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MATHEMATICAL APPROACHES FOR UNDERSTANDING HARMFUL ALGAL BLOOMS IN WEST COAST OF SABAH

(*Pendekatan Matematik untuk Memahami dan Mengurus Algal Mekar Berbahaya di Pantai Barat Sabah*)

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ABSTRACT

Massive algal growth at a certain region and period has alarmed the nation for the past years. The algal bloom is said to be harmful due to its damaging effect on marine creatures, economic sectors such as aquaculture and tourism, and human health. Mathematical modelling of Harmful Algal Blooms (HABs) is developed to describe the HAB's dynamic on the West Coast of Sabah for its annual bloom. There are four variables used in the developed model which are nutrient concentration, Toxic-Producing Phytoplankton (TPP) population, Non-Toxic Phytoplankton (NTP) population, and Zooplankton (Z) population. There exist interaction between all the variables where the nutrient is the food source for the TPP and NTP population. TPP population released toxin while the NTP population does not secrete out toxin chemicals, but it gives competition to the TPP population in terms of food. Zooplankton population act as a predator toward TPP and NTP population. The model is modified from Lotka-Volterra prey-predator model and the stability of the model is also examined. TPP species released toxin to the environment and the effects of toxin on the stability of steady states is investigated. The results show that the system is unstable when TPP species released toxin to the environment and lead to the occurrence of HAB. Zooplankton population are harmed if they consumed TPP species if the toxin content is at peak. Besides, the results also shows that interspecies competition exist between NTP species and TPP species in order to get food. Therefore, this research gives information on how HAB could occur with the biological factors involved.

Keywords: harmful algal bloom (HAB), modelling, stability, toxin

ABSTRAK

Pertumbuhan alga yang besar di rantau dalam tempoh tertentu membimbangkan negara sejak beberapa tahun lalu. Alga dikatakan berbahaya kerana kesannya kepada makhluk laut, sektor ekonomi seperti akuakultur dan pelancongan, dan juga kesihatan manusia. Pemodelan matematik Pertumbuhan Alga Berbahaya (HABs) dijalankan untuk menggambarkan dinamik HAB di Pantai Barat Sabah. Terdapat empat pembolehubah yang digunakan dalam model yang dibangunkan iaitu kepekatan nutrien, populasi Fitoplankton Penghasil Toksik (TPP), populasi Fitoplankton Tidak Toksik- (NTP), dan populasi Zooplankton (Z). Terdapat interaksi antara semua pembolehubah di mana nutrien adalah sumber makanan untuk populasi TPP dan NTP. TPP-mengeluarkan toksin manakala NTP tidak merembeskan bahan kimia toksin, tetapi ia memberi saingan kepada penduduk TPP dari segi makanan. Populasi zooplankton bertindak sebagai pemangsa terhadap populasi TPP dan NTP. Model ini diubah suai daripada model pemangsa mangsa Lotka-Volterra, dan seterusnya kestabilan model juga dianalisa. Spesies TPP membebaskan toksin kepada alam sekitar dan kesan toksin terhadap kestabilan dianalisa. Hasil menunjukkan bahawa sistem menjadi tidak stabil apabila spesies TPP melepaskan toksin ke alam sekitar dan membawa kepada kejadian HAB. Populasi zooplankton tergugat jika mereka memakan spesies TPP yang mempunyai kandungan toksin berada di puncak. Selain itu, hasil menunjukkan bahawa persaingan antara spesies wujud antara spesies NTP dan spesies TPP untuk mendapatkan makanan. Oleh itu, kajian ini memberi maklumat bagaimana HAB boleh berlaku dengan faktor biologi yang terlibat.

Kata kunci: pertumbuhan alga memudaratkan (HAB), permodelan, kestabilan, toksin

1. Introduction

The abundance of algal cell density at a place in coastal water is called an algal bloom and usually consists of a single species or a few species. The algal bloom is said to be harmful when the bloom has an adverse effect, especially on human health and also aquatic ecosystem (Skvortsova 2021; Ralston & Moore 2020; Sharma 2018; Hennon & Dyhrman 2020; Sarkar & Chattopadhayay 2003; Sarkar *et al.* 2005; Sarkar & Malchow 2005; Chattopadhayay *et al.* 2002a, b). This is because Toxin-Producing Phytoplankton (TPP) will released out toxin chemicals to the water and may harm other marine creatures. Several studies have explored the interactions between toxic phytoplankton (TPP), other plankton species, and zooplankton, especially in the context of harmful algal blooms (HABs). Chattopadhayay *et al.* (2002b) found that toxins released by TPP help terminate plankton blooms by reducing competition from other species, acting as a natural control. Roy & Chattopadhyay (2007) showed that these toxins can reduce competition between different phytoplankton species, giving TPP an advantage. Chakraborty *et al.* (2012) developed a model that looked at how zooplankton avoid toxic phytoplankton in the presence of non-toxic ones, helping toxic and non-toxic phytoplankton, as well as zooplankton, coexist.

