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SECOND HANKEL DETERMINANT FOR NEW SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH *q*-DERIVATIVE OPERATOR

(*Penentu Hankel Kedua untuk Subkelas Baharu bagi Fungsi Analisis Bersekutu dengan Operator Pembeza-q*)

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ABSTRACT

This paper utilizes the concept of subordination and q-derivative operator to introduce a new subclass of analytic functions in the open unit disk. The primary objective of this paper is to obtain the upper bounds for the second Hankel determinant for the functions belonging to this class.

Keywords: analytic functons; *q*-derivative operator; Hankel determinant

ABSTRAK

Kajian ini menggunakan konsep subordinasi dan operator pembeza-*q* untuk memperkenalkan satu subkelas baharu bagi fungsi analisis dalam cakera unit terbuka. Objektif utama adalah untuk memperoleh had atas bagi penentu Hankel kedua bagi fungsi yang tergolong dalam kelas ini.

Kata kunci: fungsi analisis; operator pembeza-*q*; penentu Hankel

1. Introduction

For $\mathcal{U} = \{\xi \in \mathcal{C}: |\xi| < 1\}$ where $\mathcal{C}$ contains all complex numbers, we consider $\mathcal{A}$ as the class of analytic functions given by

$$\mathcal{k}(\xi) = \xi + \sum_{i=2}^{\infty} a_i \xi^i \tag{1}$$

and we let $\mathcal{S}$ be the subclass of $\mathcal{A}$ which also univalent in $\mathcal{U}$. The well-known starlike functions and convex functions are the main subclasses of $\mathcal{S}$. If there exists an analytic self-map $\omega(\xi)$ as known as the Schwarz function in $\mathcal{U}$ for $\omega(0) = 0$, then we can say that $\mathcal{k} \prec \mathcal{g}$ which equivalent to $\mathcal{k} = \mathcal{g}(\omega(\xi))$.

The Hankel determinant for function $\mathcal{k} \in \mathcal{S}$ defined by Pommerenke (1966) has a series expansion where

$$H_{i,k}(\mathcal{k}) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+i-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+i} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k+i-1} & a_{k+i} & \cdots & a_{k+2i-2} \end{vmatrix}, i, k \in \mathcal{N} = \{1,2,\dots\}$$

and is further proved by Hayman (1968). We note that different values of i and k are obtained from different orders of the Hankel determinant. The generalized form of the Hankel determinant $H_{2,1}(\mathcal{k}) = a_3 - a_2^2$ is the classical Fekete-Szegö functional $|a_3 - \eta a_2^2|$ where $\eta \in \mathcal{C}$. Meanwhile, the second Hankel determinant is given by $H_{2,2}(\mathcal{k}) = a_2 a_4 - a_3^2$. The bounds for different orders of Hankel determinant have become a popular trend among mathematicians such as Khan *et al.* (2022), Lasode and Opoola (2022), Olatunji and Panigrahi (2022), Srivastava *et al.* (2023a), Srivastava *et al.* (2023b) and Srivastava *et al.* (2021).

The q-calculus operator theory has gained significant attention in the fields of applied science such as signal processing, quantum mechanics and complex systems. It extends traditional calculus by incorporating the q-difference operators which allow for the modeling of systems with discrete, fractal, or non-integer behaviors. Jackson (1910) was the earliest mathematician to study the q-derivative operator, and since then, it has been further explored by other mathematicians, such as Cheng *et al.* (2022), Hu *et al.* (2022), Hern *et al.* (2022), Srivastava and Zayed (2019), Yie and Janteng (2024), Yang *et al.* (2024) and Yie *et al.* (2024), in relation to the classes of analytic functions, continuing to be a topic of study to this day.

For $q \to 1^-$ and $i \in \mathcal{N}$, the q-analogue of i, or q-integer number i, is defined by

$$[i]_q = \frac{1-q^i}{1-q} = 1 + q + q^2 + \cdots + q^{i-1} \tag{2}$$

where $\lim_{q\to 1^-} [i]_q = i$. Aral *et al.* (2013) defined q-derivative operator of $\mathcal{k}$ as

$$D_q \mathcal{k}(\xi) = \begin{cases} \dfrac{\mathcal{k}(q\xi) - \mathcal{k}(\xi)}{(q-1)\xi} & , \quad \xi \neq 0; \\ \mathcal{k}'(0) & , \quad \xi = 0 \end{cases} \tag{3}$$

where $\lim_{q\to 1^-} D_q \mathcal{k}(\xi) = \mathcal{k}'(\xi)$. Let $\mathcal{k} \in \mathcal{A}$ be given by Eq. (1) and considering Eq. (3), we have

$$D_q \mathcal{k}(\xi) = 1 + \sum_{i=2}^{\infty} [i]_q a_i \xi^{i-1}. \tag{4}$$

By utilizing the q-derivative operator and the concept of subordination, we define the new subclass of analytic function as follows.

