6

ORDINARY DIFFERENTIAL EQUATIONS

- Introduction
- Runge-Kutta Methods
- Multi-step Methods
- System of Equations
- Boundary Value Problems
- Characteristic Value Problems

6.1 Introduction

- In many engineering applications, there are laws of physics, which can be mathematically written in form of *differential* (or, *rate*) *equations*, e.g.

$$\frac{dv}{dt} = -\frac{c}{m} v \tag{6.1}$$

- Eq. 6.1 is known as *ordinary differential equation* (ODE) since there is **only one** independent variable, i.e. t, otherwise it is a *partial differential equation* (PDE).

- Eq. 6.1 is a *first order* ODE, whereas an example of a *second order* ODE is

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0 \tag{6.2}$$

- Solving Eqs. 6.1-2 needs a set of *conditions*, which can be divided into:

  1. **Initial value problems** — if only *one* condition is required,
  2. **Boundary value problems** — if only *more than one* conditions are required.

6.2 Runge-Kutta Methods

- The **Runge-Kutta** method is a *one-step* method which requires *one initial condition*:

$$\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0$$

- A general numerical formula berangka to solve the above equation is

$$\text{(New value)} = \text{(old value)} + \text{(Gradient)} \times \text{(Step size)}$$

$$y_{i+1} = y_i + \phi \times h \tag{6.3}$$

where $h = x_{i+1} - x_i$ is the *step size*.

- If the estimation of gradient $\phi = f(x_i, y_i)$ is taken from the first derivative at x_i, the technique is known as the **Euler** (or, *gradient point*) **method**, and

$$y_{i+1} = y_i + f(x_i, y_i) \times h \tag{6.4}$$

- In solving an ODE, there are two types of errors:

1. **Global error** — relative to exact solution,
2. **Local error** — relative to the previous numerical solution.

- To calculate the local error, consider a Taylor series expansion:

$$y_{i+1} = y_i + y_i' h + \frac{y_i''}{2!} h^2 + \cdots + \frac{y_i^n}{n!} h^n + R_n$$

$$R_n = \frac{y^{n+1}(\xi)}{(n+1)!} h^{n+1}$$

where R_n is the residual term. If $y' = f(x, y)$:

$$y_{i+1} = y_i + f(x_i, y_i) h + \frac{f'(x_i, y_i)}{2!} h^2 + \cdots + \frac{f^{n-1}(x_i, y_i)}{n!} h^n + O(h^{n+1})$$

Hence, the local truncated error is:

$$E_t = \frac{f'(x_i, y_i)}{2!} h^2 + \cdots + O(h^{n+1}) \tag{6.5}$$

Example 6.1

Use the Euler method to obtain the values of y numerically for the following differential equation:

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

Given that the initial condition is $y(0) = 1$, perform the calculation from $x = 0$ to $x = 4$ with a step size of $h = 0.5$. For comparison, use the analytical solution: $y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$.

Solution

The ODE can be written as:

$$\frac{dy}{dx} = f(x, y) = -2x^3 + 12x^2 - 20x + 8.5$$

For the first step ($x = 0.5$):

$$f(0,1) = -2(0)^3 + 12(0)^2 - 20(0) + 8.5 = 8.5$$

Value using the Euler method:

$$y(0.5) = y(0) + f(0,1) \times h = 1 + 8.5(0.5)$$
$$= 5.25$$

Exact value:

$$y(0.5) = -0.5(0.5)^4 + 4(0.5)^3 - 10(0.5)^2 + 8.5(0.5) + 1.0$$
$$= 3.21875$$

True error:

$$E_t = 3.21875 - 5.25 = -2.03125 \quad , \quad \varepsilon_t = -63.1\%$$

For the second step ($x = 0.5$):

$$y(1.0) = y(0.5) + f(0.5, 5.25) \times h$$
$$= 5.25 + \left[-2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5\right](0.5)$$
$$= 5.875$$

x	y_{Euler}	y_{exact}	Global error	Local error
0.0	1.00000	1.00000	-	-
0.5	5.25000	3.21875	−63.1	−63.1
1.0	5.87500	3.00000	−95.8	−28.1
1.5	5.12500	2.21875	−131.0	−1.4
2.0	4.50000	2.00000	−125.0	20.3
2.5	4.75000	2.71875	−74.7	17.2
3.0	5.87500	4.00000	−46.9	3.9
3.5	7.12500	4.71875	−51.0	−11.3
4.0	7.00000	3.00000	−133.3	−53.1