Braselton & Braselton (2004) suggested that adding more prey or predators to the system could help control TPP and reduce the impact of HABs. Sarkar *et al.* (2005) studied the effect of toxic phytoplankton on zooplankton through a food chain model, focusing on how HABs influence zooplankton populations. Finally, Xiang *et al.* (2018) explored how an optimal control strategy in a three-species plankton model could help manage HABs. Together, these studies highlight the complex interactions in aquatic ecosystems and the role of natural mechanisms in managing harmful algal blooms.

Therefore, there exist a research gap where none of the previous research study the role of non-toxic phytoplankton in the model. It is believed that non-toxic phytoplankton could act as the blooming control agent for HAB. This model consists of four variables which are Nutrient (N), Non-Toxin Phytoplankton (NTP), Toxin-Producing Phytoplankton (TPP) and Zooplankton (Z) population. Nutrients act as a food source for the TPP and NTP populations. TPP and NTP populations were harmed by Z, and both compete with each other to obtain food (nutrient). TPP secrete out the toxic to the environment and becomes harmful to Z population.

Let the concentration of N, NTP, TPP and Z population at time t be denoted as $N(t)$, $P_1(t)$, $P_2(t)$ and $P_3(t)$. The initial value for each variables is [1.3;0.5;0.1;1]. N_0 is a constant N flow into the system. The model was developed based on ecological interactions among nutrients (N), non-toxic phytoplankton (P_1), toxic phytoplankton (P_2), and zooplankton (P_3), incorporating known biological processes such as nutrient uptake, grazing, mortality, competition, and dilution. Each differential equation represents the rate of change over time for one component, built from established interactions in the literature:

Mathematical Model

$$\begin{aligned}
\frac{dN}{dt} &= D(N_0 - N) - \alpha_1 P_1 N - \alpha_2 P_2 N,\\
\frac{dP_1}{dt} &= \theta_1 P_1 N - \beta_1 P_1 P_3 - m_1 P_1 - e_1 P_1 P_2 - D_1 P_1,\\
\frac{dP_2}{dt} &= \theta_2 P_2 N - \beta_2 P_2 P_3 - m_2 P_2 - e_2 P_1 P_2 - D_2 P_2,\\
\frac{dP_3}{dt} &= \gamma_1 P_1 P_3 - \gamma_2 P_2 P_3 - m_3 P_3 - D_3 P_3,
\end{aligned} \tag{1}$$

where

α_1 = NTP nutrient uptake rate

α_2 = TPP nutrient uptake rate
θ_1 = Nutrient conversion rate of NTP
θ_2 = Nutrient conversion rate of TPP
β_1 = NTP predation rate of zooplankton
β_2 = TPP predation rate of zooplankton
γ_1 = NTP zooplankton conversion rate
γ_2 = Rate of death due to TPP consumption
m_1 = NTP death rate
m_2 = TPP death rate
m_3 = Zooplankton death rate
D = Nutrient dilution rate
D_1 = NTP dilution rate
D_2 = TPP dilution rate
D_3 = Zooplankton dilution rate
e_1 = NTP competition coefficient
e_2 = TPP competition coefficient

Some assumptions that are used to develop the model are outlined as follows:

(1) Holling type I functional response is used for interaction of phytoplankton to N. This is because it is applied to unicellular and lower organism like algal (Ma 1996; Holling 1959; Das & Ray 2008; Yussof *et al.* 2022). Hence, the Holling type I functional response of phytoplankton to N is applied in this study.
(2) Zooplankton predation rate is shown by using linear mass law due to its simplicity (Bairagi *et al.* 2008; Yussof *et al.* 2020).
(3) Existence of interspecies competition to obtain N (Bairagi *et al.* 2008; Jana *et al.* 2012; Barton *et al.* 2010; Yussof *et al.* 2021).
(4) Natural toxin released by TPP does not harm NTP population (Usup *et al.* 2002, 2012; Holmes & Teo 2002; Sarkar *et al.* 2006).
(5) TPP harms the Z whenever it is consumed when the toxin content is produced at a high level because they accumulate in the filter feeder and cause the rigid cell wall to break and release toxin (Holmes & Teo 2002; Rehim *et al.* 2016; Wang *et al.* 2015; Al-Azad *et al.* 2016; Chakraborty & Das 2015).

1.1. *Existence of equilibrium*

The equilibrium points N^*, P_1^*, P_2^*, P_3^* are obtained for system (1):

(1) Extinction of NTP

$$E_1(N_1^*, 0, P_{12}^*, P_{13}^*)$$

(2) Extinction of TPP

$$E_2(N_2^*, P_{21}^*, 0, P_{23}^*)$$

(3) Extinction of zooplankton

$$E_3(N_3^*, P_{31}^*, P_{32}^*, 0)$$

(4) Four species coexistence equilibrium

$$E_4(N_4^*, P_{41}^*, P_{42}^*, P_{43}^*),$$

where $N(t) > 0, P_1(t) > 0, P_2(t) > 0, P_3(t) > 0$.

