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NOTES ON MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS DEFINED BY BESSEL FUNCTION

(Catatan Berkenaan Fungsi Meromorfi Berpekali Positif yang Ditakrif oleh Fungsi Bessel)

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ABSTRACT

In this article, a new subclass of univalent meromorphic under Bessel mapping function is defined. Some properties such as coefficient estimates, starlikeness and convexity radii, and extreme points are derived. Finally, partial sums and neighbourhood properties are given.

Keywords: meromorphic; extreme point; partial sums; neighbourhood

ABSTRAK

Dalam kertas kerja ini, suatu subkelas fungsi meromorfi univalen baharu dibawah pemetaan Bessel ditakrif. Beberapa sifat seperti anggaran pekali, jejari kebakbintangan dan kecembungan, dan titik ekstrim diperoleh. Akhir sekali, hasil tambah separa dan sifat kejiranan diberi.

Kata kunci: meromorfi; titik ekstrim; hasil tambah separa; kejiranan

1. Introduction

Let Ω signify the type's meromorphic mapping class

$$\vartheta(\o) = \frac{1}{\o} + \sum_{\ell=1}^{\infty} \varrho_\ell \o^\ell \tag{1}$$

which are defined in the punctured disc

$$\Delta^* = \{\o : \o \in \mathbb{C} \text{ and } 0 < |\o| < 1\} = \Delta \setminus \{0\},$$

are holomorphic. Also, let Ω_p indicate the subclass of Ω of mapping of Eq. (1) with $\varrho_\ell \geq 0$.

A mapping $\vartheta \in \Omega$ is known to be meromorphic starshaped of order ϖ if it fulfils

$$\Re\left\{-\frac{\o\vartheta'(\o)}{\vartheta(\o)}\right\} > \varpi$$

for some ϖ, $(0 \leq \varpi < 1)$ and for all $\o \in \Delta^*$. Further, a mapping $\vartheta \in \Omega$ is known to be meromorphically convex of order ϖ if it fulfils

$$\Re\left\{-1 - \frac{\o\vartheta''(\o)}{\vartheta'(\o)}\right\} > \varpi$$

for some ϖ, $(0 \leq \varpi < 1)$ and for all $\o \in \Delta^*$.

Pommerenke (1963), Miller (1970), Mogra *et al.* (1985), Cho (1990), Cho *et al.* (2003), Aouf (1989, 1991), and Venkateswarlu *et al.* (2019) have described and examined some subclasses of Ω.

We remember the first-order generalised Bessel mapping γ (see Deniz *et al.* (2011)), which is symbolised by

$$w(\o) = \sum_{\ell=0}^{\infty} \frac{(-\iota)^{\ell}}{\ell!\Gamma(\gamma+\ell+\frac{b+1}{2})}\left(\frac{\o}{2}\right)^{2\ell+\gamma} \quad (\o \in \Delta),$$

the Euler Gamma mapping is represented as Γ. This is the second-order linear homogeneous differential equation's solution,

$$\o^2 w''(\o) + b\o w'(\o) + [\iota\o^2 - \gamma^2 + (1-b)\gamma]w(\o) = 0,$$

where $\iota, \gamma, b \in \mathbb{R}^+$. For more information, see Watson (1994). In general, when it comes to the Bessel mapping w, the mapping is added φ as

$$\varphi(\o) = 2^{\gamma}\Gamma\left(\gamma + \frac{b+1}{2}\right)\o^{\ell},$$

by using Pochhammer symbol for the Euler gamma mapping $(a)_\mu$ exact for $a \in \mathbb{C}$ as

$$(\xi)_\mu = \frac{\Gamma(\xi+\mu)}{\Gamma(\xi)} = \begin{cases} 1, & (\mu = 0); \\ \xi(\xi+1)(\xi+2)\cdots(\xi+\ell-1), & (\mu = \ell \in \{1,2,3\cdots\} = \mathbb{N}). \end{cases}$$

