

## A CLASS OF TWO-SEX POPULATION QUADRATIC STOCHASTIC OPERATORS: *b*-BISTOCHASTIC-VOLTERRA

(Kelas Operator Kuadratik Stokastik bagi Populasi Dua Jantina : *b*-Bistokastik-Volterra)

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### ABSTRACT

This paper aims to investigate the simplest non-linear Markov operators which is the quadratic. Study of quadratic stochastic operators (QSOs) is not an easy task as linear operators. Thus, researchers introduced classes of QSOs such as Volterra QSOs, strictly non-Volterra QSOs, Orthogonal preserving QSOs, Centered QSOs etc. However, all the introduced classes were not yet cover the whole set of QSOs. Therefore, we introduce a new class of QSOs, namely *b*-bistochastic-Volterra QSOs or simply *bV*-QSOs. In this paper, we describe the canonical form of *bV*-QSO on two-dimensional simplex. This helps understanding the dynamical behaviours of *bV*-QSOs for future works.

**Keywords:** quadratic operator; Markov operator; *b*-bistochastic; Volterra; two-sex population; bisexual population

### ABSTRAK

Fokus kajian ini adalah untuk menyiasat pengendali Markov bukan linear yang paling mudah iaitu kuadratik. Kajian tentang operator kuadratik stokastik (QSO) bukan satu tugas mudah seperti operator linear. Oleh itu, penyelidik memperkenalkan beberapa kelas QSO seperti QSO Volterra, QSO bukan Volterra, QSO mengawet ortogon, QSO berpusat dan lain-lain. Walau bagaimanapun, semua kelas tersebut tidak meliputi keseluruhan set QSO. Oleh itu, kami memperkenalkan satu kelas baru QSO iaitu *b*-bistokastik QSO Volterra atau secara ringkas *bV*-QSO. Dalam kajian ini, kami memperkenalkan bentuk berkanun *bV*-QSO yang ditakrifkan pada simpleks dua dimensi. Hal ini membantu memahami tingkah laku dinamik *bV*-QSO bagi kerja masa hadapan.

**Kata kunci:** operator kuadratik; operator Markov; *b*-bistokastik; Volterra; populasi dua jantina; populasi dwi-jantina

### References

- Abdulghafor R., Abdullah S.S., Turaev S., Zeki A. & Al-Shaikhli I. 2020. Linear and nonlinear stochastic distribution for consensus problem in multi-agent systems. *Neural Computing and Applications* **32**: 261–277.
- Abdurakhimova S.B. & Rozikov U.A. 2022. Dynamical system of a quadratic stochastic operator with two discontinuity points. *Mathematical Notes* **111**(5): 676–687.
- Badocha M. & Bartoszek W. 2018. Quadratic stochastic operators on Banach lattices. *Positivity* **22**(2): 477–492.
- Bernstein S.N. 1924. Solution of a mathematical problem related to the theory of inheritance. *Uch. Zap. n.-i. kaf. Ukrainy* **1**: 83–115.
- Boxonov Z. 2023. On dynamical systems of quadratic stochastic operators constructed for bisexual populations. *arXiv preprint arXiv:2303.03884*.
- Castanos O., Jamilov U.U. & Rozikov U.A. 2018. On Volterra quadratic stochastic operators of a two-sex population on  $S^1 \times S^1$ . *Uzbek Mathematical Journal* **2018**(3): 37–50.
- Chye N.N.L.B., Lim M.H. & Embong A.F. 2021. The fixed points of *b*-bistochastic-Volterra quadratic stochastic operators on  $S^1 \times S^1$ . *Applied Mathematics and Computational Intelligence* **10**(1): 340–350.

