Journal of Quality Measurement and Analysis
Jurnal Pengukuran Kualiti dan Analisis
JQMA 9(2) 2013, 59-71

NEW CRITERIA FOR CERTAIN CLASSES CONTAINING GENERALISED DIFFERENTIAL OPERATOR

(Kriterium Baharu untuk Kelas Tertentu yang Mengandungi Pengoperasi Pembezaan Teritlak)

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ABSTRACT

The main objective of this study is to generalise a differential operator. This generalised differential operator reduces to many known operators studied by various authors. New subclasses containing generalisation of differential operator are introduced. Characterisation and other properties of these subclasses are given.

Keywords: analytic function; Sălăgean differential operator; Ruscheweyh differential operator

ABSTRAK

Objektif utama kajian ini adalah untuk mengitlak pengoperasi pembezaan. Pengoperasi teritlak ini menurun kepada banyak pengoperasi terkenal yang dikaji oleh pelbagai penulis. Subkelas baharu yang mengandungi pengoperasi pembezaan teritlak ini diperkenalkan. Cirian dan sifat lain bagi subkelas ini diberi.

Kata kunci: fungsi analisis; pengoperasi pembezaan Sălăgean; pengoperasi pembezaan Ruscheweyh

1. Introduction

Let H be the class of functions analytic in $U=\left\{z\in C:\left|z\right|<1\right\}$ and $H[a,n]$ be the subclass of H consisting of functions of the form $f(z)=a+a_nz^n+a_{n+1}z^{n+1}+\ldots$. Let A be the subclass of H consisting of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1.1}$$

Now we introduce a differential operator defined as follows: $D^n_{\alpha,\beta,\lambda,\delta}:A\to A$ by

$$D^k_{\alpha,\beta,\delta,\lambda}f(z)=z+\sum_{n=2}^{\infty}\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\,a_nz^n \tag{1.2}$$

for k = 0,1,2,…, $0<\alpha\le 1$, $0<\beta\le 1$, $\lambda\ge 0,\delta\ge 0$, and

$$C(\delta,n)=\binom{n+\delta-1}{\delta}=\frac{\Gamma(n+\delta)}{\Gamma(n)\Gamma(n+1)}.$$

Remark 1.1. When $\delta = 0, \alpha = \beta = \lambda = 1$ in (1.2), we get the Sălăgean differential operator (Sălăgean 1983). For $k = 0$ in (1.2), it gives Ruscheweyh operator (Ruscheweyh 1975), while $\delta = 0, \alpha = \beta = 1$ in (1.2), implies Al-Oboudi differential operator of order (k) (Al-oboudi 2004), also when in (1.2) $\alpha = \beta = \lambda = 1$, it reduces to Al-Shaqsi and Darus differential operator (Shaqsi & Darus 2008) and finally, for $\alpha = \beta = 1$ in (1.2), it reduces to Ibrahim and Darus differential operator (Darus & Ibrahim 2008).

Some of the relations for the differential operator (1.2) are discussed in the next lemma.

Lemma 1.1. (Sălăgean 1983). *Let $f \in A$. Then*

(i) $D^{0}_{\alpha,\beta,0,\lambda} f(z) = f(z),$

(ii) $D^{1}_{\alpha,\beta,0,0} f(z) = zf'(z).$

In the following definitions, new subclasses of analytic functions containing the differential operator (1.2) are introduced:

Definition 1.1. Let $f \in A$. Then $f(z) \in S^{k}_{\alpha,\beta,\delta,\lambda}(\eta)$ if and only if

$$\operatorname{Re}\left\{ \frac{z\left[D^{k}_{\alpha,\beta,\delta,\lambda} f(z) \right]'}{D^{k}_{\alpha,\beta,\delta,\lambda} f(z)} \right\} > \eta \quad 0 \le \eta < 1, 0 < \beta \le 1, 0 < \lambda \le 1, z \ge 0, z \in U.$$

Definition 1.2. Let $f \in A$. Then $f(z) \in C^{k}_{\lambda,\delta,\alpha,\beta}(\eta)$ if and only if

$$\operatorname{Re}\left\{ \frac{\left[z\left(D^{k}_{\alpha,\beta,\delta,\lambda} f(z) \right)' \right]'}{\left(D^{k}_{\alpha,\beta,\delta,\lambda} f(z) \right)'} \right\} > \eta, 0 \le \eta < 1, 0 < \beta \le 1, 0 < \alpha \le 1, \lambda \ge 0, z \in U.$$

Next, we study the characterisation properties of these classes.

