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EXPLORING SIXTH-ORDER COMPACT FINITE DIFFERENCE SCHEMES FOR HYPERBOLIC PDE: A CASE STUDY ON THE NONLINEAR GOURSAT PROBLEM

(Meneroka Skema Beza Terhingga Padat Tertib Keenam untuk Persamaan Pembezaan Separa Hiperbolik: Kajian Kes Mengenai Masalah Goursat Tak Linear)

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ABSTRACT

This study underscores the significant potential of using sixth-order compact finite difference method to address the nonlinear Goursat problem, a major challenge in mathematical modeling. This approach paves the way for advancement in numerical studies, particularly through the linearization of the nonlinear Goursat problem, which facilitates the implementation of linear scheme. The research employs numerical analysis to assess the accuracy of this new scheme and discusses the theoretical outcomes of multiple series of numerical experiments. Through error analysis, the Von Neumann method, and Taylor series expansion, the results demonstrate that the proposed method achieves superior accuracy compared to traditional approaches and is unconditionally stable and consistent. Additionally, the study confirms the convergence of the new scheme through the Lax equivalence theorem. Moreover, the research highlights the cost-effectiveness of this new method, showcasing its ability to achieve high levels of accuracy with fewer computational resources. By thoroughly examining the underlying principles and real-world applications, the significant benefits of utilizing advanced high-order compact finite difference methods are emphasized.

Keywords: higher-order compact finite difference method; nonlinear Goursat problem; hyperbolic partial differential equation; accuracy; convergence

ABSTRAK

Kajian ini menekankan potensi besar penggunaan kaedah beza terhingga padat tertib keenam untuk menangani masalah Goursat tak linear, yang merupakan cabaran utama dalam pemodelan matematik. Pendekatan ini membuka jalan untuk kemajuan dalam kajian berangka, terutamanya melalui proses linearisasi masalah Goursat tak linear yang memudahkan pelaksanaan skema linear. Penyelidikan ini menggunakan analisis berangka untuk menilai ketepatan skema baru ini dan membincangkan hasil teori daripada pelbagai siri eksperimen berangka. Melalui analisis ralat, kaedah Von Neumann, dan pengembangan siri Taylor, hasil menunjukkan bahawa kaedah yang dicadangkan mencapai ketepatan yang lebih tinggi berbanding pendekatan tradisional dan bersifat stabil tanpa syarat serta konsisten. Tambahan pula, kajian ini mengesahkan penumpuan skema baru melalui teorem kesetaraan Lax. Selain itu, penyelidikan ini menekankan kos keberkesanan kaedah baru ini, dengan menunjukkan keupayaannya untuk mencapai tahap ketepatan yang tinggi dengan sumber pengiraan yang lebih sedikit. Dengan meneliti prinsip asas dan aplikasi dunia sebenar, manfaat penting penggunaan kaedah perbezaan terhingga padat tertib tinggi yang maju turut ditekankan.

Kata kunci: kaedah beza terhingga padat tertib tinggi; masalah Goursat tak linear; persamaan pembezaan separa hiperbolik; ketepatan; penumpuan

1. Introduction

The Goursat problem pertains to initial value problems involving second-order hyperbolic partial differential equations (PDE). The general form of the Goursat problem can be expressed as (Wazwaz 2009):

$$\begin{aligned} &u_{xy} = f\left(x, y, u, u_x, u_y\right), \\ &u(x,0) = g(x), \quad u(0,y) = m(y), \\ &g(0) = m(0) = u(0,0), \\ &0 \le x \le a, \quad 0 \le y \le b, \end{aligned} \tag{1}$$

where u_{xy} is the mixed derivative in space *x* any *y* while the term $f\left(x, y, u, u_x, u_y\right)$ is a function of independent variables *x* and *y*, dependent variable *u*, and the derivative terms u_x and u_y.

The Goursat problem arises in various areas of science and technology. Researchers (Son & Thao 2019; Tian *et al*. 2020; Mokdad 2022) have explored its applications in economic dynamics, global optimal scheduling, geoscience, biomedical engineering, and even in the context of Nordström-like black holes.

