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ON STRONGLY *h*-CONVEX STOCHASTIC PROCESSES

(Mengenai Proses Stokastik *h*-Cembung Kuat)

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ABSTRACT

In this paper, the concept of strongly *h*-convex stochastic processes is introduced. New inequality related to Hermite-Hadamard type for strongly *h*-convex stochastic processes is obtained. Some properties of convex stochastic processes are presented. The results given in this work are generalizations of the known results.

Keywords: convex stochastic process; strongly *h*-convex stochastic process; Hermite-Hadamard type inequality

ABSTRAK

Dalam makalah ini, diperkenalkan konsep proses stokastik *h*-cembung kuat. Diperoleh ketaksamaan baharu berkait dengan ketaksamaan jenis Hermite-Hadamard untuk proses stokastik *h*-cembung kuat. Dipersembahkan beberapa sifat proses stokastik *h*-cembung kuat. Hasil-hasil yang diperoleh adalah pengitlakan daripada hasil yang diketahui.

Kata kunci: proses stokastik cembung; proses stokastik *h*-cembung kuat; ketaksamaan jenis Hermite-Hadamard

1. Introduction

Stochastic process is a research area in probability theory dealing with probabilistic models and stochastic convexity is of great importance in optimizations and also useful for numerical approximation when there exist probabilistic quantities.

The notion of convex stochastic processes was initiated by Nagy in 1974 (Nagy 1974). After that, Nikodem (1980) introduced the convex (Jensen-convex) stochastic processes in his work. Later, Shake and Shantikumar (1988), Skowronski (1992; 1995) generalized this type of convexity stochastic processes and obtained some further results on these processes.

Many inequalities have been established for convex stochastic processes and one of the famous is the Hermite-Hadamard inequality. In 2012, Kotrys extended this classical inequality to this type of convexity. For more information and recent studies on Hermite-Hadamard type inequalities for some type of convexities of stochastic processes, please refer to Set *at al*. (2014), Tomar *et al*. (2014), Gonzalez *et al.* (2015), Materano *et al.* (2015), Maden *et al*. (2015), Sarikaya *et al*. (2016), Li and Hao (2017), Budak and Sarikaya (2019), Okur *et al*. (2019), Ozcan (2019) and the references cited therein.

2. Preliminaries

Suppose (Ω, Δ, P) be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $\Delta-$ measurable. A function $X : I \times \Omega \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, is called a stochastic process if the function $X(t,\cdot)$ is a random variable for all $t \in I$.

Let $P-\lim$ and $E[X(s,\cdot)]$ denote the limit in probability and the expectation value of random variable $X(t,\cdot)$, respectively. Then, a stochastic process $X : I \times \Omega \to \mathbb{R}$ has

(a) continuity in probability in I if for every $s_0 \in I$, $P-\lim_{s\to s_0} X(s,\cdot) = X(s_0,\cdot)$,

(b) mean-square continuity in I if for every $s_0 \in I$, $\lim_{s\to s_0} E\left[\left(X(s,\cdot)-X(s_0,\cdot)\right)^2\right]=0$,

(c) mean-square differentiability at a point $s \in I$ if there exists a random variable $X'(s,\cdot): I \times \Omega \to \mathbb{R}$ such that $X'(s,\cdot) = P-\lim_{s\to s_0} \dfrac{X(s,\cdot)-X(s_0,\cdot)}{s-s_0}$.

Basic properties of the mean-square integral can be read in Sobczyk (1991).

We now state the definition of convex, strongly convex and *h*-convex of stochastic processes.

Definition 2.1. The stochastic process $X : I \times \Omega \to \mathbb{R}$ is said to be convex if for all $\lambda \in [0,1]$ and $u, v \in I$ with $u < v$, the following inequality holds true,

$$X\left(\lambda u + (1-\lambda)v,\cdot\right) \le \lambda X\left(u,\cdot\right) + (1-\lambda) X\left(v,\cdot\right). \tag{1}$$

Definition 2.2. Let $C : \Omega \to \mathbb{R}$ denotes a positive random variable. The stochastic process $X : I \times \Omega \to \mathbb{R}$ is said to be strongly convex with modulus $C(\cdot)$ if for all $\lambda \in [0,1]$ and $u, v \in I$ with $u < v$, the following inequality holds true,

$$X\left(\lambda u + (1-\lambda)v,\cdot\right) \le \lambda X\left(u,\cdot\right) + (1-\lambda) X\left(v,\cdot\right) - C(\cdot)\lambda(1-\lambda)(u-v)^2. \tag{2}$$