2. General properties of $D^{k}_{\alpha,\beta,\delta,\lambda}$

In this section we study the characterisation of the function $f \in A$ to belong to the classes $S^{k}_{\alpha,\beta,\delta,\lambda}(\eta)$ and $C^{k}_{\alpha,\beta,\delta,\lambda}(\eta)$ by obtaining the coefficient bounds.

Theorem 2.1. *Let the function* $f \in A$. *If*

$$\sum_{n=2}^{\infty}(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k} C(\delta,n)\left|a_{n}\right| \leq 1-\eta, \tag{2.1}$$

$$0 \leq \eta < 1, 0 < \beta \leq 1, 0 < \alpha \leq 1, \lambda \geq 0,$$

then $f(z) \in S_{\alpha,\beta,\delta,\lambda}^{k}(\eta)$. *The result* (2.1) *is sharp.*

Proof. Suppose that (2.1) holds. Since

$$\begin{aligned}
1-\eta \geq & \sum_{n=2}^{\infty}(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k} C(\delta,n)\left|a_{n}\right| \\
= & \sum_{n=2}^{\infty} n\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k} C(\delta,n)\left|a_{n}\right| \\
- & \sum_{n=2}^{\infty} \eta\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k} C(\delta,n)\left|a_{n}\right| \\
\geq & \sum_{n=2}^{\infty} \eta\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k} C(\delta,n)\left|a_{n}\right| \\
- & \sum_{n=2}^{\infty} n\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k} C(\delta,n)\left|a_{n}\right|,
\end{aligned}$$

then this implies that

$$\frac{1+\sum_{n=2}^{\infty} n\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k} C(\delta,n)\left|a_{n}\right|}{1+\sum_{n=2}^{\infty}\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k} C(\delta,n)\left|a_{n}\right|} > \eta.$$

Hence

$$\operatorname{Re}\left\{\frac{z\left[D_{\alpha,\beta,\delta,\lambda}^{k} f(z)\right]'}{D_{\alpha,\beta,\delta,\lambda}^{k} f(z)}\right\} > \eta,\ 0 \leq \eta < 1.$$

Note that the assertion (2.1) is sharp and the extremal function is given by

$$f(z)=z+\sum_{n=2}^{\infty} \frac{1-\eta}{(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k} C(\delta,n)\left|a_{n}\right|} z^{n}.$$

Corollary 2.1. *Let the hypothesis of Theorem* 2.1 *be satisfied. Then*

$$\left|a_n\right| \le \sum_{n=2}^{\infty} \frac{1-\eta}{(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)}, \forall n \ge 2. \tag{2.2}$$

Thus, the function f belongs to the class $S^k_{\alpha,\beta,\delta,\lambda}(\eta)$.

Corollary 2.2. *Let the hypothesis of Theorem* 2.1 *be satisfied. Then for* $\delta = \eta = 0$, *and* $\alpha = \beta = \lambda = 1$,

$$\left|a_n\right| \le \frac{1}{n^k}, \forall n \ge 2, k \in N_0. \tag{2.3}$$

Corollary 2.3. *Let the hypothesis of Theorem* 2.1 *be satisfied. Then for* $k = \delta = \eta = 0$,

$$\left|a_n\right| \le \frac{1}{n}, \forall n \ge 2. \tag{2.4}$$

Theorem 2.2. *Let the function* $f \in A$. *If*

$$\sum_{n=2}^{\infty} n(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right| \le 1-\eta, 0 \le \eta < 1, \tag{2.5}$$

then $f(z) \in C^k_{\alpha,\beta,\delta,\lambda}(\eta)$. *The result* (2.5) *is sharp.*