Several techniques have been proposed to address the Goursat problem, including Newton-Cotes Integration (Deraman & Nasir 2015), reduction differential transforms (Iftikhar *et al.* 2022), the fuzzy transform (Saharizan & Zamri 2019; Kim Son *et al*. 2021), iterative regularization (Meziani *et al*. 2021), Signature Kernel (Salvi *et al*. 2021), method of transmutation operators (Sitnik & Karimov 2023; Karimov & Yulbarsov 2023), and Taylor collocation (Birem *et al*. 2024). Among these, the finite difference method (FDM) is considered the most suitable for solving hyperbolic PDE due to its simplicity, high accuracy, and strong theoretical foundation (Nasir & Md Ismail 2013). The established FDM scheme is given by:

$$\frac{u_{i+1,j+1} + u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = \frac{1}{4}\left(f_{i+1,j+1} + f_{i,j} + f_{i+1,j} + f_{i,j+1}\right), \tag{2}$$

where *h* denotes the grid size (used to set the width, height, and dimensions of the grid in determining its initial size).

Previously, the FDM has been employed (Nasir & Md Ismail 2012; 2013; Pandey 2014a; 2014b) to solve second-order linear and nonlinear hyperbolic Goursat PDE. Enhancing and deepening the understanding of these schemes will improve the mathematical modeling of problems where the Goursat problem is relevant. High-order compact finite difference schemes, which utilize a compact stencil (Nasir & Md Ismail 2013), have demonstrated higher accuracy at grid points. However, numerical schemes for solving nonlinear Goursat problems face challenges in linearizing approximate numerical solutions. This motivates the development of a more accurate method that can convert to a linear Goursat scheme. Consequently, a new approach, the sixth-order compact finite difference method, has been introduced to effectively solve homogeneous nonlinear Goursat PDE problems.

The primary objective of this paper is to explore the numerical analysis and theoretical aspects of this new scheme for solving standard nonlinear problems. Beyond focusing on accuracy, the study will also address stability using Von Neumann stability analysis, consistency through Taylor series, and convergence by applying the Lax equivalence theorem. These theoretical results will be validated with data from computational experiments using Maple 18 and MATLAB 2014 software.

The proposed scheme will be detailed in Section 2 and Section 3 will apply the standard nonlinear Goursat problems to investigate both accuracy and theoretical aspects. Finally, the numerical and theoretical results of the new scheme will be presented, with conclusions drawn in the last section.

2. Sixth-Order Compact Finite Difference Scheme

The compact finite difference method is a fundamental formula within FDM for approximating higher-order derivatives of a function. The second-order partial derivatives using FDM are derived from the Taylor series expansion (up to the seventh term) and can be extended to sixth-order as follows:

$$
\begin{aligned}
u_{xy} = \frac{1}{4h^2}\begin{bmatrix} u(x+h,y+h)-u(x+h,y-h) \\ -u(x-h,y+h)+u(x-h,y-h) \end{bmatrix} - \frac{2}{12}h^2\left(u_{xxxy}+u_{xyyy}\right) \\
-\frac{3}{360}h^4\left(u_{xxxxxy}+u_{xyyyyy}\right)-\frac{1}{36}h^4u\left(x,y\right).
\end{aligned} \tag{3}
$$

known as compact difference formulas for $\partial^2 u/\partial x\partial y$ at (x,y) with the truncation error $O\left(3h^8/4h^2\right)$ and step size, h.

The next step is to adopt difference formulas (3) into left-hand side of the standard form of Goursat problem (1). Hence, the derivation of Goursat schemes can be written as follows (Deraman *et al*. 2024):

$$
\begin{aligned}
\frac{1}{4h^2}\begin{bmatrix} u(x+h,y+h)-u(x+h,y-h) \\ -u(x-h,y+h)+u(x-h,y-h) \end{bmatrix} - \frac{2}{12}h^2\left(u_{xxxy}+u_{xyyy}\right) \\
-\frac{3}{360}h^4\left(u_{xxxxxy}+u_{xyyyyy}\right)-\frac{1}{36h^4}u\left(x,y\right)+O\left(\frac{3h^8}{4h^2}\right) = f(x,y,u,u_x,u_y),
\end{aligned} \tag{4}
$$

where u_{xxxy} ,u_{xyyy}, u_{xxxxxy} and u_{xyyyyy} can be implemented by differentiating general (1) to be f_{xx} , f_{yy} , f_{xxxx} and f_{yyyy} respectively.