- Dzhumadil'daev A., Omirov B.A. & Rozikov U.A. 2016. Constrained evolution algebras and dynamical systems of a bisexual population. *Linear Algebra and its Applications* **496**: 351–380.
- Embong A.F. & Chye@ Mohd Hairie Lim N.N.L.B. 2022. Regularity of  $b$ -bistochastic-Volterra quadratic stochastic operators. *AIP Conference Proceedings*, p. 020014.
- Embong A.F. & Mukhamedov F. 2023. Lyapunov functions and dynamics of infinite dimensional Volterra operators. *Chaos, Solitons & Fractals* **173**: 113625.
- Embong A.F., Zulkifly M.I.E. & Arifin D.N.A.A. 2023. Genetic algebras generated by  $b$ -bistochastic quadratic stochastic operators: The character and associativity. *Malays. J. Math. Sci* **17**: 25–41.
- Eshmatov F.F., Jamilov U.U. & Khudoyberdiev K.O. 2023. Discrete time dynamics of a sird reinfection model. *International Journal of Biomathematics* **16**(05): 2250104.
- Ganikhodjaev N.N., Ganikhodjaev R.N. & Jamilov U.U. 2015. Quadratic stochastic operators and zero-sum game dynamics. *Ergodic Theory and Dynamical Systems* **35**(5): 1443–1473.
- Ganikhodjaev N.N. & Jamilov U.U. 2015. Contracting quadratic operators of bisexual population. *Applied Mathematics & Information Sciences* **9**(5): 2645–2650.
- Ganikhodjaev N.N., Jamilov U.U. & Ladra M. 2021. Evolutionary dynamics of zero-sum games with degenerate payoff matrix and bisexual population. *Discontinuity, Nonlinearity, and Complexity* **10**(1): 43–60.
- Ganikhodzhaev N., Zhamilov U. & Mukhitdinov R. 2014. Nonergodic quadratic operators for a two-sex population. *Ukrainian Mathematical Journal* **65**(8): 1282–1291.
- Ganikhodzhaev R., Mukhamedov F. & Rozikov U. 2011. Quadratic stochastic operators and processes: results and open problems. *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **14**(02): 279–335.
- Ganikhodzhaev R.N. 1993a. On the definition of bistochastic quadratic operators. *Russian Mathematical Surveys* **48**(4): 244–246.
- Ganikhodzhaev R.N. 1993b. Quadratic stochastic operators, Lyapunov functions, and tournaments. *Sbornik: Mathematics* **76**(2): 489–506.
- Hamzah N.Z.A., Karim S.N., Selvarajoo M. & Sahabudin N.A. 2022. Dynamics of nonlinear operator generated by lebesgue quadratic stochastic operator with exponential measure. *Mathematics and Statistics* **10**(4): 861–867.
- Hardy G.H., Littlewood J.E. & Polya G. 1952. *Inequalities*. London: Cambridge University Press.
- Jamilov U., Mukhamedov F. & Mukhamedova F. 2023. Discrete time model of sexual systems. *Heliyon* **9**(7): e17913.
- Jamilov U.U., Khudoyberdiev K.O. & Ladra M. 2020. Quadratic operators corresponding to permutations. *Stochastic Analysis and Applications* **38**(5): 929–938.
- Kesten H. 1970. Quadratic transformations: a model for population growth. I. *Advances in Applied Probability* **2**(1): 1–82.
- Mukhamedov F. & Embong A.F. 2015. On  $b$ -bistochastic quadratic stochastic operators. *Journal of Inequalities and Applications* **2015**: 226.
- Mukhamedov F. & Embong A.F. 2016a.  $b$ -bistochastic quadratic stochastic operators and their properties. *Journal of Physics: Conference Series*, p. 012010.
- Mukhamedov F. & Embong A.F. 2016b. On mixing of Markov measures associated with  $b$ -bistochastic QSOs. *AIP Conference Proceedings*, p. 020090.
- Mukhamedov F. & Embong A.F. 2017. Extremity of  $b$ -bistochastic quadratic stochastic operators on 2d simplex. *Malaysian Journal of Mathematical Sciences* **11**(2): 119–139.
- Mukhamedov F. & Embong A.F. 2018. On stable  $b$ -bistochastic quadratic stochastic operators and associated non-homogenous Markov chains. *Linear and Multilinear Algebra* **66**(1): 1–21.
- Mukhamedov F. & Fadillah Embong A. 2021. Infinite dimensional orthogonality preserving nonlinear Markov operators. *Linear and Multilinear Algebra* **69**(3): 526–550.
- Mukhamedov F. & Ganikhodjaev N. 2015. *Quantum Quadratic Operators and Processes*. Berlin: Springer.
- Parker D.S. & Ram P. 1996. Greed and majorization. *Tech. Rep.*. University of California.
- Rozikov U. & Xudayarov S. 2022. Quadratic non-stochastic operators: examples of splitted chaos. *Annals of Functional Analysis* **13**(1): 17.
- Rozikov U.A. & Zhamilov U.U. 2011. Volterra quadratic stochastic operators of a two-sex population. *Ukrainian Mathematical Journal* **63**(7): 1136–1153.

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