The mapping $\varphi(\o)$, is represented belows,

$$\varphi(\o) = \frac{1}{\o} + \sum_{\ell=0}^{\infty} \frac{(-\iota)^{\ell+1}}{4^{\ell+1}(\ell+1)!(\tau)_{\ell+1}}\o^{\ell}\left(\tau = \gamma + \frac{b+1}{2} \notin \mathbb{Z}_0^- = \{0,-1,-2,\cdots\}\right).$$

For $\vartheta \in \Omega$ indicate by Eq. (1) and $g \in \Omega$,

$$g(\o) = \frac{1}{\o} + \sum_{\ell=1}^{\infty} b_\ell \o^{\ell},$$

we've described the Hadamard product of ϑ and g by

$$(\vartheta * g)(\o) = \frac{1}{\o} + \sum_{\ell=1}^{\infty} \varrho_\ell b_\ell \o^{\ell}.$$

The Bessel operator $\mathscr{S}\tau^{\ \iota}$ by the convolution that corresponds to the mapping φ is defined by

$$\begin{aligned} \mathscr{S}_{\tau}{}^{\iota}\vartheta(\o) = (\varphi * \vartheta)(\o) &= \frac{1}{\o} + \sum_{\ell=0}^{\infty} \frac{\left(\frac{-\iota}{4}\right)^{\ell+1}\varrho_\ell}{(\ell+1)!(\tau)_{\ell+1}}\o^{\ell} \\ &= \frac{1}{\o} + \sum_{\ell=1}^{\infty} \phi(\ell,\tau,\iota)\varrho_\ell \o^{\ell}, \end{aligned} \tag{2}$$

where $\phi(\ell,\tau,\iota) = \frac{\left(\frac{-\iota}{4}\right)^{\ell}}{(\ell)!(\tau)_\ell}$.

Its easy to check from Eq. (2) that

$$\o[\mathscr{S}_{\tau+1}{}^{\iota}\vartheta(\o)] = \tau\mathscr{S}_{\tau}{}^{\iota}\vartheta(\o) - (\tau+1)\mathscr{S}_{\tau+1}{}^{\iota}\vartheta(\o). \tag{3}$$

We are going to create subclasses $\sigma_p^*(\aleph, \wp, \tau,\ \iota)$ of Ω_p based on Sivaprasad *et al.* (2005), Atshan *et al.* (2007) and Venkateswarlu *et al.* (2019).

Definition 1.1. Let $\sigma_p^*(\aleph, \wp, \tau, \iota)$ be the subclass of Ω_p comprising of mapping of the type Eq. (1) and fulfilling

$$-\Re\left(\frac{\o(\mathscr{S}_\tau{}^\iota\vartheta(\o))'}{\mathscr{S}_\tau{}^\iota\vartheta(\o)}+\aleph\right) > \wp\left|\frac{\o(\mathscr{S}_\tau{}^\iota\vartheta(\o))'}{\mathscr{S}_\tau{}^\iota\vartheta(\o)}+1\right|. \tag{4}$$

Lemma 1.2. *To demonstrate our arguments, we must use the associated lemmas introduced by Aqlan et al. (2004). Let $\aleph \in \mathbb{R}$ and $d \in \mathbb{C}$. Then*

(i) $\Re(d) \geq \aleph \Leftrightarrow |d+1-\aleph| - |d-1+\aleph| \geq 0.$

(ii) $-\Re(d) \geq \wp|d+1| + \aleph \Leftrightarrow -\Re\left[d(1+\wp e^{i\theta}) + \wp e^{i\theta}\right] \geq \aleph, -\pi \leq \theta \leq \pi.$

The primary objective of this research is to analyze some objects or items mapping theory characteristics for the class $\sigma_p^*(\aleph, \wp, \tau, \iota)$.

2. Coefficient estimates

In this section, we establish the necessary and sufficient conditions for a mapping $\vartheta \in \sigma_p^*(\aleph, \wp, \tau, \iota)$.