In the next section, the approximation (4) will be tested into standard nonlinear Goursat PDE problem.

3. Numerical Experiment

Consider the following nonlinear homogeneous Goursat problem

$$
\begin{aligned}
&u_{xy} = e^{2u}, \\
&u(x,0) = \frac{x}{2} - \ln\left(1+e^x\right), \\
&u(0,y) = \frac{y}{2} - \ln\left(1+e^y\right), \\
&0 \le x \le 1,\ 0 \le y \le 1.
\end{aligned} \tag{5}
$$

The exact solution to the problem stated in Eq. (5) is $u(x,y)=\frac{x+y}{2}-\ln\left(e^{x}+e^{y}\right)$ by Pandey 2014a. By rearranging and applying scheme (4) into problem (5), it can be rewritten as:

$$\begin{aligned}
&u(x+h,y+h)=u(x+h,y-h)+u(x-h,y+h)-u(x-h,y-h)\\
&+4h^{2}\left[\frac{2}{12}h\left(u_{xxxy}+u_{xyyy}\right)+\frac{1}{120}h^{4}\left(u_{xxxxxy}+u_{xyyyyy}\right)+\frac{1}{36}h^{4}\left(u_{xxxyyy}\right)\right]\\
&+4h^{2}e^{2u(x,y)}.
\end{aligned}\tag{6}$$

Then, differentiating Eq. (5) with respect to *x* yields

$$\begin{aligned}
u_{xy}&=e^{2u},\\
u_{xxy}&=\frac{\partial}{\partial x}u_{xy}=\frac{\partial}{\partial x}e^{2u}=2\left(\frac{\partial}{\partial x}u\right)e^{2u}=2e^{2u}u_{x},\\
u_{xxxy}&=\frac{\partial}{\partial x}u_{xxy}=\frac{\partial}{\partial x}\left(2e^{2u}u_{x}\right)=2\left(\frac{\partial}{\partial x}e^{2u}u_{x}\right)=2e^{2u}\left[\left(u_{xx}\right)+2\left(u_{x}\right)^{2}\right],\\
u_{xxxxxy}&=\frac{\partial}{\partial x}u_{xxxy}=\frac{\partial}{\partial x}2e^{2u}\left[\left(u_{xx}\right)+2\left(u_{x}\right)^{2}\right]=2\frac{\partial}{\partial x}\left\{e^{2u}\left[\left(u_{xx}\right)+2\left(u_{x}\right)^{2}\right]\right\}\\
&=2e^{2u}\left[u_{xxxx}+8u_{xxx}u_{x}+6\left(u_{xx}\right)^{2}+24u_{xx}\left(u_{x}\right)^{2}+8\left(u_{x}\right)^{4}\right].
\end{aligned}\tag{7}$$

In a like manner,

$$\begin{aligned}
u_{xy}&=e^{2u},\\
u_{xyy}&=\frac{\partial}{\partial y}u_{xy}=\frac{\partial}{\partial y}e^{2u}=2\left(\frac{\partial}{\partial y}u\right)e^{2u}=2e^{2u}u_{y},\\
u_{xyyy}&=\frac{\partial}{\partial y}u_{xyy}=\frac{\partial}{\partial y}\left(2e^{2u}u_{y}\right)=2\left(\frac{\partial}{\partial y}e^{2u}u_{y}\right)=2e^{2u}\left[\left(u_{yy}\right)+2\left(u_{y}\right)^{2}\right],\\
u_{xyyyyy}&=\frac{\partial}{\partial y}u_{xyyy}=\frac{\partial}{\partial y}2e^{2u}\left[\left(u_{yy}\right)+2\left(u_{y}\right)^{2}\right]=2\frac{\partial}{\partial y}\left\{e^{2u}\left[\left(u_{yy}\right)+2\left(u_{y}\right)^{2}\right]\right\}\\
&=2e^{2u}\left[u_{yyyy}+8u_{yyy}u_{y}+6\left(u_{yy}\right)^{2}+24u_{yy}\left(u_{y}\right)^{2}+8\left(u_{y}\right)^{4}\right].
\end{aligned}\tag{8}$$