2. Stability Results

Type of dynamics that can arise is determined in this section. The analysis begins with the simplest condition: a plankton interaction model in the absence of plankton.

2.1. *Plankton interaction model in the absence of non-toxic phytoplankton (NTP)*

In this subsection, the model of plankton interaction with the absence of NTP is investigated. The necessary and sufficient conditions of stability around NTP free equilibrium point, $E_1 = (N_1^*, 0, P_{12}^*, P_{13}^*)$ is obtained.

The Jacobian matrix of the system (1) around $E_1 = (N_1^*, 0, P_{12}^*, P_{13}^*)$ can be written as

$$J(E_1) = \begin{bmatrix} -A_1 & -\alpha_1 N_1^* & -\alpha_2 N_1^* & 0 \\ 0 & -A_2 + N_1^*\theta_1 & 0 & 0 \\ P_{12}^*\theta_2 & -e_2 P_{12}^* & -A_3 + N_1^*\theta_2 & -\beta_2 P_{12}^* \\ 0 & \gamma_1 P_{13}^* & -\gamma_2 P_{13}^* & -A_4 \end{bmatrix}, \tag{2}$$

Eigenvalues of Eq. (2) is computed as follows:

$$J(\lambda - E_1) = \begin{bmatrix} \lambda + A_1 & -\alpha_1 N_1^* & -\alpha_2 N_1^* & 0 \\ 0 & \lambda + A_2 - N_1^*\theta_1 & 0 & 0 \\ P_{12}^*\theta_2 & -e_2 P_{12}^* & \lambda + A_3 - N_1^*\theta_2 & -\beta_2 P_{12}^* \\ 0 & \gamma_1 P_{13}^* & -\gamma_2 P_{13}^* & \lambda + A_4 \end{bmatrix} = 0 \tag{3}$$

where

$$\begin{aligned} A_1 &= -D - \alpha_2 P_{12}^*, \\ A_2 &= -D_1 - m_1 - e_1 P_{12}^* - \beta_1 P_{13}^*, \\ A_3 &= -D_2 - m_2 - \beta_2 P_{13}^*, \\ A_4 &= -D_3 - m_3 - \gamma_2 P_{12}^*. \end{aligned} \tag{4}$$

The corresponding characteristic equation of E_1 is as follows:

$$(\lambda + A_2 - N_1^*\theta_1)(\lambda^3 + B_1\lambda^2 + B_2\lambda + B_3) = 0, \tag{5}$$

where

$$\begin{aligned} B_1 &= A_1 + A_3 + A_4 - N_1^*\theta_2, \\ B_2 &= A_1A_3 + A_1A_4 + A_3A_4 - \beta_2\gamma_2 P_{12}^* P_{13}^* - A_1 N_1^*\theta_2 - A_4 N_1^*\theta_2 + \alpha_2 N_1^* P_{12}^*\theta_2, \\ B_3 &= A_1A_3A_4 - A_1\beta_2\gamma_2 P_{12}^* P_{13}^* - A_1A_4 N_1^*\theta_2 + \alpha_2 N_1^* P_{12}^*\theta_2. \end{aligned} \tag{6}$$

Lemma 2.1. *At point $E_1 = (N_1^*, 0, P_{12}^*, P_{13}^*)$, the system (1) is locally asymptotically stable if $N_1^*\theta_1 < A_2$, $B_1 > 0, B_2 > 0$ and $\Delta = B_1B_2 - B_3 > 0$.*

Proof. Assume that $N_1^*\theta_1 < A_2$. We now substitute $N_1^*\theta_1 < A_2$ into Eq. (3). Then, we obtain $\lambda_1 = N_1^*\theta_1 - A_2$. From Eq. (5), λ_1 is a negative value where $\lambda_1 < 0$. Furthermore, for the other eigenvalues, according to Routh-Hurwitz criterion, the roots of the characteristic equation have negative real part if $B_1 > 0, B_2 > 0$ and $\Delta = B_1B_2 - B_3 > 0$. Therefore, the system is locally asymptotically stable since the values of all eigenvalues are negative value. □

Biological Interpretation:
In Lemma (1), the NTP population will become extinct if the maximal growth rate $N_1^*\theta_1$ is smaller than the total loss rate A_2. Total loss rate is the death rate due to the dilution, competition, predator (zooplankton) and natural death. However, if it is in contrast, the NTP population could survive.

2.2. *Plankton interaction model in the absence of toxic phytoplankton (TPP)*

In this subsection, the model of plankton interaction with the absence of TPP is determined. The necessary and sufficient conditions of stability around TPP free equilibrium point, $E_2 = (N_2^*, P_{21}^*, 0, P_{23}^*)$ is obtained.