Theorem 2.1. *Let $\vartheta \in \Omega_p$ be indicate by Eq. (1). Then $\vartheta \in \sigma_p^*(\aleph, \wp, \tau, \iota)$*

$$\Leftrightarrow \sum_{\ell=1}^{\infty}[\ell(1+\wp) + (\wp+\aleph)]\phi(\ell,\tau,\iota)\, \varrho_\ell \leq 1-\aleph. \tag{5}$$

Proof. Let $\vartheta \in \sigma_p^*(\aleph, \wp, \tau, \iota)$. Then by Definition 1.1 and applying Lemma 1.2 (ii), it should suffice to say that

$$-\Re\left\{\left(\frac{\o(\mathscr{S}_\tau{}^\iota\vartheta(\o))'}{\mathscr{S}_\tau{}^\iota\vartheta(\o)}\right)(1+\wp e^{i\theta}) + \wp e^{i\theta}\right\} > \aleph. \tag{6}$$

For simply, let

$$C(\o) = -\left[\o(\mathscr{S}_\tau{}^\iota\vartheta(\o))'\right](1+\wp e^{i\theta}) - \wp e^{i\theta}\mathscr{S}_\tau{}^\iota\vartheta(\o),$$
$$D(\o) = \mathscr{S}_\tau{}^\iota\vartheta(\o).$$

Then Eq. (6) is equivalent to

$$-\Re\left(\frac{C(\o)}{D(\o)}\right) \geq \aleph.$$

In view of Lemma 1.2 (i), we have

$$|C(\o) + (1-\aleph)D(\o)| - |C(\o) - (1-\aleph)D(\o)| \geq 0.$$

Now

$$|C(\o) + (1-\aleph)D(\o)| \geq (2-\aleph)|\o|^{-1} - \sum_{\ell=1}^{\infty}[\ell(1+\wp) + (\aleph+\wp-1)]\phi(\ell,\tau,\iota)\, \varrho_\ell|\o|^\ell,$$

and

$$|C(\o) - (1-\aleph)D(\o)| \leq \aleph|\o|^{-1} + \sum_{\ell=1}^{\infty}[\ell(1+\wp) \ + (\aleph+\wp+1)]\phi(\ell,\tau,\iota)\ \varrho_\ell|\o|^\ell.$$

It's indicate that

$$\begin{aligned}&|C(\o)+(1-\aleph)D(\o)| - |C(\o)-(1+\aleph)D(\o)|\\ \geq& 2(1-\aleph)|\o|^{-1} - 2\sum_{\ell=1}^{\infty}[\ell(1+\wp) \ + (\wp+\aleph)]\phi(\ell,\tau,\iota)\ \varrho_\ell|\o|^\ell\\ \geq& 0, \text{ by the indicate condition Eq. (5).}\end{aligned}$$

On the other hand $\vartheta \in \sigma_p^*(\aleph,\wp,\tau,\iota)$. Then by Lemma 1.2 (i), we have Eq. (6).

Selecting the values of $\o$ on the positive x-axis the inequality Eq. (6) decreases to

$$\Re\left\{\frac{(1-\aleph\ )\o^{-1} - \sum\limits_{\ell=1}^{\infty}[\ell\ (1+\wp e^{i\theta}) + (\aleph+\wp e^{i\theta})]\phi(\ell,\tau,\iota)\varrho_\ell\o^\ell}{\o^{-1}+\sum\limits_{\ell=1}^{\infty}\phi(\ell,\tau,\iota)\varrho_\ell\o^\ell}\right\} \geq 0.$$

Since $\Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$, the aforementioned difference is reduced to

$$\Re\left\{\frac{1-\aleph-\sum\limits_{\ell=1}^{\infty}[\ell(1+\wp) \ + (\wp+\aleph)]\phi(\ell,\tau,\iota)\varrho_\ell r^{\ell+1}}{1+\sum\limits_{\ell=1}^{\infty}\phi(\ell,\tau,\iota)\ \varrho_\ell r^{\ell+1}}\right\} \geq 0.$$