Therefore,

$$\begin{aligned}
u_{xxxy}+u_{xyyy}&=2e^{2u}\left[\left(u_{xx}\right)+2\left(u_{x}\right)^{2}\right]+2e^{2u}\left[\left(u_{yy}\right)+2\left(u_{y}\right)^{2}\right]\\
&=2e^{2u}\left\{\left(u_{xx}+u_{yy}\right)+2\left[\left(u_{x}\right)^{2}+\left(u_{y}\right)^{2}\right]\right\},
\end{aligned}$$

$$
\begin{aligned}
u_{xxxxxy} + u_{xyyyyy} &= 2e^{2u}\left[u_{xxxx} + 8u_{xxx}u_x + 6\left(u_{xx}\right)^2 + 24u_{xx}\left(u_x\right)^2 + 8\left(u_x\right)^4\right] \\
&\quad + 2e^{2u}\left[u_{yyyy} + 8u_{yyy}u_y + 6\left(u_{yy}\right)^2 + 24u_{yy}\left(u_y\right)^2 + 8\left(u_y\right)^4\right] \\
&= 2e^{2u}\left\{\begin{array}{l}\left(u_{xxxx} + u_{yyyy}\right) + 8\left(u_{xxx}u_x + u_{yyy}u_y\right) + 6\left[\left(u_{xx}\right)^2 + \left(u_{yy}\right)^2\right] \\ +24\left[u_{xx}\left(u_x\right)^2 + u_{yy}\left(u_y\right)^2\right] + 8\left(u_x{}^4 + u_y{}^4\right)\end{array}\right\}, \\
u_{xxxyyy} &= e^{2u}.
\end{aligned} \tag{9}
$$

The approximation of $u\left(x+h, y+h\right)$ is obtained by substituting terms in Eq. (9) into Eq. (6) as:

$$
\begin{aligned}
u\left(x+h,y+h\right) &= u\left(x+h,y-h\right) + u\left(x-h,y+h\right) - u\left(x-h,y-h\right) \\
&\quad + 4h^3 e^{2u(x,y)}\left[\begin{array}{l}\dfrac{4}{12}\left\{\left(u_{xx}+u_{yy}\right) + 2\left[\left(u_x\right)^2 + \left(u_y\right)^2\right]\right\} \\ + \dfrac{2}{120}h^3\left\{\begin{array}{l}\left(u_{xxxx}+u_{yyyy}\right) + 8\left(u_{xxx}u_x + u_{yyy}u_y\right) \\ +6\left[\left(u_{xx}\right)^2 + \left(u_{yy}\right)^2\right] \\ +24\left[u_{xx}\left(u_x\right)^2 + u_{yy}\left(u_y\right)^2\right] \\ +8\left(u_x{}^4 + u_y{}^4\right)\end{array}\right\} + \dfrac{1}{36}h^3\end{array}\right] + 4h^2 e^{2u(x,y)}.
\end{aligned} \tag{10}
$$

Summing of the Taylor series expansions, we get

$$
\begin{aligned}
2h^2\left(u_{xx}+u_{yy}\right) &= u\left(x+h,y+h\right) + u\left(x+h,y-h\right) + u\left(x-h,y+h\right) \\
&\quad + u\left(x-h,y-h\right) - 4u\left(x,y\right) + O\left(h\right)^4, \\
4hu_x &= u\left(x+h,y+h\right) + u\left(x+h,y-h\right) - u\left(x-h,y+h\right) \\
&\quad - u\left(x-h,y-h\right) + O\left(h\right)^3, \\
4hu_y &= u\left(x+h,y+h\right) - u\left(x+h,y-h\right) + u\left(x-h,y+h\right) \\
&\quad - u\left(x-h,y-h\right) + O\left(h\right)^3, \\
\frac{h^4}{6}\left(u_{xxxx}+u_{yyyy}\right) &= u\left(x+h,y+h\right) + u\left(x+h,y-h\right) \\
&\quad + u\left(x-h,y+h\right) + u\left(x-h,y-h\right) + 4u\left(x,y\right) \\
&\quad - h^4u\left(x,y\right) - 2u\left(x-h,y\right) - 2u\left(x,y-h\right) \\
&\quad - 2u\left(x+h,y\right) - 2u\left(x,y+h\right) + O\left(h\right)^6, \\
\frac{2h^3}{3}\left(u_{xxx}\right) &= u\left(x+h,y+h\right) + u\left(x+h,y-h\right) - u\left(x-h,y+h\right) \\
&\quad - u\left(x-h,y-h\right) + 2u\left(x-h,y\right) + h^2u\left(x,y-h\right) \\
&\quad - 2u\left(x+h,y\right) - h^2u\left(x,y+h\right) + O\left(h\right)^5,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\frac{2h^3}{3}\left(u_{yyy}\right) &= u\left(x+h,y+h\right)-u\left(x+h,y-h\right)+u\left(x-h,y+h\right)\\
&\quad -u\left(x-h,y-h\right)+2u\left(x,y-h\right)+h^2u\left(x-h,y\right)\\
&\quad -2u\left(x,y+h\right)-h^2u\left(x+h,y\right)+O\left(h\right)^5,\\
h^2u_{xx} &= u\left(x-h,y\right)+u\left(x+h,y\right)-2u\left(x,y\right)+O\left(h\right)^4,\\
h^2u_{yy} &= u\left(x,y-h\right)+u\left(x,y+h\right)-2u\left(x,y\right)+O\left(h\right)^4.
\end{aligned}
$$