Definition 2.3. Let $h:(0,1) \to \mathbb{R}$, $h \neq 0$ be a non-negative function. We say that a stochastic process $X : I \times \Omega \to \mathbb{R}$ is an *h*-convex stochastic process if for every $u, v \in I$, $u < v$ and $\lambda \in [0,1]$, the following inequality is satisfied,

$$X\left(\lambda u + (1-\lambda)v,\cdot\right) \le h(\lambda) X\left(u,\cdot\right) + h(1-\lambda) X\left(v,\cdot\right). \tag{3}$$

Obviously, if we take in (3) $h(\lambda)=\lambda$ then it reduces to the definition of classical convex stochastic process (1). If we take in (3) $h(\lambda)=\lambda^s$, $s \in (0,1]$, then we have the concept of s-convexity of stochastic processes, which was studied by Maden *et al.* (2015). Moreover, a stochastic process X is called Godunova-Levin stochastic processes and *P*-Stochastic processes if we take $h(\lambda)=\frac{1}{\lambda}$ and $h(\lambda)=1$ in (3), respectively.

The concept of *h*-convexity of stochastic processes, which was introduced by Barraez *et al.* in 2015 (Barraez *et al.* 2015) have been studied by many researchers (Li & Hao 2017; Budak & Sarikaya 2019). Recently, Zhou *et al.* (2020) studied the generalization of *h*-convex stochastic process and provided some inequalities related to Jensen, Hermite-Hadamard, Fajer and Ostrowski types.

Now, we state some results proved by Kotrys (2013) and Barraez *et al*. (2015) about Hermite-Hadamard inequality for convex stochastic processes and *h*-convex stochastic processes.

Theorem 2.4. (Kotrys 2013) *Let $X: I \times \Omega \to \mathbb{R}$ be a convex stochastic process and mean-square continuous in I. Then for every $u, v \in I$, $u < v$, the inequality*

$$X\left(\frac{u+v}{2},\cdot\right) \leq \frac{1}{v-u}\int_u^v X(s,\cdot)ds \leq \frac{X(u,\cdot)+X(v,\cdot)}{2} \tag{4}$$

is satisfied almost everywhere.

Theorem 2.5. (Barraez *et al*. 2015) *Let $h:(0,1) \to \mathbb{R}$, $h \neq 0$ be a non-negative function and $X: I \times \Omega \to \mathbb{R}$ a non-negative, h-convex, mean-square integrable stochastic process. Then for every $u, v \in I$, $u < v$, the following inequality is satisfied almost everywhere,*

$$\frac{1}{2h\left(\frac{1}{2}\right)} X\left(\frac{u+v}{2},\cdot\right) \leq \frac{1}{v-u}\int_u^v X(s,\cdot)ds \leq \left(X(u,\cdot)+X(v,\cdot)\right)\int_0^1 h(z)dz\,. \tag{5}$$

Motivated by (2) and (3), in this study, we propose a new type of stochastic convexity, which is called strongly *h*-convex stochastic processes. The definition is given as follows.

Definition 2.6. Let $C: \Omega \to \mathbb{R}$ denotes a positive random variable and $h:(0,1) \to \mathbb{R}$, $h \neq 0$ a non-negative function. We say that a stochastic process $X: I \times \Omega \to \mathbb{R}$ is a strongly *h*-convex stochastic process with modulus $C(\cdot)$ if for every $u, v \in I$, $u < v$ and $\lambda \in [0,1]$, the following inequality is satisfied,

$$X\left(\lambda u + (1-\lambda)v,\cdot\right) \leq h(\lambda)X\left(u,\cdot\right) + h(1-\lambda)X\left(v,\cdot\right) - C(\cdot)\lambda(1-\lambda)(u-v)^2\,. \tag{6}$$

Note that, by omitting the term $C(\cdot)\lambda(1-\lambda)(u-v)^2$ in the cases (2) and (6), we immediately get the definition of convex function and *h*-convex stochastic processes, respectively.

The aim of this paper is to provide a new inequality for Hermite-Hadamard type for strongly *h*-convex stochastic process with modulus $C(\cdot)$. We also give some properties of *h*-convex stochastic processes that obtained from the main result.