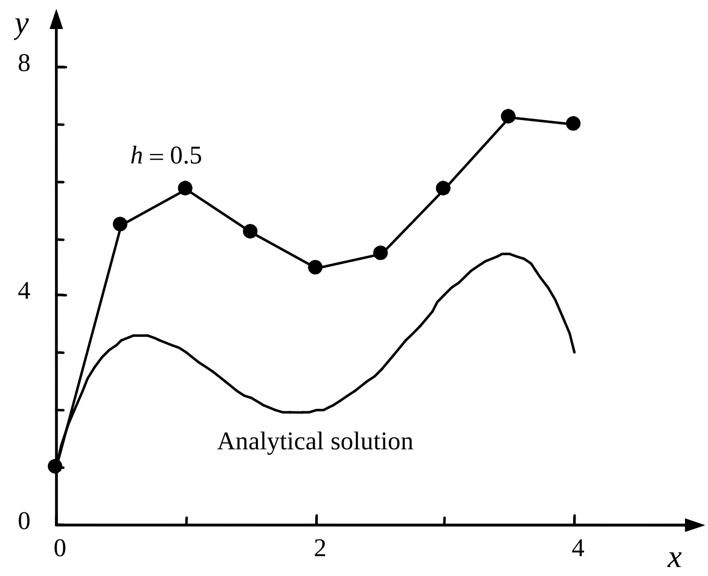


FIGURE 6.1 Graph for Ex. 6.1

- For Ex. 6.1, the truncated local error can be calculated at $x = 0$as followed:

$$E_t = \frac{f'(x_i, y_i)}{2!}h^2 + \frac{f''(x_i, y_i)}{3!}h^3 + \frac{f'''(x_i, y_i)}{4!}h^4$$

$$f'(x, y) = -6x^2 + 24x - 20$$
$$f''(x, y) = -12x + 24$$
$$f'''(x, y) = -12$$

$$E_t = \frac{-6(0)^2 + 24(0) - 20}{2}(0.5)^2 + \frac{-12(0) + 24}{6}(0.5)^3 + \frac{-12}{24}(0.5)^4$$
$$= -2.03125.$$

- The Euler method can be improved using a smaller step size.

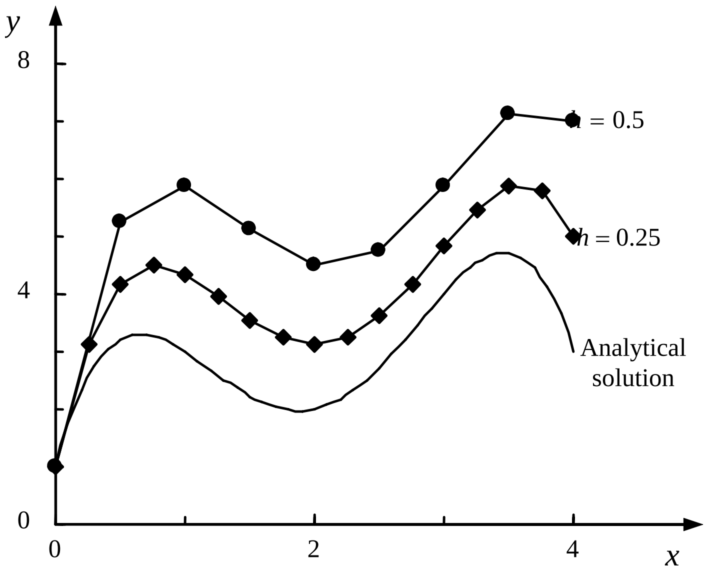


FIGURE 6.2 Effect of using a smaller step size for Ex. 6.1

- To further improve the technique, the Euler method can be used for prediction which is to be corrected via iteration mechanism.

- In the **predictor-corrector method**, the *predictor equation* is

$$y_{i+1}^{0} = y_i + f(x_i, y_i)h \tag{6.6}$$

Use y_{i+1}^{0} to obtain the gradient at x_{i+1}, and then, take the average:

$$\phi_{i+1} = f\left(x_{i+1}, y_{i+1}^{0}\right)$$

$$\bar{\phi} = \frac{\phi_i + \phi_{i+1}}{2}$$

Hence, the *corrector equation* is

$$y_{i+1}^{1} = y_i + \bar{\phi}h = y_i + \frac{\phi_i + \phi_{i+1}}{2}h$$

or,

$$y_{i+1}^{1} = y_i + \frac{f(x_i, y_i) + f\left(x_{i+1}, y_{i+1}^{0}\right)}{2}h \tag{6.7}$$

- By taking the average of gradients, Eq. 6.7 is known as the *Heun method*, which can be solved iteratively and generalised as

$$y_{i+1}^{j} = y_i + \frac{f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{j-1})}{2} h$$

$$|\varepsilon_a| = \left| \frac{y_{i+1}^{j} - y_{i+1}^{j-1}}{y_{i+1}^{j}} \right| \times 100\%$$

Example 6.2

Solve the following equation for the range of $x = 0$ to $x = 4$:

$$\frac{dy}{dx} = 4e^{0.8x} - 0.5y$$

with an initial condition of $y(0) = 2$ and a step size of $h = 1$. Perform calculation for 15 iterations.