The new scheme for the Goursat PDE problem Eq. (5) can be generated by substituting terms in Eq. (11) into Eq. (10). The indexing form is as follows:

$$
\begin{aligned}
&u_{i+2,j+2} = \\
&\frac{1}{6}\left[3h^2e^{2u_{i+1,j+1}} \left(\begin{aligned}
&-12\left(u_{i,j+1}\right)^2-\left(u_{i,j+1}\right)^4-8\left(u_{i,j+1}\right)^2-\left(u_{i+1,j}\right)^4-12\left(u_{i+1,j}\right)^3-8\left(u_{i+1,j}\right)^2\\
&-\left(u_{i+2,j+1}\right)^4-12\left(u_{i+2,j+1}\right)^3-8\left(u_{i+2,j+1}\right)^2-12\left(u_{i+1,j+2}\right)^3-\left(u_{i+1,j+2}\right)^4\\
&-8\left(u_{i+1,j+2}\right)^2-32\left(u_{i+2,j}\right)-32\left(u_{i+2,j}\right)-32\left(u_{i,j}\right)+32\left(u_{i+1,j+1}\right)+24\left(u_{i,j+1}\right)\\
&+24\left(u_{i+1,j}\right)+24\left(u_{i+2,j+1}\right)+24\left(u_{i+1,j+2}\right)-96\left(u_{i+1,j+1}\right)^2+12h^4\left(u_{i+1,j+1}\right)\\
&-12\left(u_{i,j}\right)\left(u_{i,j+1}\right)+24\left(u_{i+1,j+1}\right)\left(u_{i,j+1}\right)^2+48\left(u_{i+1,j+1}\right)\left(u_{i,j+1}\right)\\
&+12\left(u_{i+2,j}\right)\left(u_{i,j+1}\right)-12\left(u_{i,j+2}\right)\left(u_{i,j+1}\right)-12\left(u_{i,j}\right)\left(u_{i+1,j}\right)\\
&+24\left(u_{i+1,j+1}\right)\left(u_{i+1,j}\right)^2+48\left(u_{i+1,j+1}\right)\left(u_{i+1,j}\right)-12\left(u_{i+2,j}\right)\left(u_{i+1,j}\right)\\
&+12\left(u_{i,j+2}\right)\left(u_{i+1,j}\right)+12\left(u_{i+1,j+1}\right)\left(u_{i,j}\right)+24\left(u_{i+1,j+1}\right)\left(u_{i+1,j+1}\right)^2\\
&+48\left(u_{i+1,j+1}\right)\left(u_{i+2,j+1}\right)+4\left(u_{i+2,j+1}\right)\left(u_{i,j+1}\right)^3+4\left(u_{i+2,j+1}\right)^3\left(u_{i,j+1}\right)\\
&-6\left(u_{i+2,j+1}\right)^2\left(u_{i,j+1}\right)^2+12\left(u_{i+2,j+1}\right)^2\left(u_{i,j+1}\right)+12\left(u_{i+2,j+1}\right)\left(u_{i,j+1}\right)^2\\
&-32\left(u_{i,j+1}\right)\left(u_{i+2,j+1}\right)-12\left(u_{i+2,j+1}\right)\left(u_{i+2,j}\right)+12\left(u_{i+2,j+1}\right)\left(u_{i,j+2}\right)\\
&-12\left(u_{i,j+2}\right)\left(u_{i+1,j+2}\right)+12\left(u_{i,j}\right)\left(u_{i+1,j+2}\right)+48\left(u_{i+1,j+1}\right)\left(u_{i+1,j+2}\right)\\
&+24\left(u_{i+1,j+1}\right)\left(u_{i+1,j+2}\right)^2+4\left(u_{i+1,j}\right)^3\left(u_{i+1,j+2}\right)-6\left(u_{i+1,j}\right)^2\left(u_{i+1,j+2}\right)^2\\
&+4\left(u_{i+1,j}\right)\left(u_{i+1,j+2}\right)^3+12\left(u_{i+1,j}\right)^2\left(u_{i+1,j+2}\right)+12\left(u_{i+1,j}\right)\left(u_{i+1,j+2}\right)^2\\
&-32\left(u_{i+1,j}\right)\left(u_{i+1,j+2}\right)+12\left(u_{i+2,j}\right)\left(u_{i+1,j+2}\right)-24h^2\left(u_{i,j+1}\right)\left(u_{i+1,j+2}\right)\\
&-48\left(u_{i+1,j+1}\right)\left(u_{i+2,j+1}\right)\left(u_{i,j+1}\right)-48\left(u_{i+1,j+1}\right)\left(u_{i+1,j}\right)\left(u_{i+1,j+2}\right)\\
&+24h^2\left(u_{i,j+1}\right)\left(u_{i+1,j}\right)+24h^2\left(u_{i+2,j+1}\right)\left(u_{i+1,j+2}\right)-24h^2\left(u_{i+2,j+1}\right)\left(u_{i+1,j}\right)
\end{aligned}\right)\right] \qquad (12)\\
&\left[-10h^6\left(u_{i+1,j+1}\right)-90\left(u_{i+2,j}\right)-90\left(u_{i,j+2}\right)+90\left(u_{i,j}\right)\right]\\
&\left[-15+2h^2e^{2u_{i+1,j+1}}\left(3u_{i+1,j+1}-3u_{i,j+1}-3u_{i+1,j}+3u_{i+1,j+2}+8\right)\right]^{-1},
\end{aligned}
$$