The Jacobian matrix of the system (1) around $E_2 = (N_2^*, P_{21}^*, 0, P_{23}^*)$ is as follows:

$$J(E_2) = \begin{bmatrix} -C_1 & -\alpha_1 N_2^* & -\alpha_2 N_2^* & 0 \\ P_{21}^*\theta_1 & -C_2 + N_2^*\theta_1 & -e_1 P_{21}^* & -\beta_1 P_{21}^* \\ 0 & 0 & -C_3 + N_2^*\theta_2 & 0 \\ 0 & \gamma_1 P_{23}^* & -\gamma_2 P_{23}^* & -C_4 \end{bmatrix}, \tag{7}$$

Eigenvalues of Eq. (7) is computed as follows:

$$J(\lambda - E_2) = \begin{bmatrix} \lambda + C_1 & -\alpha_1 N_2^* & -\alpha_2 N_2^* & 0 \\ P_{21}^*\theta_1 & \lambda + C_2 - N_2^*\theta_1 & -e_1 P_{21}^* & -\beta_1 P_{21}^* \\ 0 & 0 & \lambda + C_3 - N_2^*\theta_2 & 0 \\ 0 & \gamma_1 P_{23}^* & -\gamma_2 P_{23}^* & \lambda + C_4 \end{bmatrix} = 0 \tag{8}$$

where

$$\begin{aligned} C_1 &= -D - \alpha_1 P_{21}^*, \\ C_2 &= -D_1 - m_1 - \beta_1 P_{23}^*, \\ C_3 &= -D_2 - m_2 - e_2 P_{21}^* - \beta_2 P_{23}^*, \\ C_4 &= -D_3 - m_3 \end{aligned} \tag{9}$$

The characteristic equation of equilibrium point E_2 can be written as

$$(\lambda + C_3 - N_2^*\theta_2)(\lambda^3 + D_1\lambda^2 + D_2\lambda + D_3) = 0, \tag{10}$$

where

$$\begin{aligned} D_1 &= C_1 + C_2 + C_4 - N_2^*\theta_1, \\ D_2 &= C_1C_2 + C_1C_4 + C_2C_4 + \beta_1\gamma_1 P_{21}^* P_{23}^* - C_1 N_2^*\theta_1 - \\ &\qquad C_4 N_2^*\theta_1 + \alpha_1 N_2^* P_{21}^*\theta_1, \\ D_3 &= C_1C_2C_4 + \beta_1 C_1\gamma_1 P_{21}^* P_{23}^* - C_1C_4 N^*\theta_1 + \alpha_1 C_4 N_2^* P_{21}^*\theta_1 \end{aligned} \tag{11}$$

Lemma 2.2. *System (1) is locally asymptotically stable around TPP free equilibrium point* $E_2 = (N_2^*, P_{21}^*, 0, P_{23}^*)$ *if* $N_2^*\theta_2 < C_3$, $D_1 > 0, D_2 > 0$ *and* $\Delta = D_1D_2 - D_3 > 0$.

Proof. Assume that $N_2^*\theta_2 < C_3$. We now substitute $N_2^*\theta_2 < C_3$ into Eq. (9). Then, we obtain $\lambda_1 = -C_3 + N_2^*\theta_2$. From Eq. (9), λ_1 is negative where $\lambda_1 < 0$. Furthermore, for other eigenvalues, according to Routh-Hurwitz criterion, the roots of the characteristic equation have negative real part if $D_1 > 0, D_2 > 0$ and $\Delta = D_1D_2 - D_3 > 0$. Therefore, the system is locally asymptotically stable since the value for all eigenvalues are negative values. □

Biological Interpretation:
In Lemma (2), TPP population will go to extinct if the total loss rate C_3 is more than the maximal growth rate $N_2^*\theta_2$. In contrast, TPP population could survive if maximal growth rate $N_2^*\theta_2$ is greater than the total loss rate C_3.

2.3. *Plankton interaction model in the absence of zooplankton*

In this subsection, the model of plankton interaction without the presence of Z is studied. Several stability conditions for the model around the Z free equilibrium point, $E_3 = (N_3^*, P_{31}^*, P_{32}^*, 0)$ is obtained.

The Jacobian matrix of system (1) at equilibrium point $E_3 = (N_3^*, P_{31}^*, P_{32}^*, 0)$ can be written as:

$$J(E_3) = \begin{bmatrix} -F_1 & -\alpha_1 N_3^* & -\alpha_2 N_3^* & 0 \\ P_{31}^*\theta_1 & -F_2 + N_3^*\theta_1 & -e_1 P_{31}^* & -\beta_1 P_{31}^* \\ P_{32}^*\theta_2 & -e_2 P_{32}^* & -F_3 + N_3^*\theta_2 & -\beta_2 P_{32}^* \\ 0 & 0 & 0 & -F_4 + \gamma_1 P_{31}^* \end{bmatrix}, \tag{12}$$

