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BERNSTEIN NEURAL NETWORKS METHOD FOR SOLVING VARIABLE ORDER FRACTIONAL MIXED VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

(*Kaedah Rangkaian Neural Bernstein untuk Menyelesaikan Persamaan Kamir-Beza Volterra-Fredholm Tercampur Pecahan Berperingkat Pembolehubah*)

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ABSTRACT

This paper deals with the drawback of the semi-group properties in variable order fractional mixed Volterra-Fredholm integro-differential equations ($\mathcal{VC}$-FVFIDEs) under Caputo derivative operator. The variable order of the equation is converted to piecewise constant functions by partition it into sub-intervals. The existence and uniqueness of solutions are investigated. A novel technique using Bernstein Neural Network (BernsteinNN) method is proposed to obtain the approximate solution for the FVFIDEs, which used the basis of Bernstein polynomials instead of the activation function in the artificial neural network method. The loss function is developed by adding the $L2$ regularization for parameter terms and the hyper-parameter λ to ensure the stability of training process and to control the regularization strength, respectively. Adam optimization approach is applied to training the neural networks and the model performance is computed using the mean square error. The validity of the presented method is demonstrated through the presented example.

Keywords: fractional variable order; Caputo-fractional derivative; Bernstein neural network method; artificial neural network method; Adam optimization method

ABSTRAK

Kertas kerja ini membincangkan kekurangan sifat semi-kumpulan bagi persamaan kamir-beza Volterra-Fredholm tercampur pecahan berperingkat pembolehubah (FVFIDEs) di bawah pengoperasi pembeza Caputo. Peringkat pembolehubah bagi persamaan ditukarkan kepada fungsi pemalar cebis demi cebis dengan membahagikannya kepada sub-selang. Kewujudan dan keunikan penyelesaian disiasat. Suatu kaedah baharu menggunakan kaedah rangkaian neural Bernstein dicadangkan untuk mendapatkan penyelesaian hampiran bagi FVFIDEs, dengan menggunakan asas bagi polynomial Beinstein selain dari fungsi keaktifan di dalam kaedah rangkaian neural buatan. Fungsi terhilang dibina dengan menambah regularisasi $L2$ bagi sebutan parameter dan hiper-parameter λ untuk memastikan kestabilan proses latihan dan mengawal kekuatan regularisasi, masing-masing. Kaedah pengoptimalan Adam digunakan untuk melatih rangkaian neural. Seterusnya, prestasi model dikira menggunakan ralat kuasa dua min. Keputusan berangka membuktikan keabsahan kaedah yang dicadangkan melalui contoh yang dibentangkan.

Kata kunci: pesanan berubah pecahan; terbitan pecahan Caputo; kaedah rangkaian neuron Bernstein; kaedah rangkaian neuron tiruan; kaedah pengoptimuman Adam

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