Letting $r \to 1^-$, we've found the disparity Eq. (5). □

Corollary 2.2. *If* $\vartheta \in \sigma_p^*(\aleph,\wp,\tau,\iota)$*, then*

$$\varrho_\ell \leq \frac{1-\aleph}{[n(1+\wp) \ + (\wp+\aleph)]\phi(n,\tau,\iota)}. \tag{7}$$

Theorem 2.3. *If* $\vartheta \in \sigma_p^*(\aleph,\wp,\tau,\ \iota)$*, then for* $0<|\o|=r<1$,

$$\frac{1}{r} - \frac{1-\aleph}{(2\wp+\aleph+1)\phi(1,\tau,\iota)}\ r \leq |\vartheta(\o)| \leq \frac{1}{r} + \frac{1-\aleph}{(2\wp+\aleph+1)\phi(1,\tau,\iota)}\ r. \tag{8}$$

The mapping has produced an accurate findings

$$\vartheta(\o) = \frac{1}{\o} + \frac{1-\aleph}{(2\wp+\aleph+1)\phi(1,\tau,\iota)}\ \o. \tag{9}$$

Proof. Since $\vartheta(\o) = \frac{1}{\o} + \sum\limits_{\ell=1}^{\infty}\varrho_\ell\o^\ell$, we have

$$|\vartheta(\o)| = \frac{1}{r} + \sum_{\ell=1}^{\infty}\varrho_\ell r^\ell \leq \frac{1}{r} + r\sum_{\ell=1}^{\infty}\varrho_\ell. \tag{10}$$

Since

$$(2\wp+\aleph+1)\phi(1,\tau,\iota)\leq[\ell(1+\wp)+(\aleph+\wp)]\phi(\ell,\tau,\iota),\ \ell\geq 1,$$

applying Theorem 2.1, we have

$$\begin{aligned}(2\wp+\aleph+1)\phi(1,\tau,\iota)\sum_{\ell=1}^{\infty}\varrho_\ell &\leq \sum_{\ell=1}^{\infty}[\ell(\wp+1)\ +(\wp+\aleph)]\phi(\ell,\tau,\iota)\varrho_\ell\\ &\leq 1-\aleph\\ \Rightarrow \sum_{\ell=1}^{\infty}\varrho_\ell &\leq \frac{1-\aleph}{(2\wp+\aleph+1)\phi(1,\tau,\iota)}.\end{aligned}$$

From Eq. (10), we have

$$|\vartheta(\o)|\leq\frac{1}{r}+\frac{1-\aleph}{(2\wp+\aleph+1)\phi(1,\tau,\iota)}\,r$$

and

$$|\vartheta(\o)|\geq\frac{1}{r}-\frac{1-\aleph}{(2\wp+\aleph+1)\phi(1,\tau,\iota)}\,r.$$

The mapping has produced an accurate findings $\vartheta(\o)=\frac{1}{\o}+\frac{1-\aleph}{(2\wp+\aleph+1)\phi(1,\tau,\iota)}\,\o$. □

Corollary 2.4. *If* $\vartheta\in\sigma_p^*(\aleph,\wp,\tau,\iota)$ *then*

$$\frac{1}{r^2}-\frac{1-\aleph}{(2\wp+\aleph+1)\phi(1,\tau,\iota)}\ \leq|\vartheta'(\o)|\leq\frac{1}{r^2}+\frac{1-\aleph}{(2\wp+\aleph+1)\phi(1,\tau,\iota)}$$

and Eq. (9) is sharp.

3. Extreme points

Theorem 3.1. *Let* $\vartheta_0(\o)=\frac{1}{\o}$ *and*

$$\vartheta_\ell(\o)=\frac{1}{\o}+\sum_{\ell=1}^{\infty}\frac{1-\aleph}{[\ell(\wp+1)\ +(\wp+\aleph)]\phi(\ell,\tau,\iota)}\,\o^\ell,\ \ \ell\geq 1. \tag{11}$$

Then $\vartheta\in\sigma_p^*(\aleph,\wp,\tau,\ \iota)\Leftrightarrow$ *it can be written in the type*

$$\vartheta(\o)=\sum_{\ell=0}^{\infty}v_\ell\vartheta_\ell(\o);v_\ell\geq 0\ \text{and}\ \sum_{\ell=0}^{\infty}v_\ell=1. \tag{12}$$