Eigenvalues of Eq. (12) is computed as follows:

$$J(\lambda - E_3) = \begin{bmatrix} \lambda + F_1 & -\alpha_1 N_3^* & -\alpha_2 N_3^* & 0 \\ P_{31}^*\theta_1 & \lambda + F_2 - N_3^*\theta_1 & -e_1 P_{31}^* & -\beta_1 P_{31}^* \\ P_{32}^*\theta_2 & -e_2 P_{32}^* & \lambda + F_3 - N_3^*\theta_2 & -\beta_2 P_{32}^* \\ 0 & 0 & 0 & \lambda + F_4 - \gamma_1 P_{31}^* \end{bmatrix} = 0 \tag{13}$$

where

$$\begin{aligned} F_1 &= D + \alpha_1 P_{31}^* + \alpha_2 P_{32}^*, \\ F_2 &= D_1 + m_1 + e_1 P_{32}^*, \\ F_3 &= D_2 + m_2 + e_2 P_{31}^*, \\ F_4 &= D_3 + m_3 + \gamma_2 P_{32}^*. \end{aligned} \tag{14}$$

The characteristic equation about E_3 is given by

$$(\lambda_1 + F_4 - \gamma_2 P_{32}^*)(\lambda^3 + G_1\lambda^2 + G_2\lambda + G_3) = 0, \tag{15}$$

where

$$\begin{aligned} G_1 &= F_1 + F_2 + F_3 - N_3^*\theta_1 - N_3^*\theta_2, \\ G_2 &= F_1F_2 + F_1F_3 + F_2F_3 - e_1e_2P_1^*P_{32}^*F_1N_3^*\theta_1 - \\ &\quad F_3N_3^*\theta_1 - F_1N_3^*\theta_2 - F_2N_3^*\theta_2 + \alpha_2N_3^*P_{32}^*\theta_2 + N_3^*\theta_1\theta_2 + \alpha_1N_3^*P_{31}^*\theta_1, \\ G_3 &= F_1F_2F_3 - e_1e_2F_1P_{31}^*P_{32}^* - F_1F_3N_3^*\theta_1 - F_1F_2N_3^*\theta_2 + \alpha_2F_2N_3^*P_{32}^*\theta_2 - \\ &\quad \alpha_1e_1N_3^*P_{31}^*P_{32}^*\theta_2 + F_1N_3^*\theta_1\theta_2 - \alpha_2N_3^*P_{32}\theta_1\theta_2 + \alpha_1F_3N_3^*P_{31}^*\theta_1 - \\ &\quad \alpha_2e_2N_3^*P_{31}^*P_{32}^*\theta_1 - \alpha_1N_3^{2*}P_{31}\theta_2\theta_1. \end{aligned} \tag{16}$$

Lemma 2.3. *System (1) is locally asymptotically stable around Z free equilibrium point $E_3 = (N_3^*, P_{31}^*, P_{32}^*, 0)$ if $\gamma_1 P_{31}^* < F_4$, $G_1 > 0, G_2 > 0$ and $\Delta = G_1G_2 - G_3 > 0$.*

Proof. Assume that $\gamma_1 P_{31}^* < F_4$. We now substitute $\gamma_1 P_{31}^* < F_4$ into Eq. (14). Then, we obtain $\lambda_1 = -F_4 + \gamma_1 P_{31}^* < 0$. From Eq. (14), λ_1 is a negative value where $\lambda_1 < 0$. Furthermore, for other eigenvalues, according to Routh-Hurwitz criterion, the roots of the characteristic equation have negative real part if $G_1 > 0, G_2 > 0$ and $\Delta = G_1G_2 - G_3 > 0$. Since all the eigenvalues are negative, the system is said to be asymptotically stable. □

Biological Interpretation:
In Lemma (3), the Z population goes extinct if total loss rate F_4 is more than maximal growth rate $\gamma_1 P_{31}^*$ or otherwise.

2.4. *Plankton interaction model of all populations*

In this subsection, the model of plankton interaction of all populations is studied. Some of the conditions of stability at $E_4 = (N_4^*, P_{41}^*, P_{42}^*, P_{43}^*)$ is obtained.

The Jacobian matrix of system (1) at equilibrium point $E_4 = (N_4^*, P_{41}^*, P_{42}^*, P_{43}^*)$ can be written as:

$$J(E_4) = \begin{bmatrix} R_1 & -\alpha_1 N_4^* & -\alpha_2 N_4^* & 0 \\ P_{41}^*\theta_1 & R_2 & -e_1 P_{41}^* & -\beta_1 P_{41}^* \\ P_{42}^*\theta_2 & -e_2 P_{42}^* & R_3 & -\beta_2 P_{42}^* \\ 0 & \gamma_1 P_{43}^* & -\gamma_2 P_{43}^* & R_4 \end{bmatrix}, \tag{17}$$

where

$$\begin{aligned} R_1 &= -d - \alpha_1 P_{41}^* - \alpha_2 P_{42}^*, \\ R_2 &= -d_1 - m_1 - e_1 P_{42}^* - \beta_1 P_{43}^* + N_3^*\theta_1, \\ R_3 &= -d_2 - m_2 - e_2 P_{41}^* - \beta_2 P_{43}^* + N_3^*\theta_2, \\ R_4 &= -d_3 - m_3 + \gamma_1 P_{41}^* - \gamma_2 P_{42}^*. \end{aligned} \tag{18}$$