Solution

At $x = 0$:

Predictor:

$$\phi_i = \frac{dy}{dx} = 4e^{0.8(0)} - 0.5(2) = 3 \quad \Rightarrow \quad y_{i+1}^{0} = 2 + (3)1 = 5$$

Gradient averaging:

$$\phi_{i+1} = f(x_{i+1}, y_{i+1}^{0}) = f(1,5)$$
$$= 4e^{0.8(1)} - 0.5(5) = 6.402164$$

$$\bar{\phi} = \frac{3 + 6.402164}{2} = 4.701082$$

Corrector:

$$y_{i+1}^{1} = 2 + (4.701082)1 = 6.701082$$

$$\varepsilon_a = \frac{5 - 6.701082}{6.701082} = -25.4\%$$

In the next iteration:

$$y_{i+1}^{2} = 2 + \frac{3 + [4e^{0.8(1)} - 0.5(6.701082)]}{2}(1) = 6.275811$$

$$\varepsilon_a = 6.78\%$$

x	y_{exact}	After 1 iteration		After 15 iterations	
		y_{Heun}	$\lvert\varepsilon_t\rvert$ %	y_{Heun}	$\lvert\varepsilon_t\rvert$ %
0	2.00000	2.00000	0.00	2.00000	0.00
1	6.19463	6.70108	8.18	6.36087	2.68
2	14.84392	16.31978	9.94	15.30224	3.09
3	33.67717	37.19925	10.46	34.74328	3.17
4	75.33896	83.33777	10.62	77.73510	3.18

- The general equation for the **Runge-Kutta methods** is

$$y_{i+1} = y_i + \phi(x_i, y_i, h) \times h \tag{6.8}$$

- The gradient term $\phi(x_i, y_i, h)$ is known as *iteration function* which can be written as

$$\phi = a_1 k_1 + a_2 k_2 + \cdots + a_n k_n$$

where a_i are constants and k_i are defined as

$$\begin{aligned} k_1 &= f(x_i, y_i) \\ k_2 &= f(x_i + p_1 h, y_i + q_{11} k_1 h) \\ k_3 &= f(x_i + p_2 h, y_i + q_{21} k_1 h + q_{22} k_2 h) \\ &\vdots \\ k_n &= f(x_i + p_{n-1} h, y_i + q_{n-1,1} k_1 h + q_{n-1,2} k_2 h + \cdots + q_{n-1,n-1} k_{n-1} h) \end{aligned}$$

- When $n = 1$, the Euler method can be formed, which represents the first order Runge-Kutta method.

- For $n = 2$, the second order Runge-Kutta method can be formed:

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) \times h \tag{6.9}$$

where

$$\begin{aligned} k_1 &= f(x_i, y_i) \\ k_2 &= f(x_i + p_1 h, y_i + q_{11} k_1 h) \end{aligned}$$

TABLE 6.1 The second order Runge-Kutta methods

Method	a_1	a_2	$p_1 = q_{11}$	**Equation**
Heun	$\frac{1}{2}$	$\frac{1}{2}$	1	$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right)h$ $k_1 = f(x_i, y_i)$ $k_2 = f(x_i + h, y_i + k_1 h)$
Midpoint	0	1	$\frac{1}{2}$	$y_{i+1} = y_i + k_2 h$ $k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$
Ralston	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{4}$	$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right)h$ $k_1 = f(x_i, y_i)$ $k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$

Example 6.3

Use the Heun, midpoint and Ralston methods to solve Ex. 6.1.

Solution

The Heun method:

$$k_1 = f(x_i, y_i)$$
$$= -2(0)^3 + 12(0)^2 - 20(0) + 8.5 = 8.5$$
$$k_2 = f(x_i + h, y_i + k_1 h)$$
$$= -2(0+0.5)^3 + 12(0+0.5)^2 - 20(0+0.5) + 8.5 = 1.25$$

Hence,

$$y(0) = 1 + \left[\tfrac{1}{2}(8.5) + \tfrac{1}{2}(1.25)\right]0.5 = 3.4375$$
$$\varepsilon_t = -6.8\%$$

The midpoint method:

$$k_2 = f\left(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}k_1 h\right)$$
$$= -2(0+0.25)^3 s + 12(0+0.25)^2 - 20(0+0.25) + 8.5$$
$$= 4.21875$$

Hence,

$$y(0.5) = 1 + 4.21875(0.5) = 3.109375$$
$$\varepsilon_t = -3.4\%$$

The Ralston method:

$$k_2 = f\left(x_i + \tfrac{3}{4}h, y_i + \tfrac{3}{4}k_1 h\right)$$
$$= -2(0+0.375)^3 + 12(0+0.375)^2 - 20(0+0.375) + 8.5$$
$$= 2.58203125$$