Proof. Suppose $\vartheta(\o)$ can be written as in Eq. (12). Then

$$\begin{aligned}\vartheta(\o)=&\sum_{\ell=0}^{\infty}v_\ell\vartheta_\ell(\o)=v_0\vartheta_0(\o)+\sum_{\ell=1}^{\infty}v_\ell\vartheta_\ell(\o)\\ =&\frac{1}{\o}+\sum_{\ell=1}^{\infty}v_\ell\,\frac{1-\aleph}{[\ell(\wp+1)\ +(\wp+\aleph)]\phi(\ell,\tau,\iota)}\,\o^\ell.\end{aligned}$$

Therefore

$$\sum_{\ell=1}^{\infty} v_\ell \frac{1-\aleph}{[\ell(\wp+1)+(\wp+\aleph)]\phi(\ell,\tau,\iota)} \frac{[\ell(\wp+1)+(\wp+\aleph)]\phi(\ell,\tau,\iota)}{1-\aleph} \o^\ell$$
$$=\sum_{\ell=1}^{\infty} v_\ell = 1 - v_0 \leq 1.$$

So from Theorem 2.1, $\vartheta \in \sigma_p^*(\aleph,\wp,\tau,\iota)$.

Conversely, $\vartheta \in \sigma_p^*(\aleph,\wp,\tau,\iota)$. Since

$$\varrho_\ell \leq \frac{1-\aleph}{[\ell(\wp+1)+(\wp+\aleph)]\phi(\ell,\tau,\iota)}\ ,\ \ell \geq 1.$$

We set

$$v_\ell = \frac{[\ell(1+\wp)+(\aleph+\wp)]\phi(\ell,\tau,\iota)}{1-\aleph}\ \varrho_\ell,\ \ell \geq 1 \text{ and } v_0 = 1 - \sum_{\ell=1}^{\infty} v_\ell.$$

Then we have

$$\vartheta(\o) = \sum_{\ell=0}^{\infty} v_\ell \vartheta_\ell(\o) = v_0\vartheta_0(\o) + \sum_{\ell=1}^{\infty} v_\ell\vartheta_\ell(\o). \quad \square$$

4. Properties of Radii

Theorem 4.1. *Let $\vartheta \in \sigma_p^*(\aleph,\wp,\tau,\iota)$. Then ϑ is meromorphically starshaped of order $\hbar$, $(0 \leq \hbar < 1)$ in the unit disc $|\o| < r_1$, where*

$$r_1 = \inf_\ell \left[\frac{(1-\hbar)[n(1+\wp)\ +(\aleph+\wp)]\phi(\ell,\tau,\iota)}{(1-\aleph\ )(\ell+2-\hbar)}\right]^{\frac{1}{\ell+1}},\ \ \ell \geq 1.$$

The mapping has produced an accurate findings $\vartheta(\o)$ indicated by Eq. (11).

Proof. The mapping $\vartheta \in \sigma_p^*(\aleph,\wp,\tau,\iota)$ of the type Eq. (1) is meromorphically starshaped of order $\hbar$ in the disc $|\o| < r_1 \Leftrightarrow$ it fulfils the condition

$$\left|\frac{\o\vartheta'(\o)}{\vartheta(\o)} + 1\right| < 1 - \hbar. \tag{13}$$

Since

$$\left|\frac{\o\vartheta'(\o)}{\vartheta(\o)} + 1\right| \leq \left|\frac{\sum\limits_{\ell=1}^{\infty}(\ell+1)\varrho_\ell \o^{\ell+1}}{1+\sum\limits_{\ell=1}^{\infty}\varrho_\ell \o^{\ell+1}}\right| \leq \frac{\sum\limits_{\ell=1}^{\infty}(\ell+1)\varrho_\ell|\o|^{\ell+1}}{1-\sum\limits_{\ell=1}^{\infty}\varrho_\ell|\o|^{\ell+1}}.$$

The calculation above is less than $(1-\hbar)$ if $\sum\limits_{\ell=1}^{\infty} \frac{(\ell+2-\hbar)}{(1-\hbar)}\varrho_\ell|\o|^{\ell+1} < 1$.