Eigenvalues of Eq. (17) is computed as follows:

$$J(\lambda - E_4) = \begin{bmatrix} \lambda - R_1 & -\alpha_1 N_4^* & -\alpha_1 N_4^* & -\alpha_2 N_4^* \\ P_{41}^*\theta_1 & \lambda - R_2 & -e_1 P_{41}^* & -\beta_1 P_{41}^* \\ P_{42}^*\theta_2 & -e_2 P_{42}^* & \lambda - R_3 & -\beta_2 P_{42}^* \\ 0 & 0 & 0 & \lambda - R_4 \end{bmatrix} = 0 \tag{19}$$

The characteristic equation about E_4 is given by

$$(\lambda^4 + S_1\lambda^3 + S_2\lambda^2 + S_3\lambda + S_4) = 0, \tag{20}$$

where

$$\begin{aligned} S_1 &= -(R_1 + R_2 + R_3 + R_4), \\ S_2 &= -e_1e_2P_{41}^*P_{42}^* + \beta_1\gamma_1P_{41}^*P_{43}^* - \beta_2\gamma_2P_{42}^*P_{43}^* + R_1(R_2 + R_3 + R_4) \\ &\quad + R_2(R_3 + R_4) + R_3R_4 + \alpha_1N_4^*P_{41}^*\theta_1 + \alpha_2N_4^*P_{42}^*\theta_2, \\ S_3 &= (\beta_2e_1\gamma_1 - \beta_1e_2\gamma_2)P_{41}^*P_{42}^*P_{43}^*R_1 - \beta_2\gamma_2P_{42}^*P_{43}^*R_1R_2 + \beta_1\gamma_1P_{41}^*P_{43}^*R_1R_3 \\ &\quad - e_1e_2P_{41}^*P_{42}^*R_1R_4 + R_1R_2R_3R_4 - (\alpha_2\gamma_1 - \alpha_1\gamma_2)\beta_2\gamma_1N_4^*P_{41}^*P_{42}^*P_{43}^*\theta_1 \\ &\quad + (\alpha_2e_2\theta_1 + \alpha_1e_1\theta_2)N_4P_{41}^*P_{42}^*R_4 + \alpha_1N_4P_{41}^*R_3R_4\theta_1 + (\alpha_2\gamma_1 + \alpha_1\gamma_2)\beta_1N_4^*P_{41}^*P_{42}^*P_{43}^*\theta_2 \\ &\quad + \alpha_2N_4^*P_{42}^*R_2R_4\theta_2, \\ S_4 &= -\beta_2e_1\gamma_1P_{41}^*P_{42}^*P_{43}^* + \beta_1e_2\gamma_2P_{41}^*P_{42}^*P_{43}^* + (R_1 + R_4)e_1e_2P_{41}^*P_{42}^* \\ &\quad - (R_1 + R_3)\beta_1\gamma_1P_{41}^*P_{43}^* + (R_1 + R_2)\beta_2\gamma_2P_{42}^*P_{43}^* - R_1R_2R_3 - R_1R_2R_4 \\ &\quad - (R_1 + R_2)R_3R_4 - \alpha_2e_2N_4^*P_{41}^*P_{42}^*\theta_1 + (R_3 - R_4)\alpha_1N_4^*P_{41}^*\theta_1 \\ &\quad - \alpha_1e_1N_4P_{41}^*P_{42}^*\theta_2 - (R_2 - R_4)\alpha_2N_4^*P_{42}^*\theta_2. \end{aligned}$$

Lemma 2.4. *System (1) is unstable around* $E_4 = (N_4^*, P_{41}^*, P_{42}^*, P_{43}^*)$ *if* $S_1 < 0, S_1S_2 - S_0S_3 < 0, (S_1S_2 - S_0S_3)S_3 - S_1^2S_4 < 0$ *and* $S_4 < 0$.

Proof. From Eq. (20), S_1 is a negative value. According to Routh-Hurwitz criterion, the conditions for the roots to have negative real part are $S_1 > 0, S_1S_2 - S_0S_3 > 0, (S_1S_2 - S_0S_3)S_3 - S_1^2S_4 > 0$ and $S_4 > 0$. However, value of S_1 is always negative. Therefore, equilibrium E_4 is unstable since Eq. (20) does not follow Routh-Hurwitz criterion where $S_1 < 0$. □

Biological Interpretation:
In Lemma (5), the system is always unstable around E_4, which shows that HAB events always occur if all the variables exist.

3. Results and Discussion

Table 1 shows the parameter values from the literature (Al-Azad *et al.* 2016; Chakraborty & Das 2015) used in the numerical simulation to obtain the analytical results.