where

$i = 0,1,2, ...,(c/k-2)$ and $j = 0,1,2, ...,(d/h-2)$. The parameter *c*, *d*, *k* and *h* are the range of *x*, *y*, the grid size for the domain *x* and the grid size for the domain *y*, respectively. Likewise, the starting points of the scheme were acquired through the utilization of the established method (2).

Thus, the nonlinear problem (5) performs linear scheme as (12), it is proven the new scheme is achieve linearization.

4. Numerical Result

The results presented below are approximation numerical solutions and average relative errors at several selected grid points for problem (5) involving standard scheme (2) versus new scheme (12).

Table 1: Approximate numerical solution at *h* = 0.125

	$u(x,y)$			
Scheme	*u*(0.25,0.25)	*u*(0.5,0.5)	*u*(0.75,0.75)	*u*(1,1)
Exact [8]	-6.9314e-01	-6.9314e-01	-6.9314e-01	-6.9314e-01
Standard (2)	-6.9316e-01	-6.9322e-01	-6.9332e-01	-6.9346e-01
New scheme (10)	-6.9314e-01	-6.9314e-01	-6.9315e-01	-6.9315e-01

Table 2: Average relative errors at $u(1,1)$

	Scheme	
Step size (*h*)	Standard scheme (2)	New scheme (12)
0.125	1.3796e-04	2.2502e-05
0.050	1.9302e-05	1.6114e-06
0.025	4.6050e-06	2.0746e-07
0.020	2.9194e-06	1.0680e-07

The approximate solution and average relative errors at selected step sizes are illustrated in Table 1 and Table 2. The average relative error in Table 2 becomes smaller as the grid size decreases for both schemes. The numerical result for the new scheme is accurate because it shows a smaller error value compared to the standard scheme

5. Theoretical Analysis

This section explores the theoretical foundations of a new scheme (12) for solving nonlinear Goursat problems (5). This new extended scheme was chosen due to the higher accuracy compared to standard scheme (2). Testing has shown it perform better, making the optimal choice for achieving consistency, stability, and convergence.