Hence,

$$y(0) = 1 + \left[\tfrac{1}{3}(8.5) + \tfrac{2}{3}(2.58203125)\right]0.5 = 3.27734375$$
$$\varepsilon_t = -1.82\%$$

x	y_{sebenar}	Heun		Midpoint		Ralston	
		y	$\lvert\varepsilon_t\rvert$ %	y	$\lvert\varepsilon_t\rvert$ %	y	$\lvert\varepsilon_t\rvert$ %
0.0	1.00000	1.00000	0.0	1.00000	0.0	1.00000	0.0
0.5	3.21875	3.43750	6.8	3.10937	3.4	3.27734	1.8
1.0	3.00000	3.37500	12.5	2.81250	6.3	3.10156	3.4
1.5	2.21875	2.68750	21.1	1.98438	10.6	2.34766	5.8
2.0	2.00000	2.50000	25.0	1.75000	12.5	2.14062	7.0
2.5	2.71875	3.18750	17.2	2.48438	8.6	2.85547	5.0
3.0	4.00000	4.37500	9.4	3.81250	4.7	4.11719	2.9
3.5	4.71875	4.93750	4.6	4.60938	2.3	4.80078	1.7
4.0	3.00000	3.00000	0.0	3.00000	0.0	3.03125	1.0

- For $n = 3$, the third order Runge-Kutta method can be formed:

$$y_{i+1} = y_i + \tfrac{1}{6}(k_1 + 4k_2 + k_3)h \qquad (6.10)$$

where

$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}k_1 h\right)$$
$$k_3 = f(x_i + h, y_i - k_1 h + 2k_2 h)$$

- For $n = 4$, the fourth order Runge-Kutta method can be formed:

$$y_{i+1} = y_i + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \qquad (6.11)$$

where

$$
\begin{aligned}
k_1 &= f(x_i, y_i) \\
k_2 &= f\left(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}k_1h\right) \\
k_3 &= f\left(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}k_2h\right) \\
k_4 &= f\left(x_i + h, y_i + k_3h\right)
\end{aligned}
$$

- For $n = 5$, the fifth order Runge-Kutta method (also known as the *Butcher method*) can be formed:

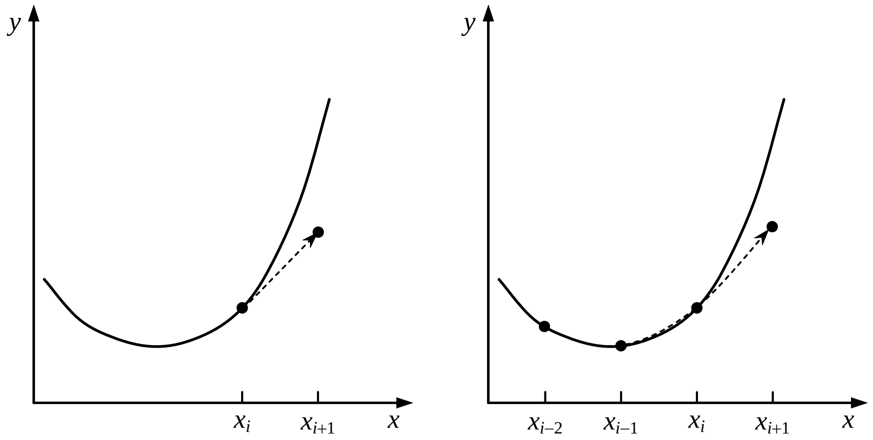
$$y_{i+1} = y_i + \tfrac{1}{90}\left(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6\right)h \tag{6.12}$$

where

$$
\begin{aligned}
k_1 &= f(x_i, y_i) \\
k_2 &= f\left(x_i + \tfrac{1}{4}h, y_i + \tfrac{1}{4}k_1h\right) \\
k_3 &= f\left(x_i + \tfrac{1}{4}h, y_i + \tfrac{1}{8}k_1h + \tfrac{1}{8}k_2h\right) \\
k_4 &= f\left(x_i + \tfrac{1}{2}h, y_i - \tfrac{1}{2}k_2h + k_3h\right) \\
k_5 &= f\left(x_i + \tfrac{3}{4}h, y_i + \tfrac{3}{16}k_1h + \tfrac{9}{16}k_4h\right) \\
k_6 &= f\left(x_i + h, y_i - \tfrac{3}{7}k_1h + \tfrac{2}{7}k_2h + \tfrac{12}{7}k_3h - \tfrac{12}{7}k_4h + \tfrac{8}{7}k_5h\right)
\end{aligned}
$$

6.3 Multi-step Methods

- In the **multi-step** methods, the calculation is based on *more than one* points, and a popular version is the *Adams formula*.



(a) One-step method (b) Multi-step method

FIGURE 6.3 Comparison between the one-step method and the multi-step method

- It can use the predictor and corrector approaches which are termed as the *open* and *closed* formulae.