3. Main Results

First, we present the following result for strongly *h*-convex stochastic processes related to Hermite-Hadamard type inequality.

Theorem 3.1. *Let* $h:(0,1)\to\mathbb{R}$, $h\neq 0$ *be a non-negative function and* $X: I\times\Omega\to\mathbb{R}$ *be a non-negative, strongly h-convex with modulus* $C(\cdot)>0$, *mean square integrable stochastic process. For every* $a,b\in I$ *with* $a<b$, *we have the following inequality*

$$\frac{1}{2h(\frac{1}{2})}\left[X\left(\frac{a+b}{2},\cdot\right)+\frac{C(\cdot)}{12}(a-b)^2\right]\leq\frac{1}{b-a}\int_a^b X(x,\cdot)dx$$
$$\leq\left[X(a,\cdot)+X(b,\cdot)\right]\int_0^1 h(t)dt-\frac{C(\cdot)}{6}(b-a)^2. \qquad (7)$$

Proof. Take $u=za+(1-z)b$ and $v=(1-z)a+zb$ for the fixed $a,b\in I$ with $a<b$, we get that $a+b=u+v$. Then, the strong *h*-convexity stochastic process of *X* with $C(\cdot)>0$ implies

$$X\left(\frac{a+b}{2},\cdot\right)=X\left(\frac{u+v}{2},\cdot\right)$$
$$\leq h\left(\frac{1}{2}\right)\left[X(u,\cdot)+X(v,\cdot)\right]-\frac{C(\cdot)}{4}(b-a)^2$$
$$=h\left(\frac{1}{2}\right)\left[X(ta+(1-t)b,\cdot)+X((1-t)a+tb,\cdot)\right]-\frac{C(\cdot)}{4}(1-2t)^2\left(a-b\right)^2.$$

Integrating the above inequality with respect to t, we obtain

$$X\left(\frac{a+b}{2},\cdot\right)\leq h\left(\frac{1}{2}\right)\left[\int_0^1 X(ta+(1-t)b,\cdot)dt+\int_0^1 X((1-t)a+tb,\cdot)dt\right]$$
$$-\frac{C(\cdot)}{4}\left(a-b\right)^2\int_0^1(1-2t)^2dt$$
$$=h\left(\frac{1}{2}\right)\left(\frac{2}{b-a}\right)\int_a^b X(x,\cdot)dx-\frac{C(\cdot)}{12}\left(a-b\right)^2$$

which gives the left-hand side of inequalities (7). Also, from the equality that

$$\frac{1}{b-a}\int_a^b X(x,\cdot)dx=\int_0^1 X\left((1-t)a+tb,\cdot\right)dt,$$

then we have

$$\frac{1}{b-a}\int_a^b X(x,\cdot)dx \le X(a,\cdot)\int_0^1 h(1-t)dt + X(b,\cdot)\int_0^1 h(t)dt$$
$$- C(\cdot)(b-a)^2\int_0^1 t(1-t)dt$$
$$= \left[X(a,\cdot)+X(b,\cdot)\right]\int_0^1 h(t)dt - \frac{C(\cdot)}{6}(b-a)^2.$$

Thus, we get the second inequalities (7), which completes the proof of Theorem 3.1. □

Remark 3.2

(1) In the case of $C(\cdot)=0$, the inequalities (7) coincide with the Hermite-Hadamard-type inequalities for *h*-convex stochastic processes function, which was proved by Barraez *et al*. (2015).

(2) If $h(t)=t,\ t\in(0,1)$, then inequalities (7) reduce to

$$X\left(\frac{a+b}{2},\cdot\right)+\frac{C(\cdot)}{12}\left(a-b\right)^2 \le \frac{1}{b-a}\int_a^b X(x,\cdot)dx$$
$$\le \frac{X(a,\cdot)+X(b,\cdot)}{2} - \frac{C(\cdot)}{6}(b-a)^2.$$

These Hermite-Hadamard-type inequalities for strongly convex stochastic processes functions have been proved by Kotrys (2013).

(3) If $h(t)=1$ and $h(t)=1/t,\ t\in(0,1)$ in (7), then we have the inequalities of Hermite-Hadamard-type for strongly *P*-convex stochastic processes and strongly Godunova-Levin stochastic processes, respectively.