Making use of the fact that $\vartheta(\o) \in \sigma_p^*(\aleph, \wp, \tau, \iota) \Leftrightarrow$

$$\sum_{\ell=1}^{\infty} \frac{[\ell(1+\wp)\ +(\aleph+\wp)]\phi(\ell,\tau,\iota)}{1-\aleph}\ \varrho_\ell \leq 1.$$

Thus, Eq. (13) will be true if

$$\frac{\ell+2-\hbar}{1-\hbar}|\o|^{\ell+1} < \frac{[\ell(1+\wp)\ +(\aleph+\wp)]\phi(\ell,\tau,\iota)}{1-\aleph},$$

or equivalently

$$|\o|^{\ell+1} < \frac{(1-\hbar)[\ell(1+\wp)\ +(\aleph+\wp)]\phi(\ell,\tau,\iota)}{(1-\aleph\ )(\ell+2-\hbar)},$$

which gives the starshaped of the family. □

Theorem 4.2. *Let* $\vartheta \in \sigma_p^*(\aleph, \wp, \tau, \iota)$. *Then* ϑ *is meromorphically convex of order* $\hbar$, $(0 \leq \hbar < 1)$ *in the unit disc* $|\o| < r_2$, *where*

$$r_2 = \inf_{\ell}\left[\frac{(1-\hbar)[\ell(1+\wp)+(\aleph+\wp)]\phi(\ell,\tau,\iota)}{\ell(1-\aleph\ )(\ell+2-\hbar)}\right]^{\frac{1}{\ell+1}},\ \ \ell \geq 1$$

and Eq. (11) is sharp.

Because the theorem's evidence is similar to that of Theorem 4.1, we will skip the proof of Theorem 4.2

5. Partial Sums

Let $\vartheta \in\ \Omega_p$ be a maping of the type Eq. (1). The partial sums are defined as ϑ_ρ described by Silverman (1997), Silvia (1985) and Aouf Aouf *et al.* (2006)

$$\vartheta_\rho(\o) = \frac{1}{\o} + \sum_{\ell=1}^{\rho} \varrho_\ell \o^\ell, (\rho \in \mathbb{N}). \tag{14}$$

We'll look at partial sums of mapping from the class $\sigma_p^*(\aleph, \wp, \tau, \iota)$ in this part and find lower bounds for the real part of the ratios ϑ to ϑ_ρ and ϑ' to ϑ'_ρ.

Theorem 5.1. *Let* $\vartheta \in \sigma_p^*(\aleph, \wp, \tau, \iota)$ *be provided by Eq. (1) and indicate the partial sums* $\vartheta_1(\o)$ *and* $\vartheta_\rho(\o)$ *by*

$$\vartheta_1(\o) = \frac{1}{\o}\ \textit{and}\ \vartheta_\rho(\o) = \frac{1}{\o} + \sum_{\ell=1}^{\rho} \varrho_\ell \o^\ell, (\rho \in \mathbb{N} \setminus \{1\}). \tag{15}$$

Suppose also that $\sum\limits_{\ell=1}^{\infty} t_\ell \varrho_\ell \leq 1$, *where*

$$t_\ell \geq \begin{cases} 1\ , & \textit{if}\ \ \ell = 1, 2, \cdots, \rho \\ \frac{[\ell(1+\wp)\ +(\aleph+\wp)]\phi(\ell,\tau,\iota)}{(1-\aleph)}, & \textit{if}\ \ \ell = \rho+1, \rho+2, \cdots \end{cases}. \tag{16}$$

Then $\vartheta \in \sigma_p^*(\aleph, \wp, \tau, \iota)$. *Moreover*

$$\Re\left(\frac{\vartheta(\emptyset)}{\vartheta_\rho(\emptyset)}\right) > 1 - \frac{1}{t_{\rho+1}} \tag{17}$$

and

$$\Re\left(\frac{\vartheta_\rho(\emptyset)}{\vartheta(\emptyset)}\right) > \frac{t_{\rho+1}}{1+t_{\rho+1}}. \tag{18}$$