- The **open** formula is known as the *Adams-Bashforth* predictor, which can be derived from the Taylor series at x_i:

$$y_{i+1} = y_i + f_i h + \frac{f_i' h^2}{2} + \frac{f_i'' h^3}{6} + \cdots$$
$$= y_i + h\left[f_i + \frac{h}{2} f_i' + \frac{h^2}{6} f_i'' + \cdots \right]$$

Taking

$$f_i' = \frac{f_i - f_{i-1}}{h} + \frac{f_i''}{2} h + O(h^2)$$

$$y_{i+1} = y_i + h\left[f_i + \frac{h}{2}\left(\frac{f_i - f_{i-1}}{h} + \frac{f_i''}{2} h + O(h^2) \right) + \frac{h^2}{6} f_i'' + \cdots \right]$$

which can be summarised in form of a *second order* equation:

$$y_{i+1} = y_i + h\left(\tfrac{3}{2} f_i - \tfrac{1}{2} f_{i-1}\right) + \tfrac{5}{12} h^3 f_i'' + O(h^4) \qquad (6.13)$$

- The general form of the *n*-th order Adams-Bashforth predictor is:

$$y_{i+1} = y_i + h\sum_{k=0}^{n-1} \beta_k f_{i-k} + O\left(h^{n+1}\right) \tag{6.14}$$

TABLE 6.2 Coefficients for the Adams-Bashforth predictor

Order	β_0	β_1	β_2	β_3	Error
1	1	-	-	-	$\frac{1}{2}h^2 f'(\xi)$
2	$\frac{3}{2}$	$-\frac{1}{2}$	-	-	$\frac{5}{12}h^3 f''(\xi)$
3	$\frac{23}{12}$	$-\frac{16}{12}$	$\frac{5}{12}$	-	$\frac{9}{24}h^4 f'''(\xi)$
4	$\frac{55}{24}$	$-\frac{59}{24}$	$\frac{37}{24}$	$-\frac{9}{24}$	$\frac{251}{752}h^5 f^{(4)}(\xi)$

- The closed formula is known as the *Adams-Moulton* corrector, which can be derived from the Taylor series at x_{i+1}:

$$y_i = y_{i+1} - f_{i+1}h + \frac{f'_{i+1}h^2}{2} - \frac{f''_{i+1}h^3}{6} + \cdots$$

$$y_{i+1} = y_i + h\left[f_{i+1} - \frac{h}{2} f'_{i+1} + \frac{h^2}{6} f''_{i+1} + \cdots \right]$$

Taking

$$f'_{i+1} = \frac{f_{i+1} - f_i}{h} + \frac{f''_{i+1}}{2} h + O\left(h^2\right)$$

which produces a *second order* equation:

$$y_{i+1} = y_i + h\left(\tfrac{1}{2} f_{i+1} + \tfrac{1}{2} f_i\right) - \tfrac{1}{12} h^2 f''_{i+1} - O\left(h^4\right) \tag{6.15}$$

- The general form of the *n*-th order Adams-Moulton corrector is:

$$y_{i+1} = y_i + h\sum_{k=0}^{n-1} \beta_k f_{i+1-k} + O\left(h^{n+1}\right) \tag{6.16}$$

TABLE 6.2 Coefficients for the Adams-Moulton corrector

Order	β_0	β_1	β_2	β_3	Error
2	$\frac{1}{2}$	$\frac{1}{2}$	-	-	$-\frac{1}{12}h^3 f''(\xi)$
3	$\frac{5}{12}$	$\frac{8}{12}$	$-\frac{1}{12}$	-	$-\frac{1}{24}h^4 f'''(\xi)$
4	$\frac{9}{24}$	$\frac{19}{24}$	$-\frac{5}{24}$	$\frac{1}{24}$	$-\frac{19}{720}h^5 f^{(4)}(\xi)$

Example 6.4

Use the combination of the fourth order Adams-Bashforth predictor and the Adams Moulton corrector to solve the following equation at $x = 1$:

$$y' = \frac{dy}{dx} = 4e^{0.8x} - 0.5y$$

Given that the initial condition is $y(0) = 2$ and the step size is $h = 1$, and the information at the three previous points are:

$$x_{i-3} = -3: \quad y_{i-3} = -4.547302$$
$$x_{i-2} = -2: \quad y_{i-2} = -2.306160$$
$$x_{i-1} = -1: \quad y_{i-1} = -0.392995$$

As a guidance, the exact solution is:

$$y = \frac{4}{1.3}\left(e^{0.8x} - e^{-0.5x}\right) + 2e^{-0.5x}$$

Solution

The values of derivatives at all points:

$$f_i = 4e^{0.8(0)} - 0.5(2) = 3$$
$$f_{i-1} = 4e^{0.8(-1)} - 0.5(-0.392995) = 1.993814$$
$$f_{i-2} = 4e^{0.8(-2)} - 0.5(-2.306160) = 1.960667$$
$$f_{i-3} = 4e^{0.8(-3)} - 0.5(-4.547302) = 2.636523$$

Then, use the Adams-Bashforth predictor:

$$y_{i+1}^0 = y_i + h\left[\tfrac{55}{24} f_i - \tfrac{59}{24} f_{i-1} + \tfrac{37}{24} f_{i-2} - \tfrac{9}{24} f_{i-3}\right]$$
$$y_1^0 = 2 + (1)\left[\tfrac{55}{24}(3) - \tfrac{59}{24}(1.993814) + \tfrac{37}{24}(1.960667) - \tfrac{9}{24}(2.636523)\right]$$
$$= 6.007536$$