5.1. *Stability*

The finite difference scheme achieves the stability condition if it provides numerical solutions that grows at a rate that is equal to or lower than the exact solution. Concerns the behaviors of the numerical solution in the limit as the spatial increasingly smaller. The stability is difficult to analyze. Normally, it is warranted by the absence of noticeable spatial in the numerical experiments (Pozrikidis 1998). The Von Neumann method, typically applied to linear initial value problems with constant coefficients, requires that the analyzed problem meet specific criteria. If these criteria are not met, the problem can be adjusted by linearization, disregarding boundary conditions, and using average values for variable coefficients. This allows for the method's application to nonlinear problems. To measure the stability, new scheme (12) is converted into the error scheme by replacing terms $u \to \zeta$. Then, by substituting error equation $\xi_{i,j} = \lambda^j e^{\sqrt{-1}\theta i}$ produces:

$$
\begin{aligned}
&\xi_{i+2,j+2} = \lambda^{j+2} e^{\sqrt{-1}\theta(i+2)}, \\
&\xi_{i+2,j} = \lambda^{j} e^{\sqrt{-1}\theta(i+2)}, \\
&\xi_{i,j+2} = \lambda^{j+2} e^{\sqrt{-1}\theta(i)}, \\
&\xi_{i+2,j+1} = \lambda^{j+1} e^{\sqrt{-1}\theta(i+2)}, \\
&\xi_{i+1,j+2} = \lambda^{j+2} e^{\sqrt{-1}\theta(i+1)}, \\
&\xi_{i+1,j+1} = \lambda^{j+1} e^{\sqrt{-1}\theta(i+1)}, \\
&\xi_{i,j+1} = \lambda^{j+1} e^{\sqrt{-1}\theta(i)}, \\
&\xi_{i+1,j} = \lambda^{j} e^{\sqrt{-1}\theta(i+1)}, \\
&\xi_{i,j} = \lambda^{j} e^{\sqrt{-1}\theta(i)}.
\end{aligned}
\tag{13}
$$

Therefore, to manage the complexity of algebraic manipulation, the error terms in Eq. (13) are substituted into the error scheme using the Maple 18 programming language. For stability it is required that the condition of $|\lambda| \le 1, \forall\theta$ is satisfied. The stability region is illustrated in Figure 1, which shows that the new scheme (12) is unconditionally stable in approximating nonlinear problem (5).

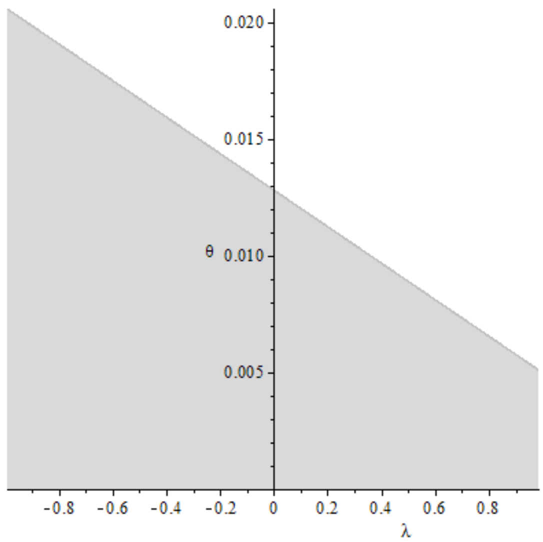

Figure 1: Stability region for scheme (12)

It can be seen that the stability region is $|\lambda| \le 1, \forall\theta$. Thus, the stability condition for scheme Eq. (12) is satisfied.