Definition 1.1. A function $\mathcal{k} \in \mathcal{A}$ is said to be in the class $\mathfrak{T}_q(\varphi)$ if it satisfies the following subordination conditions

$$D_q\big(\mathcal{k}(\xi)\big) \prec \varphi(\xi)$$

for $q \in (0,1)$, $\xi \in \mathcal{U}$ and $\varphi \in \mathcal{P}$ where $\mathcal{P}$ is the class of all functions φ that is analytic and univalent in $\mathcal{U}$, and

$$\varphi(\xi) = 1 + B_1\xi + B_2\xi^2 + B_3\xi^3 + \cdots \tag{5}$$

where $B_1 > 0$.

2. Preliminary results

The main results rely on the following lemmas.

Lemma 2.1. (Duren 1983) *Let* $l \in \mathcal{P}$ *be given by*

$$l(\xi) = 1 + c_1\xi + c_2\xi^2 + c_3\xi^3 + \cdots \qquad (6)$$

for $\xi \in \mathcal{U}$, *then* $|c_i| \leq 2$ *where* $i \in \mathcal{N}$.

Lemma 2.2. (Grenander and Szegö 1958) *If* $l \in \mathcal{P}$ *is given by Eq. (6), then*

$$2c_2 = c_1^2 + y(4 - c_1^2) \qquad (7)$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 y - c_1(4 - c_1^2)y^2 + 2(4 - c_1^2)(1 - |y|^2)\xi \qquad (8)$$

for some y, ξ *with* $|y| \leq 1$ *and* $|\xi| \leq 1$.

3. Main results

Theorem 3.1. *Let* $q \in (0,1)$ *and* $\tau = \frac{[2]_q[4]_q}{[3]_q^2}$. *Suppose* $\mathscr{k} \in \mathfrak{T}_q(\varphi)$ *is given by Eq. (1).*

1. *If* B_1, B_2 *and* B_3 *complies with the conditions*

$$8(1 - \tau)|B_2| + 4B_1(1 - 2\tau) \leq 0$$

and

$$\left|\tau B_1 - 2\tau B_2 + B_3 - \frac{\tau(B_2 - B_1)^2}{B_1}\right| - \tau B_1 \leq 0,$$

then

$$|a_2 a_4 - a_3^2| \leq \frac{B_1^2}{[3]_q^2}.$$

2. *If* B_1, B_2 *and* B_3 *complies with the conditions*

$$8(1 - \tau)|B_2| + 4B_1(1 - 2\tau) \geq 0$$

and

$$\left|\tau B_1 - 2\tau B_2 + B_3 - \frac{\tau(B_2 - B_1)^2}{B_1}\right| - (1 - \tau)|B_2| - \frac{B_1}{2} \geq 0,$$

or

$$8(1 - \tau)|B_2| + 4B_1(1 - 2\tau) \leq 0$$

and

$$\left|\tau B_1 - 2\tau B_2 + B_3 - \frac{\tau(B_2 - B_1)^2}{B_1}\right| - \tau B_1 \geq 0,$$

then

$$|a_2 a_4 - a_3^2| \leq \frac{B_1}{[2]_q[4]_q}\left|\tau B_1 - 2\tau B_2 + B_3 - \frac{\tau(B_2 - B_1)^2}{B_1}\right|.$$

3. *If* B_1, B_2 *and* B_3 *complies with the conditions*

$$8(1-\tau)|B_2| + 4B_1(1-2\tau) > 0$$

and

$$\left|\tau B_1 - 2\tau B_2 + B_3 - \frac{\tau(B_2 - B_1)^2}{B_1}\right| - (1-\tau)|B_2| - \frac{B_1}{2} \leq 0,$$

then

$$|a_2 a_4 - a_3^2| \leq \frac{B_1}{4[2]_q[4]_q}\left(\frac{M}{V}\right)$$

where

$$M = B_1\left(4\tau\left|\tau B_1 - 2\tau B_2 + B_3 - \frac{\tau(B_2 - B_1)^2}{B_1}\right| - 4|B_2|(1-\tau) - B_1(1+2\tau)\right) \\ +4\big((1-\tau)|B_2|\big)^2$$

and

$$V = \left|\tau B_1 - 2\tau B_2 + B_3 - \frac{\tau(B_2 - B_1)^2}{B_1}\right| - 2(1-\tau)|B_2| - B_1(1-\tau).$$