(4) If $h(t)=t^s,\ t\in(0,1), s\in[0,1]$, then we get the inequalities of Hermite-Hadamard-type for strongly *P*-convex stochastic processes, i.e.,

$$2^{s-1}\left[X\left(\frac{a+b}{2},\cdot\right)+\frac{C(\cdot)}{12}\left(a-b\right)^2\right] \le \frac{1}{b-a}\int_a^b X(x,\cdot)dx$$
$$\le \frac{X(a,\cdot)+X(b,\cdot)}{s+1} - \frac{C(\cdot)}{6}(b-a)^2.$$

Set *et al*. (2014) have provided the inequalities of Hermite-Hadamard-type for *s*-convex stochastic processes.

We also provide the properties for the stochastic processes $H(t),F(t):[0,1]\times\Omega\to\mathbb{R}$, which are stated as follows.

Theorem 3.3 *Let* $h:(0.1)\to\mathbb{R}$ *be a non-negative function. Suppose that* $X:I\times\Omega\to\mathbb{R}$ *stochastic process is strongly h-convex and integrable on* $[u,v]\times\Omega,\ [u,v]\subseteq I$ *. Consider the stochastic processes* $H(t,\cdot),F(t,\cdot):[0,1]\times\Omega\to\mathbb{R}_+$ *defined by*

$$H(\alpha,\cdot)=\frac{1}{v-u}\int_u^v X\left(\alpha t+(1-\alpha)\frac{u+v}{2},\cdot\right)dt \tag{8}$$

and

$$F(\alpha,\cdot)=\frac{1}{(v-u)^2}\int_u^v\int_u^v X\left(\alpha s+(1-\alpha)t,\cdot\right)dsdt \tag{9}$$

where $\alpha\in[0,1]$.

(i) *H and F are strongly h-convex stochastic processes on* $[0,1]\times\Omega$.

(ii) *We have the inequalities*

$$H(t,\cdot)\geq\frac{1}{2h(\frac{1}{2})}\left[X\left(\tfrac{u+v}{2},\cdot\right)+\tfrac{C(\cdot)}{12}(v-u)^2\right] \tag{10}$$

and

$$\begin{aligned}&2h(\tfrac{1}{2})F(\alpha,\cdot)-\frac{C(\cdot)}{24}(2\alpha-1)^2(v-u)^2\\&\qquad\geq F(\tfrac{1}{2},\cdot)=\frac{1}{(v-u)^2}\int_u^v\int_u^v X\left(\tfrac{s+t}{2},\cdot\right)dsdt.\end{aligned} \tag{11}$$

Proof. (i) Let $t\in[0,1]$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$. We have

$$\begin{aligned}H(\alpha x+\beta y,\cdot)&=\frac{1}{v-u}\int_u^v X\left((\alpha x+\beta y)t+(1-(\alpha x+\beta y))\frac{u+v}{2},\cdot\right)dt\\&=\frac{1}{v-u}\int_u^v X\left(\alpha[xt+(1-x)\frac{u+v}{2}]+\beta[yt+(1-y)\frac{u+v}{2}],\cdot\right)dt\\&\leq\frac{1}{v-u}h(\alpha)\int_u^v X\left(xt+(1-x)\frac{u+v}{2},\cdot\right)dt\\&\quad+\frac{1}{v-u}h(\beta)\int_u^v X\left(yt+(1-y)\frac{u+v}{2},\cdot\right)dt-\frac{1}{v-u}C(\cdot)\alpha\beta(b-a)^2\\&=h(\alpha)H(x,\cdot)+h(\beta)H(y,\cdot)-C(\cdot)\alpha\beta(b-a)^2.\end{aligned}$$

We also have

$$\begin{aligned}&F(\alpha x+\beta y,\cdot)\\&\quad=\frac{1}{(v-u)^2}\int_u^v\int_u^v X\left(\alpha(xs+(1-x)t+\beta(ys+(1-y)t),\cdot\right)dsdt\end{aligned}$$

$$\leq \frac{1}{(v-u)^2}\int_u^v\int_u^v \begin{bmatrix} h(\alpha)X\left(xs+(1-x)t,\cdot\right)+h(\beta)X\left(s+(1-y)t),\cdot\right) \\ -C(\cdot)\alpha\beta(b-a)^2 \end{bmatrix} dsdt$$
$$= h(\alpha)F(x,\cdot)+h(\beta)F(y,\cdot)-C(\cdot)\alpha\beta(b-a)^2.$$

These show that H and F are strongly h-convex stochastic processes.