Comparison with the exact solution:

$$y(1) = \frac{4}{1.3}\left(e^{0.8(1)} - e^{-0.5(1)}\right) + 2e^{-0.5(1)} = 6.194631$$

$$\varepsilon_t = 3.1\%$$

Next, use the Adams-Moulton corrector:

$$\begin{aligned} f_{i+1} &= 4e^{0.8x_{i+1}} - 0.5y_{i+1}^0 \\ &= 4e^{0.8(1)} - 0.5(6.007536) = 5.898394 \end{aligned}$$

Hence,

$$\begin{aligned} y_{i+1}^1 &= y_i + h\left[\tfrac{9}{24} f_{i+1} + \tfrac{19}{24} f_i - \tfrac{5}{24} f_{i-1} + \tfrac{1}{24} f_{i-2}\right] \\ y_1^1 &= 2 + (1)\left[\tfrac{9}{24}(5.898394) + \tfrac{19}{24}(3) - \tfrac{5}{24}(1.993814) + \tfrac{1}{24}(1.960667)\right] \\ &= 6.253214 \end{aligned}$$

$$\varepsilon_t = -0.96\%$$

6.4 System of Equations

- Generally, a system of ODEs containing *n* equations (thus requires *n* initial conditions) can be written as:

$$\begin{aligned}\frac{dy_1}{dx} &= f_1(x, y_1, y_2, y_3, \cdots, y_n) \\ \frac{dy_2}{dx} &= f_2(x, y_1, y_2, y_3, \cdots, y_n) \\ &\vdots \\ \frac{dy_n}{dx} &= f_n(x, y_1, y_2, y_3, \cdots, y_n)\end{aligned} \tag{6.17}$$

- The concept of a system of ODEs can also be used to solve a higher order ODE by decomposing into several first order ODEs, for example:

$$\frac{d^3y}{dx^3} + a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x, y)$$

having three initial conditions:

$$\text{At } x = 0: \quad y = \alpha, \quad \frac{dy}{dx} = \beta, \quad \frac{d^2y}{dx^2} = \gamma$$

can be converted into:

$$\begin{aligned}\frac{dy_1}{dx} &= y_2 \\ \frac{dy_2}{dx} &= y_3 \\ \frac{dy_3}{dx} &= f(x, y_1) - ay_3 - by_2 - cy_1\end{aligned}$$

and, the three initial conditions now become:

$$\text{At } x = 0: \quad y_1 = \alpha, \quad y_2 = \beta, \quad y_3 = \gamma$$

Example 6.5

Use the Euler method to solve the following system:

$$\frac{dy_1}{dx} = -0.5y_1$$

$$\frac{dy_2}{dx} = 4 - 0.3y_2 - 0.1y_1$$

from $x = 0$ to $x = 2$. Use the initial conditions of $y_1(0) = 4$ and $y_2(0) = 6$, and a step size of $h = 0.5$.

Solution

For the first step:

$$y_1(0.5) = 4 + [-0.5(4)]0.5 = 3$$

$$y_2(0.5) = 6 + [4 - 0.3(6) - 0.1(4)]0.5 = 6.9$$

The next steps are as followed:

x	y_1	y_2
0	4	6
0.5	3	6.9
1.0	2.25	7.715
1.5	1.6875	8.44525
2.0	1.265625	9.094087

Example 6.6

Repeat Ex. 6.5 using the midpoint method.

Solution

For the first step:

$$k_{1,1} = -0.5(4) = -2$$
$$k_{2,1} = 4 - 0.3(6) - 0.1(4) = 1.8$$
$$k_{1,2} = -0.5\left[4 + \tfrac{1}{2}(-2)(0.5)\right] = -1.75$$
$$k_{2,2} = 4 - 0.3\left[6 + \tfrac{1}{2}(1.8)(0.5)\right] - 0.1\left[4 + \tfrac{1}{2}(-2)(0.5)\right] = 1.715$$

Then, at $x = 0.5$:

$$y_1(0.5) = 4 + (-1.75)(0.5) = 3.125$$
$$y_2(0.5) = 6 + (1.715)(0.5) = 6.8575$$

The next steps are as followed:

i	x_i	$y_{1,i}$	$y_{2,i}$
0	0	4	6
1	0.5	3.125	6.8575
2	1.0	2.44141	7.63102
3	1.5	1.90735	8.32456
4	2.0	1.49012	8.94324

6.5 Boundary Value Problems

- An ODE of an order more than one usually require more than one condition, such a problem is referred to as the *boundary value* problem.

- For example, a second order ODE require two conditions and can be written in form of

$$\frac{d^2y}{dx^2} = f(x,y) \tag{6.18}$$

with the *boundary* conditions:

$$x = 0: \quad y = y_0$$
$$x = L: \quad y = y_L$$

- For a this problem, the *finite difference method* can be used where the second order derivatives can be discretised via central difference as:

$$\frac{d^2y}{dx^2} \cong \frac{\left(\frac{y_{i+1} - y_i}{\Delta x}\right) - \left(\frac{y_i - y_{i-1}}{\Delta x}\right)}{\Delta x}$$
$$\cong \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2}$$

Hence, Eq. (6.18) becomes

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} = f(x_i, y_i) \tag{6.20}$$

- The accuracy of this method depends on the selected step size Δx.