5.2. *Consistency*

The finite difference scheme is said to be consistent if in the limit, the grid spacings are reduced. However, in arbitrary manner that allows them to have different orders of magnitude, the finite difference equation approximates the PDE with increasing accuracy. The consistency can be achieved if all variables are continuous functions. The finite difference scheme is consistent if the modified differential equation reduces to the original partial differential equation as step size tends to zero (Nasir & Md Ismail 2013). For consistency, the new scheme (12) can be rearranged so that the right-hand side is $e^{2(u_{i,j})}$ and replacing the exact solution into the scheme will yield,

$$
\begin{aligned}
&u\left(x_{i+2}, y_{j+2}\right)=\left[u+2hu_x+2hu_y+2\left(h^2u_{xx}+2h^2u_{xy}+h^2u_{yy}\right)+\ldots\right],\\
&u\left(x_{i+2}, y_j\right)=\left(u+2hu_x+2h^2u_{xx}+\ldots\right),\\
&u\left(x_i, y_{j+2}\right)=\left(u+2hu_y+2h^2u_{yy}+\ldots\right),\\
&u\left(x_{i+2}, y_{j+1}\right)=\left[u+2hu_x+hu_y+\frac{1}{2}\left(4h^2u_{xx}+4h^2u_{xy}+h^2u_{yy}\right)+\ldots\right],\\
&u\left(x_{i+1}, y_{j+2}\right)=\left[u+hu_x+2hu_y+\frac{1}{2}\left(h^2u_{xx}+4h^2u_{xy}+4h^2u_{yy}\right)+\ldots\right],\\
&u\left(x_{i+1}, y_{j+1}\right)=\left[u+hu_x+hu_y+\frac{1}{2}\left(h^2u_{xx}+2h^2u_{xy}+h^2u_{yy}\right)+\ldots\right],\\
&u\left(x_i, y_{j+1}\right)=\left(u+hu_y+\frac{1}{2}h^2u_{yy}+\ldots\right), u\left(x_{i+1}, y_j\right)=\left(u+hu_x+\frac{1}{2}h^2u_{xx}+\ldots\right),\\
&u\left(x_i, y_j\right)=u.
\end{aligned}
\tag{14}
$$

Therefore, by substituting Eq. (14) where all terms involving u are evaluated at $\left(x_i, y_j\right)$. The Maple 18 programming language has been utilized to the simplify the equation as $h\to 0$, leading to the result $u_{xy}=e^{2u}$. Therefore, the consistency condition is satisfied.

5.3. *Convergence*

The Lax equivalent theorem has been demonstrated for linear scalar PDE and with some adaptations, has been extended to certain types of nonlinear scalar equations as well (Gerdt 2011). Thus, this theorem also can be applied for our new scheme (12). The Lax equivalent theorem (Nasir & Md Ismail 2013) states that; if a properly well-posed initial value problem and finite difference approximation satisfied the stability condition, consistency is the necessary and sufficient condition for convergence. The consistency and the stability ensure convergence and vice versa. The relationship can be illustrated as in Figure 2:

Stability + Consistency ↔ Convergence

Figure 2: The relationship between stability, consistency and convergence

Then, based on the Lax equivalence theorem, it can be concluded that the new scheme (12) is convergent.

6. Conclusions

The objective of this study was to develop a novel method for solving nonlinear Goursat hyperbolic PDE using higher-order finite difference method (FDM). This goal was successfully achieved, resulting in a new technique that demonstrated high accuracy in solving the classic nonlinear Goursat PDE problem. The theoretical aspects of solving a nonlinear Goursat problem using sixth-order compact FDM have been thoroughly explored. Additionally, numerical analysis showed that the new scheme outperformed standard method, with its most notable advantage being its effective linearization. The accuracy of the scheme indicates that it is both unconditionally stable and consistent. Furthermore, by applying the Lax equivalence theorem, the scheme was confirmed to be convergent. However, it is important to acknowledge the specific limitations of this study. While there is substantial empirical evidence supporting the effectiveness of high-order compact finite difference methods, alternative methods may yield better results in certain domains and step sizes. Despite these limitations, this study provides valuable insights into the practical application and theoretical foundation of high-order compact finite difference approaches. Future research should focus on extending the methodology to address other nonlinear Goursat cases, thereby enhancing its applicability and overcoming the constraints of numerical methods. The Goursat PDE problem has wide applications in various scientific and engineering fields. Implementing this proposed method will significantly reduce costs, particularly for engineers working in economic dynamics, global optimal scheduling, geoscience, biomedical engineering, and Nordström-like black hole research. The cost will be influenced by factors such as derivation time, running time, software development, energy consumption, and production time.

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