Proof. For the coefficient t_ℓ indicate by Eq. (16), it's not difficult to establish this

$$t_{\rho+1} > t_\rho > 1. \tag{19}$$

Therefore, we have

$$\sum_{\ell=1}^{\rho} \varrho_\ell + t_{\rho+1} \sum_{\ell=\rho+1}^{\infty} \varrho_\ell \leq \sum_{\ell=1}^{\infty} \varrho_\ell t_\rho \leq 1, \tag{20}$$

by using the hypothesis Eq. (16). By forming

$$\chi_1(\emptyset) = t_{\rho+1}\left(\frac{\vartheta(\emptyset)}{\vartheta_\rho(\emptyset)} - \left(1 - \frac{1}{t_{\rho+1}}\right)\right) = 1 + \frac{t_{\rho+1} \sum\limits_{\ell=\rho+1}^{\infty} \varrho_\ell \emptyset^{\ell-1}}{1 + \sum\limits_{\ell=1}^{\infty} \varrho_\ell \emptyset^{\ell-1}},$$

then only demonstrating that is sufficient

$$\Re\big(\chi_1(\emptyset)\big) \geq 0, (\emptyset \in \Delta^*) \text{ or } \left|\frac{\chi_1(\emptyset)-1}{\chi_1(\emptyset)+1}\right| \leq 1, \ (\emptyset \in \Delta^*),$$

and applying Eq. (20), we find that

$$\begin{aligned}\left|\frac{\chi_1(\emptyset)-1}{\chi_1(\emptyset)+1}\right| &\leq \frac{t_{\rho+1} \sum\limits_{\ell=\rho+1}^{\infty} \varrho_\ell}{2 - 2\sum\limits_{\ell=1}^{\rho} \varrho_\ell - t_{\rho+1} \sum\limits_{\ell=\rho+1}^{\infty} \varrho_\ell} \\ &\leq 1,\end{aligned}$$

which gives the assertion Eq. (17) of Theorem 5.1. In order to notice this

$$\vartheta(\emptyset) = \frac{1}{\emptyset} + \frac{\emptyset^{\rho+1}}{t_{\rho+1}}, \tag{21}$$

gives sharp result for

$$\emptyset = re^{\frac{i\pi}{\rho}} \text{ that } \frac{\vartheta(\emptyset)}{\vartheta_\rho(\emptyset)} = 1 - \frac{r^{\rho+2}}{t_{\rho+1}} \to 1 - \frac{1}{t_{\rho+1}} \text{ as } r \to 1^-.$$

Similarly, if we takes $\chi_2(\emptyset) = (1+t_{\rho+1})\left(\frac{\vartheta_\rho(\emptyset)}{\vartheta(\emptyset)} - \frac{t_{\rho+1}}{1+t_{\rho+1}}\right)$ and making use of Eq. (20), we indicate that

$$\left|\frac{\chi_2(\emptyset)-1}{\chi_2(\emptyset)+1}\right| < \frac{(1+t_{\rho+1}) \sum\limits_{\ell=\rho+1}^{\infty} \varrho_\ell}{2 - 2\sum\limits_{\ell=1}^{\rho} \varrho_\ell - (1-t_{\rho+1}) \sum\limits_{\ell=\rho+1}^{\infty} \varrho_\ell},$$

which yields the assertion Eq. (18) of Theorem 5.1.
The bound in Eq. (18) is sharp for each $\rho \in \mathbb{N}$ with extremal mapping $\vartheta(\emptyset)$ provided by Eq. (21). □