Table 1: Value of parameter used in the numerical simulation

Parameters	Symbols	Values
Nutrient dilution rate	D	$0.3(h^{-1})$
Constant input of nutrient concentration	N_0	$1.58(h^{-1})$
NTP nutrient uptake rate	α_1	$0.03(ml.h^{-1})$
TPP nutrient uptake rate	α_2	$0.022(ml.h^{-1})$
Nutrient conversion rate of NTP	θ_1	$0.02(ml.h^{-1})$
Nutrient conversion rate of TPP	θ_2	$0.02(ml.h^{-1})$
NTP death rate	m_1	$0.006(h^{-1})$
TPP death rate	m_2	$0.006(h^{-1})$
Zooplankton death rate	m_3	$0.005(h^{-1})$
NTP competition coefficient	e_1	$0.02(ml.h^{-1})$
TPP competition coefficient	e_2	$0.02(ml.h^{-1})$
NTP predation rate of zooplankton	β_1	$0.02(ml.h^{-1})$
TPP predation rate of zooplankton	β_2	$0.01(ml.h^{-1})$
NTP zooplankton conversion rate	γ_1	$0.01(ml.h^{-1})$
Rate of death due to TPP consumption	γ_2	$0.008(ml.h^{-1})$
NTP dilution rate	D_1	$0.0004(h^{-1})$
TPP dilution rate	D_2	$0.0004(h^{-1})$
Zooplankton	D_3	$0.0003(h^{-1})$

Figure 1 shows periodic oscillations of nutrients, non-toxic phytoplankton (NTP), toxic phytoplankton (TPP), and zooplankton over time, at point $E^*(1.4653,\ 0.6618,\ 0.1647,\ 0.9806)$ indicating that the system does not stabilize at a fixed equilibrium point. In this context, such instability is directly associated with the occurrence of harmful algae blooms (HAB). This is because the toxic phytoplankton (TPP) population, which continues to fluctuate, is actively producing toxins that disrupt the natural balance of the aquatic ecosystem. As the TPP population increases and releases toxins, it suppresses zooplankton through toxic effects and reduces their grazing pressure. This allows the TPP population to proliferate uncontrollably, leading to sustained or recurring blooms. Therefore, the observed instability in the system reflects the continuous occurrence of HABs, driven by the toxin-liberating behavior of the TPP population, preventing the ecosystem from reaching a stable and healthy equilibrium.

Meanwhile, the equilibrium between TPP population zooplankton population is as depicted in Figure 2. The contour plot illustrates a clear inverse relationship between TPP and zooplankton populations, which suggests a significant ecological interaction between these two

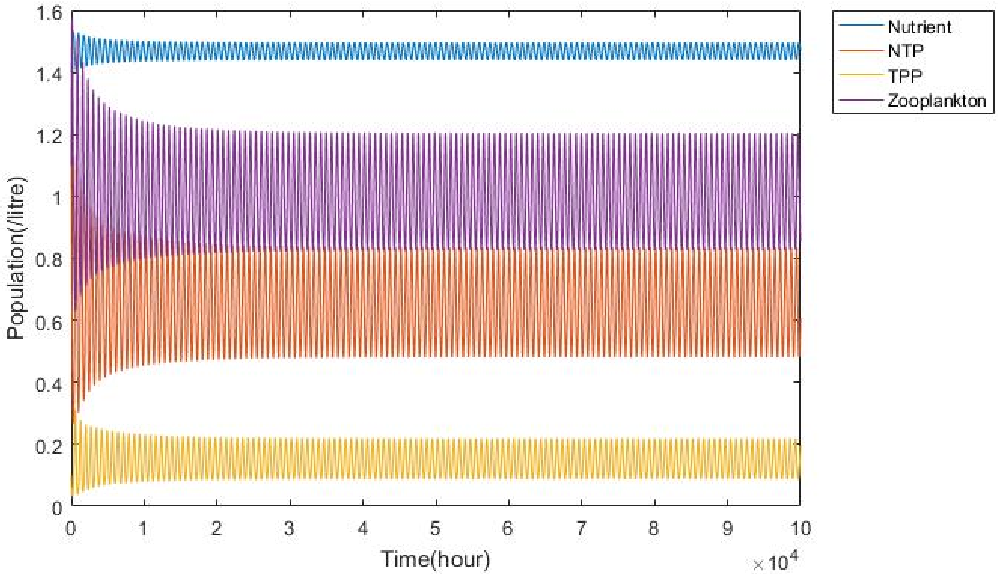

Figure 1: Simulation results of System (1)

variables, especially during the occurrence of harmful algal blooms (HABs). As TPP increases, it promotes the growth of phytoplankton, particularly harmful and toxic species. These toxic phytoplankton negatively impact zooplankton by producing toxins that reduce their feeding efficiency, reproduction, and survival. As a result, the zooplankton population declines, as shown by the downward trend in the plot with increasing TPP levels. This decline in zooplankton leads to reduced grazing on phytoplankton, allowing toxic algal blooms to persist or intensify. Therefore, the plot highlights a negative feedback loop where high nutrient levels support toxic phytoplankton growth, which in turn suppresses zooplankton, further destabilizing the aquatic ecosystem.

