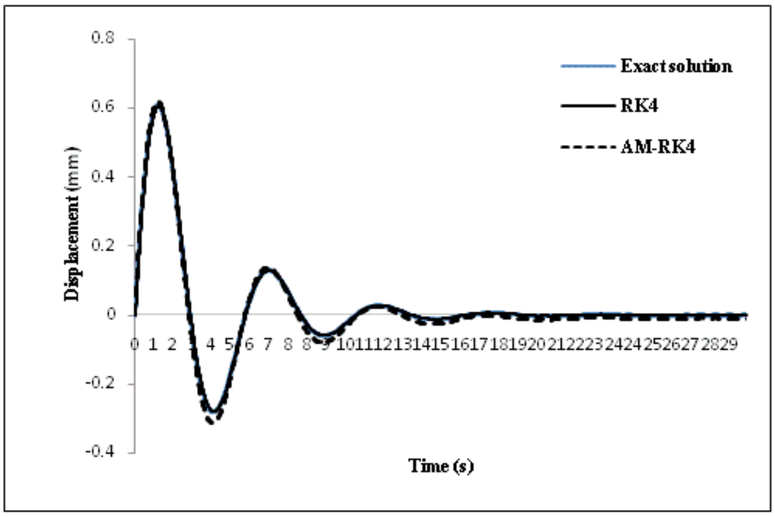
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2\xi\omega_o x_2 - \omega_o{}^2 x_1 + \sum_{n=1}^{\infty}\frac{4}{mn\pi}\sin(nt) \quad \text{for } n = 1,3,5,...$$

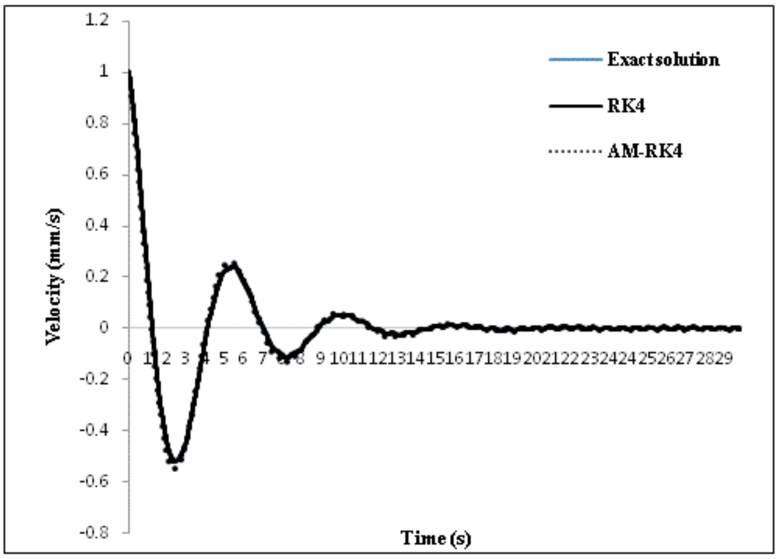
The simulated numerical and exact solutions from both methods are consumed to calculate the absolute errors.

3. Discussion on Numerical Solutions

By taking the mass of the backpack, m= 3.5 kg, as this is the average load carriage carried by primary school children (Fazrolrozi & Rambely 2008), the damping coefficient c = 2 Nsmm^{-1}, and the spring stiffness k = 5 Nmm^{-1}, the numerical solutions for the displacement and velocity of the backpack vibration system are considered over the range of time $0 \leq t \leq 30$. Using Matlab software with step size h = 0.02, the numerical solutions of every two steps using the classical RK4 and AM-RK4 methods are shown in Figure 2. The numerical results for the displacement and velocity of the backpack suspension system obtained using the classical RK4 shows excellent accuracy when compared to the AM-RK4 method. The graphs of the displacement and velocity of the backpack suspension system in Figures 2(a) and (b) also showed almost zero difference between the numerical solution from RK4 method and exact solution. However, the graph from the numerical solution of AM-RK4 method showed higher amplitudes in both the displacement and velocity of the backpack suspension system compared to the exact solution and RK4 method.



(a)



(b)

Figure 2: (a) Displacement, and (b) velocity of the backpack suspension system

For comparison, the absolute errors for displacement and velocity values are attained for both methods and are shown in Table 1 and 2. The numerical solutions of the displacement and velocity of the backpack vibration system from these two methods show that the absolute errors for RK4 method are better than the AM-RK4 method.

Table 1: Numerical solutions and absolute errors for the displacement of the backpack suspension system for each method

t	Exact Solution	RK4	AM-RK4	Absolute error RK4	Absolute error AM-RK4
0	0	0	0	0	0
2	0.357878888	0.357878891	0.351362402	3.11364E-09	6.51649E-03
4	-0.274853631	-0.274853635	-0.30496152	4.30428E-09	3.01079E-02
6	0.098365814	0.098365815	0.107320028	7.44044E-10	8.95421E-03
8	0.012198063	0.012198064	-0.004781895	1.34125E-09	1.69800E-02
10	-0.041177329	-0.04117733	-0.054209249	1.10413E-09	1.30319E-02
12	0.026974212	0.026974214	0.023093974	1.50926E-09	3.88024E-03
14	-0.008350701	-0.008350701	-0.022704164	5.19623E-11	1.43535E-02
16	-0.002256711	-0.002256711	-0.011195522	6.05362E-10	8.93881E-03
18	0.004995625	0.004995625	-0.00426804	2.89044E-10	9.26367E-03
20	-0.002256795	-0.002256796	-0.01335071	6.13065E-10	1.10939E-02
22	0.000700698	0.000700698	-0.008562213	2.46494E-11	9.26291E-03
24	4.30081E-05	4.30086E-05	-0.010061773	4.85355E-10	1.01048E-02
26	-0.000993305	-0.000993304	-0.011169815	5.55525E-10	1.01765E-02
28	-8.46049E-05	-8.46046E-05	-0.009998629	3.76494E-10	9.91402E-03
30	1.47434E-05	1.47433E-05	-0.010174996	1.23955E-10	1.01897E-02

