7

PARTIAL DIFFERENTIAL EQUATIONS

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7.1 Introduction

- *Partial Differential Equations* (PDE) are differential equations which have at least two independent variables, e.g.

$$\frac{\partial^2 u}{\partial x^2} + 4xy\frac{\partial^2 u}{\partial y^2} + u = 3 \qquad \text{second order \& linear}$$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^3 + 6\frac{\partial^3 u}{\partial x \partial y^2} = x \qquad \text{third order \& nonlinear}$$

- For a second order linear two-dimensional equation, a general equation is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \qquad (7.1)$$

which can be divided into three types.

TABLE 7.1 Types of second order linear PDEs

$B^2 - 4AC$	**Type**	**Examples**
< 0	Elliptic	$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$ Laplace equation
$= 0$	Parabolic	$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$ Heat conduction equation
> 0	Hyperbolic	$\frac{\partial^2 y}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial x^2}$ Wave equation

7.2 Elliptic Equations

- **Elliptic PDEs** are generally related to *steady-state problems with diffusivity* having boundary conditions, e.g. the Laplace equation.

- Consider a steady-state 2-D heat conduction problem:

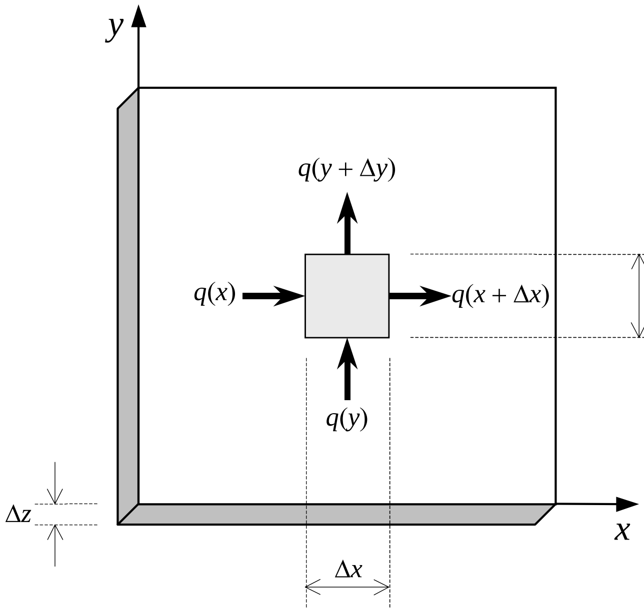


FIGURE 7.1 Thin plate having a thickness Δz with temperatures at the boundary

Taking $q(x)$ as the heat flux [J/m^2·s], the heat flow over the element can be written in a time interval of Δt as:

$$q(x)\Delta y\,\Delta z\,\Delta t + q(y)\Delta x\,\Delta z\,\Delta t = q(x+\Delta x)\Delta y\,\Delta z\,\Delta t + q(y+\Delta y)\Delta x\,\Delta z\,\Delta t$$

which can be summarised into

$$\left(q(x) - q(x+\Delta x)\right)\Delta y + \left(q(y) - q(y+\Delta y)\right)\Delta x = 0$$
$$\frac{q(x) - q(x+\Delta x)}{\Delta x} + \frac{q(y) - q(y+\Delta y)}{\Delta y} = 0$$

For $\Delta x, \Delta y \to 0$:

$$-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} = 0 \qquad (7.2)$$

- In conduction heat transfer, the relationship between heat flux and temperature is given by the Fourier Law:

$$q_x = -k\rho C\frac{\partial T}{\partial x} \quad , \quad q_y = -k\rho C\frac{\partial T}{\partial y} \tag{7.3}$$

where k is heat diffusivity [m^2/s], ρ is density [kg/m^3] and C is heat capasity [J/kg°C]. Combining Eq. (7.2) with Eq. (7.3) gives the *Laplace equation*:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{7.4}$$

- If a heat source or heat generation of $Q(x, y)$ is present in the domain, a *Poisson equation* can be formed:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + Q(x, y) = 0 \tag{7.5}$$

- To solve Eq. (7.4), use the central differencing:

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$$

Hence, Eq. (7.4) can be written in an algebraic form:

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

Taking $\Delta x = \Delta y$ gives

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0 \tag{7.6}$$

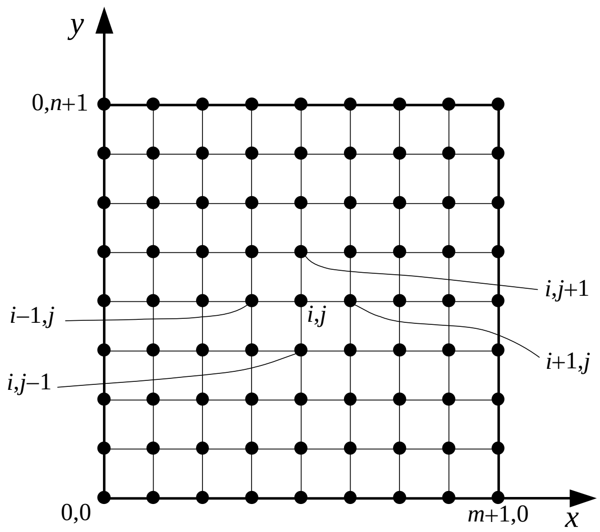


FIGURE 7.2 Finite difference grid for solving the Laplace equation

- Eq. (7.6) needs *boundary conditions* (BC), which may be in the form of:

  1. Fixed value — *Dirichlet* (see Fig. 7.3), e.g. Fixed temperature
  2. First derivative, or gradient —*Neumann*, e.g. Fix heat flux or insulated

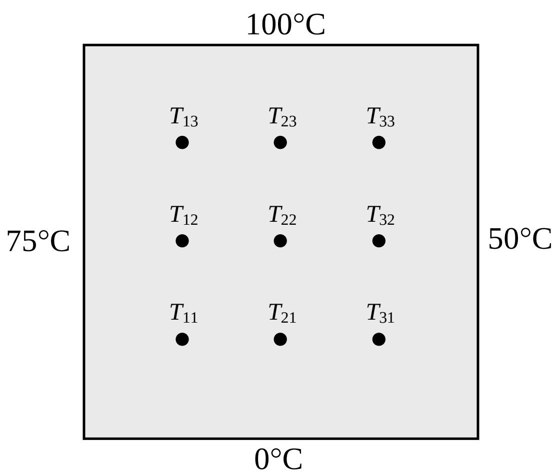


FIGURE 7.3 Grid/Node representatioan and BC for Fig. 7.1

For node (1,1) — $T_{01} = 75°C$ and $T_{10} = 0°C$:

$$T_{21} + T_{01} + T_{12} + T_{10} - 4T_{11} = 0$$

$$4T_{11} - T_{12} - T_{21} = 75$$

Hence, the complete system (can be assembled into a matrix equation) is:

$$
\begin{array}{ccccccccccc}
4T_{11} & -T_{21} & & -T_{12} & & & & & & = & 75 \\
-T_{11} & +4T_{21} & -T_{31} & & -T_{22} & & & & & = & 0 \\
& -T_{21} & +4T_{31} & & & -T_{32} & & & & = & 50 \\
-T_{11} & & & +4T_{12} & -T_{22} & & -T_{13} & & & = & 75 \\
& -T_{21} & & -T_{12} & +4T_{22} & -T_{32} & & -T_{23} & & = & 0 \\
& & -T_{31} & & -T_{22} & +4T_{32} & & & -T_{33} & = & 50 \\
& & & -T_{12} & & & +4T_{13} & -T_{23} & & = & 175 \\
& & & & -T_{22} & & -T_{13} & +4T_{23} & -T_{33} & = & 100 \\
& & & & & -T_{32} & & -T_{23} & +4T_{33} & = & 150
\end{array}
$$

- If the Gauss-Seidel method is used, at each node (i,j):

$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4} \tag{7.7}$$

which can be solve iteratively via *relaxation* approach:

$$T_{i,j}^{\text{new}} = \lambda T_{i,j}^{\text{new}} + (1-\lambda) T_{i,j}^{\text{old}} \tag{7.8}$$

where λ is the *relaxation parameter* $1 \le \lambda \le 2$, and is subjected to the termination criterion as followed:

$$\varepsilon_{a_{i,j}} = \left| \frac{T_{i,j}^{\text{new}} - T_{i,j}^{\text{old}}}{T_{i,j}^{\text{new}}} \right| \times 100\%$$

Example 7.1

Solve the problem in Fig. 7.3 using the Gauss-Seidel method using the termination criterion $\varepsilon_a \le 1\%$ dan the relaxation parameter $\lambda = 1.5$.