Proof. Suppose that $\mathcal{k} \in \mathfrak{T}_q(\varphi)$. Then a Schwarz function $\omega(\xi)$ exists with $\omega(0) = 0$ in $\mathcal{U}$ where

$$D_q\big(\mathcal{k}(\xi)\big) = \varphi\big(\omega(\xi)\big). \tag{9}$$

Define a function

$$l(\xi) = \frac{1+\omega(\xi)}{1-\omega(\xi)} = 1 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + \cdots$$

or equivalently,

$$\omega(\xi)=\frac{l(\xi)-1}{l(\xi)+1}=\frac{c_1}{2}\xi+\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\xi^2+\frac{1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)\xi^3+\cdots. \tag{10}$$

Utilizing Eq. (5) and Eq. (10), we get

$$\begin{aligned}\varphi(\omega(z))=1+\frac{B_1c_1}{2}\xi+\left(\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)B_1+\frac{c_1^2}{4}B_2\right)\xi^2\\+\left(\frac{1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)B_1+\frac{1}{2}\left(c_1c_2-\frac{c_1^3}{2}\right)B_2+\frac{c_1^3}{8}B_3\right)\xi^3+\cdots.\end{aligned} \tag{11}$$

From Eq. (2) and Eq. (4), we get

$$D_q(\mathscr{k}(\xi))=1+[2]_qa_2\xi+[3]_qa_3\xi^2+[4]_qa_4\xi^3+\cdots. \tag{12}$$

By comparing the coefficients of ξ, ξ^2 and ξ^3 in Eq. (11) and Eq. (12), respectively we get that

$$\begin{aligned}a_2&=\frac{B_1c_1}{2[2]_q},\\a_3&=\frac{1}{2[3]_q}\left(c_2-\frac{c_1^2}{2}\right)B_1+\frac{c_1^2}{4[3]_q}B_2,\end{aligned}$$

and

$$a_4=\frac{1}{2[4]_q}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)B_1+\frac{1}{2[4]_q}\left(c_1c_2-\frac{c_1^3}{2}\right)B_2+\frac{c_1^3}{8[4]_q}B_3.$$

Therefore, we have

$$a_2a_4-a_3^2=\kappa\Big(B_1c_1c_3+(-B_1+B_2-\tau(B_2-B_1))c_1^2c_2-\tau B_1c_2^2+\left(\frac{B_1}{4}-\frac{B_2}{2}+\frac{B_3}{4}-\frac{\tau(B_2-B_1)^2}{4B_1}\right)c_1^4\Big)$$

where $\kappa=\frac{B_1}{4[2]_q[4]_q}$ and $\tau=\frac{[2]_q[4]_q}{[3]_q^2}$.

Let

$$\begin{aligned}m_1&=B_1,\\m_2&=-B_1+B_2-\tau(B_2-B_1),\\m_3&=-\tau B_1,\\m_4&=\frac{1}{4}\left(B_1-2B_2+B_3-\frac{\tau(B_2-B_1)^2}{B_1}\right).\end{aligned} \tag{13}$$

Then Eq. (13) yields

$$|a_2a_4-a_3^2|=\kappa|m_1c_1c_3+m_2c_1^2c_2+m_3c_2^2+m_4c_1^4|. \tag{14}$$

By substituting for c_2 and c_3 from Eq. (7) and Eq. (8) into Eq. (14), it yields

$$\begin{aligned}|a_2a_4-a_3^2| = \tfrac{\kappa}{4}\Big|&c_1^4(m_1+2m_2+m_3+4m_4)+2yc_1^2(4-c_1^2)(m_1+m_2+m_3)+\\ &y^2(4-c_1^2)\left(-m_1c_1^2+m_3(4-c_1^2)\right)+2m_1c_1(4-c_1^2)(1-|y|^2)\xi\Big|.\end{aligned}$$

Considering the triangle inequality in the equation above with $\Omega = |y| \leq 1$, $c_1 = c \in [0,2]$, and substitute the values of m_1, m_2, m_3 and m_4 from Eq. (13), we get

$$\begin{aligned}|a_2a_4-a_3^2| &\leq \tfrac{\kappa}{4}\Big(c^4\left|\tau B_1-2\tau B_2+B_3-\tfrac{\tau(B_2-B_1)^2}{B_1}\right|+2\Omega c^2(4-c^2)(1-\tau)|B_2|+\\ &\qquad \Omega^2(4-c^2)(c^2+\tau(4-c^2)-2c)B_1+2B_1c(4-c^2)\Big)\\ &\equiv \mathcal{H}(c,\Omega).\end{aligned}$$