Table 2: Numerical solutions and absolute errors for the velocity of the backpack suspension system for each method

t	Exact Solution	RK4	AM-RK4	Absolute error RK4	Absolute error AM-RK4
0	1.000000000000	1.000000000000	1.000000000000	0	0
2	-0.492229888605	-0.492229892452	-0.517829669583	3.84738E-09	2.55998E-02
4	0.056543992795	0.056543992589	0.075276569363	2.06014E-10	1.87326E-02
6	0.116233212455	0.116233217398	0.115010343881	4.9429E-09	1.22287E-03
8	-0.100198461965	-0.100198463243	-0.111689544675	1.27806E-09	1.14911E-02
10	0.047005335716	0.047005337084	0.059526116897	1.36827E-09	1.25208E-02
12	-0.002482820144	-0.002482820686	-0.006534401045	5.41358E-10	4.05158E-03
14	-0.017650946729	-0.017650949277	-0.017784485308	2.54739E-09	1.33539E-04
16	0.006581883095	0.006581882336	0.008828749888	7.58307E-10	2.24687E-03
18	-0.005173541781	-0.005173541875	-0.008432060722	9.48112E-11	3.25852E-03
20	0.002117724421	0.002117725637	0.001957945024	1.2159E-09	1.59779E-04
22	0.006059036240	0.006059038095	0.006086936205	1.85566E-09	2.79000E-05
24	0.002402766130	0.002402767005	0.002659993710	8.74452E-10	2.57228E-04
26	0.000404794011	0.000404793555	0.001859123898	4.56194E-10	1.45433E-03
28	-0.003292322129	-0.003292323675	-0.002634417204	1.54569E-09	6.57905E-04
30	-0.004705336540	-0.004705338159	-0.004871293238	1.61909E-09	1.65957E-04

Several different step sizes had also been chosen to obtain numerical solutions and relative errors for both methods at time $t = 10s$ as shown in Table 3 and 4. The absolute errors for the AM-RK4 are higher than the classical RK4 as the step sizes get larger, in both the displacement and velocity of the backpack suspension system. Even though we did not investigate the stability region for AM-RK4 method but we assume that these maybe due to the smaller stability region of the AM-RK4 method as compared with the stability region of RK4 method.

Table 3: Exact, numerical solutions and absolute errors for displacement of the backpack suspension system for each method at time $t = 10s$

Step size	Exact solution	RK4	AM-RK4	Absolute error	
				RK4	AM-RK4
0.5	-0.041177328609	-0.040945022655	-0.177121723495	2.32306E-04	1.35944E-01
0.25	-0.041177328609	-0.041202916107	-0.125884491053	2.55875E-05	8.47072E-02
0.125	-0.041177328609	-0.041179047469	-0.114491284351	1.71886E-06	7.33140E-02
0.1	-0.041177328609	-0.041178033055	-0.102230438927	7.04447E-07	6.10531E-02
0.02	-0.041177328609	-0.041177329713	-0.054209249393	1.10413E-09	1.30319E-02
0.01	-0.041177328609	-0.041177328677	-0.047715867587	6.86738E-11	6.53854E-03

Table 4: Exact, numerical solutions and absolute errors for velocity of the backpack suspension system for each method at time $t = 10s$

Step size	Exact solution	RK4	AM-RK4	Absolute error	
				RK4	AM-RK4
0.5	0.047005335716	0.047288187698	4.522690596776	2.82852E-04	4.47569E+00
0.25	0.047005335716	0.047031130948	0.488312649085	2.57952E-05	4.41307E-01
0.125	0.047005335716	0.047007195562	0.166746862379	1.85985E-06	1.19742E-01
0.1	0.047005335716	0.047006119209	0.133061532568	7.83493E-07	8.60562E-02
0.02	0.047005335716	0.047005337084	0.059526116897	1.36827E-09	1.25208E-02
0.01	0.047005335716	0.047005335802	0.053035358202	8.64280E-11	6.03002E-03

4. Conclusions

In this study a backpack vibration system involving second order ordinary linear differential equations which has been reduced to a system of first order linear differential equations has been developed to solve a biomechanical model of a backpack vibration system. The displacement and velocity of the backpack vibration system are calculated using two numerical methods, the classical RK4 and AM-RK4 methods. The graphs of the displacement and velocity of the backpack suspension system showed a diminutive difference between the numerical solution of RK4 method and exact solution. However the graphs from AM-RK4 method showed higher amplitudes in both displacement and velocity of the backpack suspension system compared to the exact solution. We conclude that the classical RK4 is more suitable for solving the biomechanical model of a backpack vibration system.

Acknowledgments

The authors gratefully acknowledge the supports received in the form of grant by the Universiti Kebangsaan Malaysia (GUP-2012-004 and UKM-DLP-049).

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