(ii) Suppose that $t\in(0,1]$, then change the variable in (7) by $\theta=\alpha t+(1-\alpha)\frac{u+v}{2}$ gives us that

$$H(t,\cdot)=\frac{1}{\alpha(v-u)}\int_{\alpha u+(1-\alpha)\frac{u+v}{2}}^{\alpha v+(1-\alpha)\frac{u+v}{2}} X\left(\theta,\cdot\right)d\theta$$
$$=\frac{1}{q-p}\int_p^q X\left(\theta,\cdot\right)d\theta$$

where $p=\alpha u+(1-\alpha)\frac{u+v}{2}$ and $q=\alpha v+(1-\alpha)\frac{u+v}{2}$. Now, applying the left hand side of Hermite-Hadamard-type inequalities for strongly h-convex stochastic processes (7), we have

$$\frac{1}{2h(\frac{1}{2})}\left[X\left(\tfrac{p+q}{2},\cdot\right)+\frac{C(\cdot)}{12}\left(q-p\right)^2\right]\leq\frac{1}{q-p}\int_p^q X(\theta,\cdot)d\theta=H(\alpha,\cdot)\cdot$$

Thus, we obtain the desire result (10).

By the fact that X is strongly h-convex stochastic processes, we have

$$X\left(\frac{u+v}{2},\cdot\right)\leq h(\tfrac{1}{2})\left[X(u,\cdot)+X(v,\cdot)\right]-\tfrac{C(\cdot)}{4}(v-u)^2$$
$$X\left(\tfrac{tp+(1-t)q+(1-t)p+tq}{2},\cdot\right)\leq h(\tfrac{1}{2})\left[X(tp+(1-t)q,\cdot)+X\left((1-t)p+tq,\cdot\right)\right]-\tfrac{C(\cdot)}{4}(v-u)^2$$
$$X\left(\tfrac{p+q}{2},\cdot\right)\leq h(\tfrac{1}{2})\left[X(tp+(1-t)q,\cdot)+X(\left(1-t\right)p+tq,\cdot)\right]-\tfrac{C(\cdot)}{4}(2t-1)^2(q-p)^2$$

where $u=tp+(1-t)q$ and $v=\left(1-t\right)p+tq$. Integrating the above inequality on $[a,b]^2$, we have

$$\int_a^b\int_a^b X\left(\tfrac{p+q}{2},\cdot\right)dpdq\leq h(\tfrac{1}{2})\int_a^b\int_a^b\left[X(tp+(1-t)q,\cdot)+X(\left(1-t\right)p+tq,\cdot)\right]dpdq$$
$$-\int_a^b\int_a^b\tfrac{C(\cdot)}{4}(2t-1)^2(q-p)^2dpdq$$
$$=2h(\tfrac{1}{2})\int_a^b\int_a^b X(tp+(1-t)q,\cdot)\,dpdq$$
$$-\int_a^b\int_a^b\tfrac{C(\cdot)}{4}(2t-1)^2(q-p)^2dpdq$$

$$\frac{1}{(b-a)^2}\int_a^b\int_a^b X\left(\tfrac{p+q}{2},\cdot\right)dpdq \le 2h(\tfrac{1}{2})\frac{1}{(b-a)^2}\left[\int_a^b\int_a^b X(tp+(1-t)q,\cdot)\,dpdq\right.$$
$$\left. -\frac{C(\cdot)}{4(b-a)^2}(2t-1)^2\int_a^b\int_a^b (q-p)^2 dpdq\right]$$
$$= 2h(\tfrac{1}{2})F(t,\cdot)-\frac{C(\cdot)}{4(b-a)^2}(2t-1)^2\frac{(b-a)^4}{6}$$
$$= 2h(\tfrac{1}{2})F(t,\cdot)-\frac{C(\cdot)}{24}(2t-1)^2(b-a)^2$$

$$\frac{1}{(v-u)^2}\int_u^v\int_u^v X\left(\tfrac{s+t}{2},\cdot\right)dsdt = F\left(\frac{1}{2},\cdot\right) \le 2h(\tfrac{1}{2})F(\alpha,\cdot)-\frac{C(\cdot)}{24}(2\alpha-1)^2(v-u)^2.$$

The proof of Theorem 3.3 is complete. □

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