Example 6.7

Obtain a temperature distribution along a slender rod shown in Fig. 6.4, where its thermal behaviour can be represented by the following equation:

$$\frac{d^2T}{dx^2} + \alpha(T_a - T) = 0$$

Its boundary conditions are $T(0) = 40$°C and $T(L) = 200$°C. Use $L = 10$ m, $\alpha = 0.01$ m^{-2}, $T_a = 20$°C and a step size of $h = 2$ m.

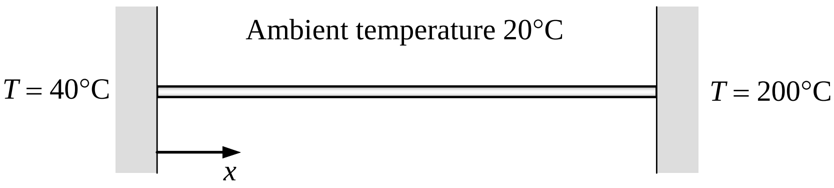


FIGURE 6.4 A slender rod having thermal loss through convection

For comparison, the analytical solution is

$$T = 73.4523e^{0.1x} + 53.4523e^{-0.1x} + 20$$

Solution

The discretised equation can be rewritten as

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{2^2} - 0.01(T_i - 20) = 0$$

$$-T_{i-1} + 2.04T_i - T_{i+1} = 0.8$$

Then, the following equations can be formed:

$$\begin{aligned} 2.04T_1 - T_2 &= 0.8 + 40 = 40.8 \\ -T_1 + 2.04T_2 - T_3 &= 0.8 \\ -T_2 + 2.04T_3 - T_4 &= 0.8 \\ -T_3 + 2.04T_4 &= 0.8 + 200 = 200.8 \end{aligned}$$

or, in a matrix form:

$$\begin{bmatrix} 2.04 & -1 & 0 & 0 \\ -1 & 2.04 & -1 & 0 \\ 0 & -1 & 2.04 & -1 \\ 0 & 0 & -1 & 2.04 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 40.8 \\ 0.8 \\ 0.8 \\ 200.8 \end{Bmatrix}$$

$$T_1 = 65.9698, \quad T_2 = 93.7785, \quad T_3 = 124.5382, \quad T_4 = 159.4795$$

Comparison with analytical solution:

x	T	T_{exact}	$\lvert\varepsilon_t\rvert$ (%)
0	40	40	-
2	65.9698	65.9518	0.0273
4	93.7785	93.7478	0.0327
6	124.5382	124.5036	0.0278
8	159.4795	159.4534	0.0164
10	200	200	-

6.6 Characteristic Value Problems

- There are problems in elasticity and vibration which are categorised as *characteristic value* problems when the second order ODEs are solved in their *homogeneous* form.

- For example, consider Eq. 6.1 for a system of spring without damping having one single degree of freedom:

$$m\frac{d^2x}{dt^2}+kx=0 \qquad x(0)=0, x(1)=0 \tag{6.21}$$

or, in another form:

$$\frac{d^2x}{dt^2}+\omega^2 x=0 \tag{6.22}$$

where $\omega=\sqrt{k/m}$ is the natural frequency of the system.

- The general solution of Eq. 6.22 is

$$x = A\sin\omega t + B\cos\omega t$$

By using the boundary conditions in Eq. 6.21:

$$\omega = \pm m\pi \qquad m=1,2,\ldots$$

Hence,

$$x = A\sin m\pi t$$

- For a system having *n* degree of freedom, the mass and stiffness properties can be represented in form of matrices, **M** and **K**, respectively:

$$\mathbf{M}\cdot\frac{d^2\mathbf{x}}{dt^2}+\mathbf{K}\cdot\mathbf{x}=\mathbf{0} \tag{6.23}$$

Using the similar general solution as above, Eq. 6.23 can be converted to

$$\left[\mathbf{K}-\omega^2\mathbf{M}\right]\cdot\{\mathbf{x}\}=\{\mathbf{0}\} \tag{6.24}$$

Pre-multiplication of Eq. 6.24 with $\mathbf{M}^{-1}$ produces a general *eigen equation*:

$$\mathbf{M}^{-1}\left[\mathbf{K}-\omega^2\mathbf{M}\right]\cdot\{\mathbf{x}\}=\left[\mathbf{M}^{-1}\mathbf{K}-\omega^2\mathbf{M}^{-1}\mathbf{M}\right]\cdot\{\mathbf{x}\}=\{\mathbf{0}\}$$

$$[\mathbf{A} - \lambda\mathbf{I}] \cdot \{\mathbf{x}\} = \{\mathbf{0}\} \tag{6.25}$$

where $\mathbf{A} = \mathbf{M}^{-1}\mathbf{K}$ is a square matrix and $\lambda = \omega^2$ is the *eigen value* for **A**.