Solution

Let all the initial values be 0°C. For the first iteration:

$$T_{11} = \frac{T_{21} + T_{01} + T_{12} + T_{10}}{4} = \frac{0 + 75 + 0 + 0}{4} = 18.75$$

$$T_{11}^{\text{new}} = 1.5(18.75) + (1-1.5)0 = 28.125$$

$$T_{21} = \frac{T_{31} + T_{11} + T_{22} + T_{20}}{4} = \frac{0 + 28.125 + 0 + 0}{4} = 7.03125$$

$$T_{21}^{\text{new}} = 1.5(7.03125) + (1 - 1.5)0 = 10.54688$$

$$T_{31} = \frac{T_{41} + T_{21} + T_{32} + T_{30}}{4} = \frac{50 + 10.54688 + 0 + 0}{4} = 15.13672$$

$$T_{31}^{\text{new}} = 1.5(15.13672) + (1 - 1.5)0 = 2.70508$$

and, for other nodes:

$$T_{12} = 38.67188 \qquad T_{22} = 18.45703 \qquad T_{23} = 34.18579$$
$$T_{13} = 80.12696 \qquad T_{23} = 74.46900 \qquad T_{33} = 96.99554$$

Error for node (1,1) — for the first iteration, all errors are 100%:

$$\varepsilon_{a_{11}} = \left| \frac{28.125 - 0}{28.125} \right| \times 100 = 100\%$$

For the second iteration:

$$T_{11} = 32.51953 \qquad T_{21} = 22.35718 \qquad T_{31} = 28.60108$$
$$T_{12} = 57.95288 \qquad T_{22} = 61.63333 \qquad T_{32} = 71.86833$$
$$T_{13} = 75.21973 \qquad T_{23} = 87.95872 \qquad T_{33} = 67.68736$$

The process is repeated until the ninth iteration in which the termination criterion is fulfilled ($\varepsilon_a < 1\%$):

$$T_{11} = 43.00061 \qquad T_{21} = 33.29755 \qquad T_{31} = 33.88506$$
$$T_{12} = 63.21152 \qquad T_{22} = 56.11238 \qquad T_{32} = 52.33999$$
$$T_{13} = 78.58718 \qquad T_{23} = 76.06402 \qquad T_{33} = 69.71050$$

7.3 Parabolic Equations

- **Parabolic PDEs** are generally related to *transient problems with diffusivity*, e.g. the 1-D heat conduction equation.

- For a transient problem, there are three approaches:

  1. **Explicit** method,
  2. **Implicit** method,
  3. **Semi-implicit** method — the *Crank-Nicolson* method.

- Consider a transient 1-D heat conduction problem:

$$\text{input} - \text{output} = \text{storage}$$

$$q(x)\Delta y\,\Delta z\,\Delta t + q(x+\Delta x)\Delta y\,\Delta z\,\Delta t = \Delta x\,\Delta y\,\Delta z\,\rho C\,\Delta T$$

FIGURE 7.4 A rod with different temperature at its ends

Dividing with the volume $\Delta x\,\Delta y\,\Delta z$ and the time interval Δt:

$$\frac{q(x) - q(x+\Delta x)}{\Delta x} = \rho C\,\frac{\Delta T}{\Delta t}$$

In the limits of $\Delta x, \Delta t \rightarrow 0$:

$$-\frac{\partial q}{\partial x} = \rho C\frac{\partial T}{\partial t} \tag{7.9}$$