Proof. Suppose that (2.5) holds. Since

$$\begin{aligned}
1-\eta &\ge \sum_{n=2}^{\infty} n(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right| \\
&= \sum_{n=2}^{\infty} n^2\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right| \\
&\quad - \sum_{n=2}^{\infty} (n).(\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right| \\
&\ge \sum_{n=2}^{\infty} (n).(\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right| \\
&\quad - \sum_{n=2}^{\infty} n^2\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right|,
\end{aligned}$$

then this implies that

$$\frac{1+\sum_{n=2}^{\infty} n^2\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right|}{1+\sum_{n=2}^{\infty} n\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right|} > \eta.$$

Hence

$$\operatorname{Re}\left\{\frac{\left[z\left(D^k_{\alpha,\beta,\delta,\lambda} f(z)\right)'\right]'}{\left(D^k_{\alpha,\beta,\delta,\lambda} f(z)\right)'}\right\} > \eta, 0 \le n < 1, z \in U.$$

Note that the assertion (2.5) is sharp and the extremal function is given by

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\eta}{n(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right|} z^n. \tag{2.6}$$

Corollary 2.4. *Let the hypothesis of Theorem* 2.2 *be satisfied. Then*

$$\left|a_n\right| \le \sum_{n=2}^{\infty} \frac{1-\eta}{n(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)}, \forall n \ge 2. \tag{2.7}$$

Thus, the function f belongs to the class $C^k_{\alpha,\beta,\delta,\lambda}(\eta)$.

Also we have the following inclusion results.

Theorem 2.3. *Let* $0 \le \eta_1 \le \eta_2 < 1$. *Then* $S^k_{\alpha,\beta,\delta,\lambda}(\eta_1) \supseteq S^k_{\alpha,\beta,\delta,\lambda}(\eta_2)$.

Proof: By Theorem 2.1, we have

$$\sum_{n=2}^{\infty}(n-\eta_2)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right|$$
$$\le \sum_{n=2}^{\infty}(n-\eta_1)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right| \le 1-\eta.$$

Theorem 2.4. *Let* $0 \le \eta_1 \le \eta_2 < 1$. *Then* $C^k_{\alpha,\beta,\delta,\lambda}(\eta_1) \supseteq C^k_{\alpha,\beta,\delta,\lambda}(\eta_2)$

Proof: By Theorem 2.2, we have

$$\sum_{n=2}^{\infty} n(n-\eta_2)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right|$$
$$\le \sum_{n=2}^{\infty} n(n-\eta_1)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right| \le 1-\eta.$$

Theorem 2.5. *Let* $0 \le \lambda_1 \le \lambda_2 < 1$. *Then* $S^k_{\alpha,\beta,\delta,\lambda_1}(\eta) \subseteq S^k_{\alpha,\beta,\delta,\lambda_2}(\eta)$.

Proof: By Theorem 2.1, we have

$$\sum_{n=2}^{\infty} (n-\eta)\left[\lambda_1(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right|$$
$$\le \sum_{n=2}^{\infty} (n-\eta)\left[\lambda_2(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right| \le 1-\eta.$$

Theorem 2.6. *Let* $0 \le \lambda_1 \le \lambda_2 < 1$. *Then* $C^k_{\alpha,\beta,\delta,\lambda_1}(\eta) \subseteq C^k_{\alpha,\beta,\delta,\lambda_2}(\eta)$.

Proof: By Theorem 2.2, we have

$$\sum_{n=2}^{\infty} n(n-\eta)\left[\lambda_1(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right|$$
$$\le \sum_{n=2}^{\infty} n(n-\eta)\left[\lambda_2(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right| \le 1-\eta.$$

Theorem 2.7.

(i) *Let* $0 \le k_1 \le k_2$ *where* $k_1, k_2 \le N_0$. *If* $\left[\lambda(\alpha+\beta-1)(n-1)+1\right] \ge 1$ *then*

$$S^{k_1}_{\alpha,\beta,\delta,\lambda}(\eta) \subseteq S^{k_2}_{\alpha,\beta,\delta,\lambda}(\eta).$$

(ii) *Let* $0 \le k_1 \le k_2$ *where* $k_1, k_2 \le N_0$. *If* $1 > \left[\lambda(\alpha+\beta-1)(n-1)+1\right] \ge 0$ *then*

$$S^{k_1}_{\alpha,\beta,\delta,\lambda}(\eta) \supseteq S^{k_2}_{\alpha,\beta,\delta,\lambda}(\eta).$$