- For a non-zero solution of **x**:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = |\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{6.26}$$

$$\mathbf{A} \cdot \mathbf{x} = \lambda\mathbf{x} \tag{6.27}$$

- Eq. 6.26 is known as the characteristic equation and its solution produces *n* eigen values (real or complex), and the corresponding solutions **x** which follows Eq. 6.27 is called *eigen vector*.

- To obtain eigen values, Eq. 6.26 can be rewritten as

$$f_n(\lambda) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} - \lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0 \tag{6.28}$$

which can be expanded into an *n*-th order polynomials of λ.

- To obtain the *maximum* real eigen value, the **power method** can be used, where Eq. 6.27 can be modified as followed

$$\lambda\mathbf{x}_{i+1} = \mathbf{A} \cdot \mathbf{x}_i \tag{6.29}$$

During the calculation, the $\mathbf{x}_{i+1}$ vector is normalised such that its maximum component is always unity.

Example 6.8

Obtain the maximum eigen values and the corresponding eigen vector using the power method:

$$\begin{pmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{pmatrix}$$

Perform iteration until the solution converges at 4 decimal places. Use an initial vector of $\mathbf{v}_0 = (1,1,1)^{\mathrm{T}}$.

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For the first iteration:

$$\begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 2 \\ -1 \\ 0 \end{Bmatrix} = 2 \begin{Bmatrix} 1 \\ -0.5 \\ 0 \end{Bmatrix}$$

For the second iteration:

$$\begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{Bmatrix} 1 \\ -0.5 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 3.5 \\ -4 \\ 0.5 \end{Bmatrix} = -4 \begin{Bmatrix} 0.875 \\ 1 \\ -0.125 \end{Bmatrix}$$

For the third iteration:

$$\begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{Bmatrix} 0.875 \\ 1 \\ -0.125 \end{Bmatrix} = \begin{Bmatrix} -3.625 \\ 6.125 \\ -1.125 \end{Bmatrix} = 6.125 \begin{Bmatrix} -0.5918 \\ 1 \\ -0.1837 \end{Bmatrix}$$

For the fourth iteration:

$$\begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{Bmatrix} -0.5918 \\ 1 \\ -0.1837 \end{Bmatrix} = \begin{Bmatrix} -2.7755 \\ 5.7347 \\ -1.1837 \end{Bmatrix} = 5.7347 \begin{Bmatrix} -0.4840 \\ 1 \\ -0.2064 \end{Bmatrix}$$

And it converges after 15 iterations:

$$\begin{bmatrix} 3 & -1 & 0 \\ -2 & 4 & -3 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{Bmatrix} -0.4037 \\ 1 \\ -0.2233 \end{Bmatrix} = \begin{Bmatrix} -2.2111 \\ 5.4774 \\ -1.2233 \end{Bmatrix} = 5.4774 \begin{Bmatrix} -0.4037 \\ 1 \\ -0.2233 \end{Bmatrix}$$

Hence, the maximum eigen value is 5.4774 and the corresponding eigen vector is $(-0.4037, 1, -0.2233)^T$.

- If there is a need to determine the *minimum* real eigen value, the power method can be modified to form the **inverse power method** from Eq. 6.27:

$$\mathbf{A}^{-1}\mathbf{A} \cdot \mathbf{x} = \lambda \mathbf{A}^{-1} \cdot \mathbf{x}$$

$$\frac{1}{\lambda}\mathbf{x}_{i+1} = \mathbf{A}^{-1} \cdot \mathbf{x}_i \qquad (6.30)$$

Exercises

1. Solve the following system of ordinary differential equations:

$$\frac{dy_1}{dx} = x + y_1 y_2 , \quad y_1(0) = 0 ,$$
$$\frac{dy_2}{dx} = x - y_1 , \quad y_2(0) = 1 ,$$

from $x = 0$ to $x = 2$ using the Heun method with the step size of $h = 0.2$. Plot the graphs of y_1 and y_2 against x, and the graph of y_2 against y_1.

2. The following equation can be used to model the deflection $y(z)$ of a fixed slender column having a wind loading:

$$\frac{d^2 y}{dz^2} = \frac{f}{2EI}(L - z)^2 , \quad y'(0) = y(0) = 0 ,$$

where E is the modulus of elasticity (1.2×10^8 N/m^2), I is the area moment of inertia (0.05 m^4), L is the column height (30 m), and f is the distribution function of wind force which varies with elevation and is given by

$$f(z) = \frac{200z}{5 + z} e^{-2z/30} \ \text{[N/m]}$$

Perform the calculation using the third order Runge-Kutta method to obtain the deflection at all position along the column. Use the step size of 5 m.

3. Obtain the largest and smallest eigen values for the following matrix, as well as the corresponding eigen vectors:

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 3 & 3 \end{bmatrix}$$

For each case, use the initial eigen vector of $\mathbf{v}_0 = (1,1,1)^T$.