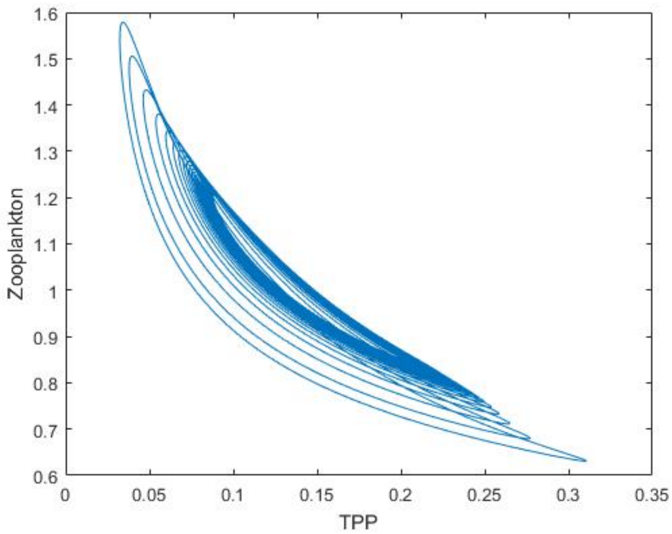

Figure 2: Equilibrium between the TPP and zooplankton populations loses its stability

In addition, Figure 3 illustrates that the equilibrium between non-toxic phytoplankton (NTP) and toxic phytoplankton (TPP) populations becomes unstable during a Harmful Algal Bloom (HAB) event. The closed, spiral-like trajectories in the phase plot indicate sustained oscillations, showing that the system does not return to a fixed point but instead cycles continuously. This instability arises from the interspecies competition between NTP and TPP, as both groups compete for the same resources whis is nutrient. During HAB events, the toxic phytoplankton (TPP) gain a competitive advantage due to their ability to produce toxins that can inhibit or outcompete the non-toxic phytoplankton (NTP). As a result, the balance between the two phytoplankton types is disrupted, leading to persistent fluctuations in their populations.

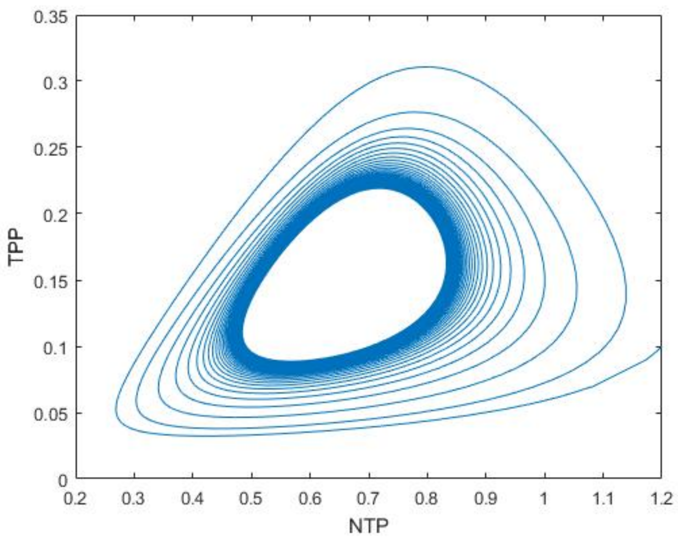

Figure 3: Equilibrium between the NTP and TPP populations loses its stability

4. Conclusion

From the results of the graph shown, it is evident that the presence of toxic phytoplankton (TPP) plays a significant role in disrupting the balance of the marine ecosystem, particularly during the occurrence of harmful algae blooms (HAB). The results of the model show that when the TPP population secretes toxic chemicals into the environment, the zooplankton population is adversely affected. These toxins can impair the respiratory systems of zooplankton, often leading to increased mortality and a decline in their population. The reduction in zooplankton not only disrupts the food web but also weakens the natural grazing pressure on phytoplankton, further allowing toxic species to dominate.

However, the presence of non-toxic phytoplankton (NTP) introduces a crucial balancing factor. NTP competes directly with TPP for essential resources such as nutrients and light. This interspecies competition helps to prevent the uncontrolled growth of TPP, as both TPP and NTP compete for the same resources. Through this natural form of biological control, some of the TPP population may be suppressed, thereby reducing the intensity or duration of HAB events. This interaction highlights the ecological importance of maintaining phytoplankton diversity in stabilizing marine ecosystems.

Furthermore, the model clearly demonstrates that elevated nutrient concentrations in water, particularly from anthropogenic sources such as agricultural runoff or wastewater discharge, serve as a key trigger for the formation of algal blooms. High availability of nutrients accelerates the growth of phytoplankton, including harmful species, making nutrient loading a critical factor in HAB dynamics.

In summary, the mathematical model developed in this research provides valuable insight into the complex biological interactions and environmental conditions that contribute to HAB occurrences. It emphasizes the dual role of TPP toxicity in harming zooplankton and the regulatory role of NTP through competition. In addition, it underscores the importance of nutrient management in preventing bloom outbreaks. Thus, this model can serve as a useful tool for predicting and mitigating HAB events, helping to maintain the sustainable management of aquatic ecosystems.

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