Proof: (i) By Theorem 2.1, we have

$$\sum_{n=2}^{\infty}(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k_1} C(\delta,n)\left|a_n\right|$$
$$\leq \sum_{n=2}^{\infty}(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k_2} C(\delta,n)\left|a_n\right| \leq 1-\eta.$$

Proof: (ii) By Theorem 2.1, we have

$$\sum_{n=2}^{\infty}(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k_2} C(\delta,n)\left|a_n\right|$$
$$\leq \sum_{n=2}^{\infty}(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k_1} C(\delta,n)\left|a_n\right| \leq 1-\eta.$$

Theorem 2.8.

(i) *Let* $0 \leq k_1 \leq k_2$ *where* $k_1, k_2 \leq N_0$. *If* $\left[\lambda(\alpha+\beta-1)(n-1)+1\right] \geq 1$ *then*

$$C^{k_1}_{\alpha,\beta,\delta,\lambda}(\eta) \subseteq C^{k_2}_{\alpha,\beta,\delta,\lambda}(\eta).$$

(ii) *Let* $0 \leq k_1 \leq k_2$ *where* $k_1, k_2 \leq N_0$. *If* $1 > \left[\lambda(\alpha+\beta-1)(n-1)+1\right] \geq 0$ *then*

$$C^{k_1}_{\alpha,\beta,\delta,\lambda}(\eta) \supseteq C^{k_2}_{\alpha,\beta,\delta,\lambda}(\eta)$$

Proof: (i) By Theorem 2.2, one can verify that

$$\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k_1} \leq \left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k_2},$$

then

$$\sum_{n=2}^{\infty} n(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k_1} C(\delta,n)\left|a_n\right|$$
$$\leq \sum_{n=2}^{\infty} n(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k_2} C(\delta,n)\left|a_n\right| \leq 1-\eta.$$

Proof: (ii) By Theorem 2.2, one can verify that

$$\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k_2} \leq \left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k_1},$$

then

$$\sum_{n=2}^{\infty} n(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k_2} C(\delta,n)\left|a_n\right|$$
$$\leq \sum_{n=2}^{\infty} n(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^{k_1} C(\delta,n)\left|a_n\right| \leq 1-\eta.$$

Theorem 2.9. *Let* $0 \le \alpha_1 \le \alpha_2$ *where* $0 < \alpha_1 \le 1$ *and* $0 < \alpha_2 \le 1$. *Then*

$$S^k_{\alpha_1,\beta,\delta,\lambda}(\eta) \subseteq S^k_{\alpha_2,\beta,\delta,\lambda}(\eta).$$

Proof: By Theorem 2.1, we have

$$\sum_{n=2}^{\infty}(n-\eta)\left[\lambda(\alpha_1+\beta-1)(n-1)+1\right]^k C(\delta,n)|a_n|$$
$$\le \sum_{n=2}^{\infty}(n-\eta)\left[\lambda(\alpha_2+\beta-1)(n-1)+1\right]^k C(\delta,n)|a_n| \le 1-\eta.$$

Theorem 2.10. *Let* $0 \le \alpha_1 \le \alpha_2$ *where* $0 < \alpha_1 \le 1$ *and* $0 < \alpha_2 \le 1$. *Then*

$$C^k_{\alpha_1,\beta,\delta,\lambda}(\eta) \subseteq C^k_{\alpha_2,\beta,\delta,\lambda}(\eta).$$

Proof: By Theorem 2.2, we have

$$\sum_{n=2}^{\infty}n(n-\eta)\left[\lambda(\alpha_1+\beta-1)(n-1)+1\right]^k C(\delta,n)|a_n|$$
$$\le \sum_{n=2}^{\infty}n(n-\eta)\left[\lambda(\alpha_2+\beta-1)(n-1)+1\right]^k C(\delta,n)|a_n| \le 1-\eta.$$

Theorem 2.11. *Let* $0 \le \beta_1 \le \beta_2$ *where* $0 < \beta_1 \le 1$ *and* $0 < \beta_2 \le 1$. *Then*

$$S^k_{\alpha,\beta_1,\delta,\lambda}(\eta) \subseteq S^k_{\alpha,\beta_2,\delta,\lambda}(\eta).$$