Assume the upper bound is attained at a point within the area enclosed by the rectangle $[0,2]\times[0,1]$. Differentiating $\mathcal{H}(c,\Omega)$ in terms of Ω, we get

$$\frac{\partial\mathcal{H}}{\partial\Omega}=\frac{\kappa}{4}\left(2c^2(4-c^2)(1-\tau)|B_2|+2\Omega(4-c^2)(c^2+\tau(4-c^2)-2c)B_1\right)$$

where $\frac{\partial\mathcal{H}}{\partial\Omega}>0$ for $\Omega\in[0,1)$ and fixed $c\in[0,2]$. It implies that $\mathcal{H}(c,\Omega)$ is an increasing function of Ω. Therefore,

$$\max\mathcal{H}(c,\Omega)=\mathcal{H}(c,1)\equiv\mathcal{G}(c)$$

for fixed $c\in[0,2]$ where

$$\begin{aligned}\mathcal{G}(c)=\frac{B_1}{16[2]_q[4]_q}\Big(&c^4\left(\left|\tau B_1-2\tau B_2+B_3-\tfrac{\tau(B_2-B_1)^2}{B_1}\right|-2(1-\tau)|B_2|-B_1(1-\tau)\right)+\\ &c^2\big(8(1-\tau)|B_2|+4B_1(1-2\tau)\big)+16\tau B_1\Big).\end{aligned}$$

Let

$$\begin{aligned}\mathcal{X}&=\left|\tau B_1-2\tau B_2+B_3-\frac{\tau(B_2-B_1)^2}{B_1}\right|-2(1-\tau)|B_2|-B_1(1-\tau),\\ \mathcal{Y}&=8(1-\tau)|B_2|+4B_1(1-2\tau),\\ \mathcal{Z}&=16\tau B_1.\end{aligned}\tag{15}$$

Since

$$\max_{0\leq t\leq 4}(\mathcal{X}t^2+\mathcal{Y}t+\mathcal{Z})=\begin{cases}\mathcal{Z}, & \mathcal{Y}\leq 0, \mathcal{X}\leq -\frac{\mathcal{Y}}{4};\\ 16\mathcal{X}+4\mathcal{Y}+\mathcal{Z}, & \mathcal{Y}\geq 0, \mathcal{X}\geq -\frac{\mathcal{Y}}{8} \text{ or } \mathcal{Y}\leq 0, \mathcal{X}\geq -\frac{\mathcal{Y}}{4};\\ \frac{4\mathcal{X}\mathcal{Z}-\mathcal{Y}^2}{4\mathcal{X}}, & \mathcal{Y}>0, \mathcal{X}\leq -\frac{\mathcal{Y}}{8};\end{cases}\tag{16}$$

then by Eq. (16), we have

$$|a_2a_4 - a_3^2| \leq \frac{B_1}{16[2]_q[4]_q}\begin{cases} \mathcal{Z}, & \mathcal{Y} \leq 0, \mathcal{X} \leq -\frac{\mathcal{Y}}{4}; \\ 16\mathcal{X} + 4\mathcal{Y} + \mathcal{Z}, & \mathcal{Y} \geq 0, \mathcal{X} \geq -\frac{\mathcal{Y}}{8} \text{ or } \mathcal{Y} \leq 0, \mathcal{X} \geq -\frac{\mathcal{Y}}{4}; \\ \frac{4\mathcal{X}\mathcal{Z} - \mathcal{Y}^2}{4\mathcal{X}}, & \mathcal{Y} > 0, \mathcal{X} \leq -\frac{\mathcal{Y}}{8}; \end{cases}$$

where $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are given by Eq. (15). The proof is complete. □

Remark 3.1 For $q \to 1^-$ and $B_1 = B_2 = B_3 = 2$, Theorem 3.1 reduces to Theorem 3.1 by Janteng *et al.* (2006).

4. Conclusion

A new subclass $\mathfrak{T}_q(\varphi)$ of analytic functions that are subordinate to $\varphi(\xi)$ has been established in this study by utilizing the q-derivative operator. The upper bounds of the second Hankel determinant for the class have been successfully determined. The primary results are stated and proved in Theorem 3.1. The general findings are primarily inspired by their various special cases and implications, one of which is highlighted in Remark 3.1. However, equality is not attained, and hence the bounds are not sharp.

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