Theorem 5.2. *If $\vartheta \in \sigma_p^*(\aleph, \wp, \tau, \ \iota)$ be provided by Eq. (1) and satisfies the condition Eq. (5) then*

$$\Re\left(\frac{\vartheta'(\emptyset)}{\vartheta'_\rho(\emptyset)}\right) > 1 - \frac{\rho+1}{t_{\rho+1}},$$

and

$$\Re\left(\frac{\vartheta'_\rho(\emptyset)}{\vartheta'(\emptyset)}\right) > \frac{t_{\rho+1}}{\rho+1+t_{\rho+1}},$$

where

$$t_\ell \geq \begin{cases} \ell\,, & \text{if } \ell = 2, 3, \cdots, \rho \\ \frac{[\ell(1+\wp)+(\aleph+\wp)]\phi(\ell,\tau,\iota)}{1-\aleph}, & \text{if } \ell = \rho+1, \rho+2, \cdots \end{cases}.$$

The mapping has produced an accurate findings $\vartheta(\emptyset)$ of the type Eq. (7).

Because the theorem's argument is similar to that of Theorem 5.1, we will skip the proof of Theorem 5.2.

6. Neighborhoods property

The definition of the neighbourhood for the class $\sigma_p^{*\xi}(\aleph, \wp, \tau, \iota)$ is as follows.

Definition 6.1. A mapping $\vartheta \in \Omega_p$ is said to be in the class $\sigma_p^{*\xi}(\aleph, \wp, \tau, \iota)$ if there exits a mapping $g \in \sigma_p^*(\aleph, \wp, \tau, \ \iota)$ such that

$$\left|\frac{\vartheta(\emptyset)}{g(\emptyset)} - 1\right| < 1 - \xi, \ \ (\emptyset \in \Delta, \ 0 \leq \xi < 1). \tag{22}$$

We define the $\hbar-$ neighbourhoods of mapping $\vartheta \in \Omega_p$ by following Goodman (1957) and Ruscheweyh (1981) earlier works on neighbourhoods of analytic mapping.

$$N_\hbar(\vartheta) = \Big\{g \in \Omega_p : g(\emptyset) = \frac{1}{\emptyset} + \sum_{\ell=1}^{\infty} b_\ell \emptyset^\ell \ \ and \ \ \sum_{\ell=1}^{\infty} \ell|\varrho_\ell - b_\ell| \leq \hbar\Big\}. \tag{23}$$

Theorem 6.2. *If $g \in \sigma_p^*(\aleph, \wp, \tau, \ \iota)$ and*

$$\xi = 1 - \frac{\hbar(2\wp+\aleph+1)\phi(1,\tau,\iota)}{(2\wp+\aleph+1)\phi(1,\tau,\iota) - (1-\aleph)}, \tag{24}$$

then $N_\hbar(g) \subset \sigma_p^{*\xi}(\aleph, \wp, \tau, \iota)$.

Proof. Let $\vartheta \in N_{\hbar}(g)$. Then we find based on Eq. (23) that

$$\sum_{\ell=1}^{\infty} \ell|\varrho_\ell - b_\ell| \leq \hbar \Rightarrow \sum_{\ell=1}^{\infty} |\varrho_\ell - b_\ell| \leq \hbar.$$

Since $g \in \sigma_p^*(\aleph, \wp, \tau, \iota)$, we've

$$\sum_{\ell=1}^{\infty} b_\ell \leq \frac{1-\aleph}{(2\wp+\aleph+1)\phi(1,\iota,\tau)}. \tag{25}$$

Now

$$\begin{aligned} \left|\frac{\vartheta(\emptyset)}{g(\emptyset)} - 1\right| &< \frac{\sum\limits_{\ell=1}^{\infty} |\varrho_\ell - b_\ell|}{1 - \sum\limits_{\ell=1}^{\infty} b_\ell} \\ &\leq \frac{\hbar(2\wp+\aleph+1)\phi(1,\iota,\tau)}{(2\wp+\aleph+1)\phi(1,\tau,\iota)-(1-\aleph)} \\ &= 1-\xi \end{aligned}$$

provided ξ is indicated by Eq. (24). Hence by definition, $\vartheta \in \sigma_p^{*\xi}(\aleph, \wp, \tau, \ \iota)$ for ξ provided by which completes the proof. □

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