Proof: By Theorem 2.1, we have

$$\sum_{n=2}^{\infty}(n-\eta)\left[\lambda(\alpha+\beta_1-1)(n-1)+1\right]^k C(\delta,n)|a_n|$$
$$\le \sum_{n=2}^{\infty}(n-\eta)\left[\lambda(\alpha+\beta_2-1)(n-1)+1\right]^k C(\delta,n)|a_n| \le 1-\eta.$$

Theorem 2.12. *Let* $0 \le \beta_1 \le \beta_2$ *where* $0 < \beta_1 \le 1$ *and* $0 < \beta_2 \le 1$. *Then*

$$C^k_{\alpha,\beta_1,\delta,\lambda}(\eta) \subseteq C^k_{\alpha,\beta_2,\delta,\lambda}(\eta).$$

Proof: By Theorem 2.2, we have

$$\sum_{n=2}^{\infty} n(n-\eta)\left[\lambda(\alpha+\beta_1-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right|$$
$$\le \sum_{n=2}^{\infty} n(n-\eta)\left[\lambda(\alpha+\beta_2-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right| \le 1-\eta.$$

Theorem 2.13. *Let the function* $f \in A$ *and* (2.3) *holds. Then for* $z \in U$ *and* $0 \le \eta < 1$,

$$\left|D^k_{\alpha,\beta,\delta,\lambda} f(z)\right| \ge |z| - \frac{(1-\eta)}{2-\eta}$$

and

$$\left|D^k_{\alpha,\beta,\delta,\lambda} f(z)\right| \le |z| + \frac{(1-\eta)}{2-\eta}.$$

Proof: By using Theorem 2.1, one can verify that

$$(2-\eta)\sum_{n=2}^{\infty}\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right|$$

$$\le \sum_{n=2}^{\infty}(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right| \le 1-\eta,$$

then

$$\sum_{n=2}^{\infty}\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right| \le \frac{1-\eta}{2-\eta}.$$

Thus we obtain

$$\left|D^k_{\alpha,\beta,\delta,\lambda} f(z)\right| \le |z| + \sum_{n=2}^{\infty}\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right|\left|z\right|^n$$

$$\le |z| + \sum_{n=2}^{\infty}\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)\left|a_n\right|\left|z\right|^2$$

$$\le |z| + \left[\frac{1-\eta}{2-\eta}\right]|z|^2.$$

On the other hand we obtain

$$\left|D^k_{\alpha,\beta,\delta,\lambda}f(z)\right| = \left|z + \sum_{n=2}^{\infty}(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)a_n z^n\right|$$
$$\geq |z| - \sum_{n=2}^{\infty}(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)|a_n||z|^n$$
$$\geq |z| - \sum_{n=2}^{\infty}(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)|a_n||z|^2$$
$$\geq |z| - \left[\frac{1-\eta}{2-\eta}\right]|z|^2.$$

This completes the proof.

In the same way we can get the following result.

Theorem 2.14. *Let the function f* $\in A$ *and* (2.5) *holds. Then for* $z \in U$ *and* $0 \leq \eta < 1$,

$$\left|D^k_{\alpha,\beta,\delta,\lambda}f(z)\right| \geq |z| - \left[\frac{1-\eta}{2(2-\eta)}\right]|z|^2$$

and

$$\left|D^k_{\alpha,\beta,\delta,\lambda}f(z)\right| \leq |z| + \left[\frac{1-\eta}{2(2-\eta)}\right]|z|^2.$$

Also, we have the following distortion results.

Theorem 2.15. *Let the function f* $\in A$ *and* (2.1) *be satisfied. Then we have for*

$(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n) \geq 1$ $0 \leq \eta < 1$,

$$|f(z)| \geq |z| - (1-\eta)|z|^2$$

and

$$|f(z)| \leq |z| + (1-\eta)|z|^2.$$

Proof: In virtue of Theorem 2.1, we have

$$\sum_{n=2}^{\infty}|a_n| \leq \sum_{n=2}^{\infty}(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)|a_n| \leq 1-\eta,$$

then

$$\sum_{n=2}^{\infty}|a_n| \leq 1-\eta.$$

Thus we obtain

$$\left| f(z) \right| = \left| z + \sum_{n=2}^{\infty} a_n z^n \right|$$
$$\leq |z| + \sum_{n=2}^{\infty} |a_n| |z|^2$$
$$\leq |z| + (1-\eta)|z|.$$

On the other hand we obtain

$$\left| f(z) \right| = \left| z + \sum_{n=2}^{\infty} a_n z^n \right|$$
$$\geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^2$$
$$\geq |z| - (1-\eta)|z|^2.$$

This completes the proof.

Theorem 2.16. *Let the function* $f \in A$ *and* (2.5) *be satisfied. Then for*

$n(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n) \geq 1\ 0 \leq \eta < 1$, *we have*

$$\left| f(z) \right| \geq |z| - \frac{(1-\eta)}{2}|z|^2$$

and

$$\left| f(z) \right| \leq |z| + \frac{(1-\eta)}{2}|z|^2.$$

The result is sharp.

Theorem 2.17. *Let the hypothesis of Theorem* 2.1 be *satisfied. Then*

$$\left| f(z) \right| \geq |z| - \frac{(1-\eta)\Gamma(\delta+1)}{(2-\eta)\Gamma(\delta+2)\left[\lambda(\alpha+\beta-1)+1\right]^k}|z|^2$$

and

$$\left| f(z) \right| \leq |z| + \frac{(1-\eta)\Gamma(\delta+1)}{(2-\eta)\Gamma(\delta+2)\left[\lambda(\alpha+\beta-1)+1\right]^k}|z|^2.$$

Proof: In virtue of Theorem 2.1, we have

$$(2-\eta)\left[\lambda(\alpha+\beta-1)+1\right]^k \frac{\Gamma(\delta+2)}{\Gamma(\delta+1)} \sum_{n=2}^{\infty} |a_n|$$

$$\leq \sum_{n=2}^{\infty} (n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n)|a_n| \leq 1-\eta,$$

then

$$\sum_{n=2}^{\infty}|a_n| \le \frac{(1-\eta)\Gamma(\delta+1)}{(2-\eta)\Gamma(\delta+2)\left[\lambda(\alpha+\beta-1)+1\right]^k}.$$

Thus we obtain

$$|f(z)| = \left|z + \sum_{n=2}^{\infty} a_n z^n\right|$$

$$\le |z| + \sum_{n=2}^{\infty}|a_n||z|^2$$

$$\le |z| + \frac{(1-\eta)\Gamma(\delta+1)}{(2-\eta)\Gamma(\delta+2)\left[\lambda(\alpha+\beta-1)+1\right]^k}|z|^2.$$

On the other hand,

$$|f(z)| \ge |z| - \sum_{n=2}^{\infty}|a_n||z|^n$$

$$\ge |z| - \sum_{n=2}^{\infty}|a_n||z|^2$$

$$\ge |z| - \frac{(1-\eta)\Gamma(\delta+1)}{(2-\eta)\Gamma(\delta+2)\left[\lambda(\alpha+\beta-1)+1\right]^k}|z|^2.$$

This completes the proof.

In the same way, we get the following results.

Theorem 2.18. *Let the hypothesis of Theorem* 2.2 be *satisfied. Then*

$$n(n-\eta)\left[\lambda(\alpha+\beta-1)(n-1)+1\right]^k C(\delta,n) \ge 1 \text{ and } 0 \le \eta < 1 \text{ poses}$$

$$|f(z)| \ge |z| - \frac{(1-\eta)\Gamma(\delta+1)}{2(2-\eta)\Gamma(\delta+2)\left[\lambda(\alpha+\beta-1)+1\right]^k}|z|^2$$

and

$$|f(z)| \le |z| + \frac{(1-\eta)\Gamma(\delta+1)}{2(2-\eta)\Gamma(\delta+2)\left[\lambda(\alpha+\beta-1)+1\right]^k}|z|^2.$$

Acknowledgement

The work here is supported by LRGS/TD/2011/UKM/ICT/03/02 and GUP-2013-004.

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