By using the Fourier law, Eq. (7.3), Eq. (7.9) becomes

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{7.10}$$

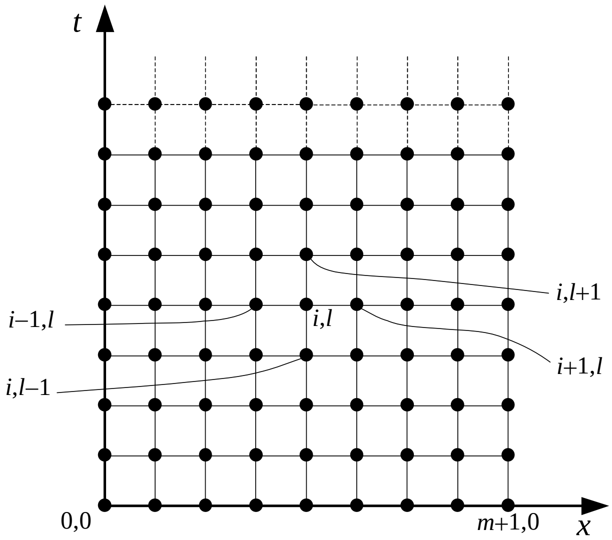


FIGURE 7.5 Finite difference grid for the heat conduction equation

- For the **explicit method**, the right-hand side of Eq. (7.9) can be discretised via *central difference* as:

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{(\Delta x)^2}$$

and, the left-hand side can be discretised via *forward difference* as:

$$\frac{\partial T}{\partial t} = \frac{T_i^{l+1} - T_i^l}{\Delta t}$$

Hence, Eq. (7.10) can be written in the algebraic form:

$$k\,\frac{T_{i+1}^l - 2T_i^l + T_{i-1}^l}{(\Delta x)^2} = \frac{T_i^{l+1} - T_i^l}{\Delta t} \qquad (7.11)$$

$$T_i^{l+1} = T_i^l + \lambda\left(T_{i+1}^l - 2T_i^l + T_{i-1}^l\right) \qquad (7.12)$$

where $\lambda = k\,\Delta t/(\Delta x)^2$.

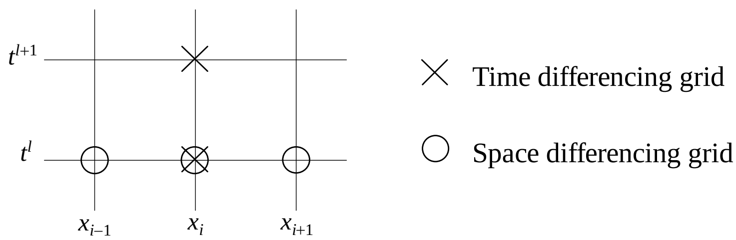


FIGURE 7.6 Computational grid for the explicit method

Example 7.2

Use the explicit method to determine the temperature distribution for a slender rod having a length of 10 cm. At time $t = 0$, the temperature of the rod is 20°C and the boundary conditions are $T(0) = 100°C$ and $T(10\text{ cm}) = 50°C$. Use the conduction coefficient $k = 0.835\text{ cm}^2/\text{s}$, the time interval $\Delta t = 0.5$ s and the step size $\Delta x = 2$ cm.

Solution

Calculate λ:

$$\lambda = \frac{k\,\Delta t}{(\Delta x)^2} = \frac{(0.835)(0.5)}{2^2} = 0.104375$$

At $t = 0$:

$$T_1^0 = T_2^0 = T_3^0 = T_4^0 = 20$$

Use Eq. (7.12). At $t = 0.5$ s:

$$T_1^1 = 20 + 0.1044[20 - 2(20) + 100] = 28.35$$
$$T_2^1 = 20 + 0.1044[20 - 2(20) + 20] = 20$$
$$T_3^1 = 20 + 0.1044[20 - 2(20) + 20] = 20$$
$$T_4^1 = 20 + 0.1044[50 - 2(20) + 20] = 23.1313$$

At $t = 1$ s:

$$T_1^2 = 28.352 + 0.1044[20 - 2(28.352) + 100] = 34.9569$$
$$T_2^2 = 20 + 0.1044[20 - 2(20) + 28.352] = 20.8715$$
$$T_3^2 = 20 + 0.1044[23.132 - 2(20) + 20] = 20.3268$$
$$T_4^2 = 23.132 + 0.1044[50 - 2(23.132) + 20] = 25.6089$$

The calculation can be continued to produce the result as in Fig. 7.7.

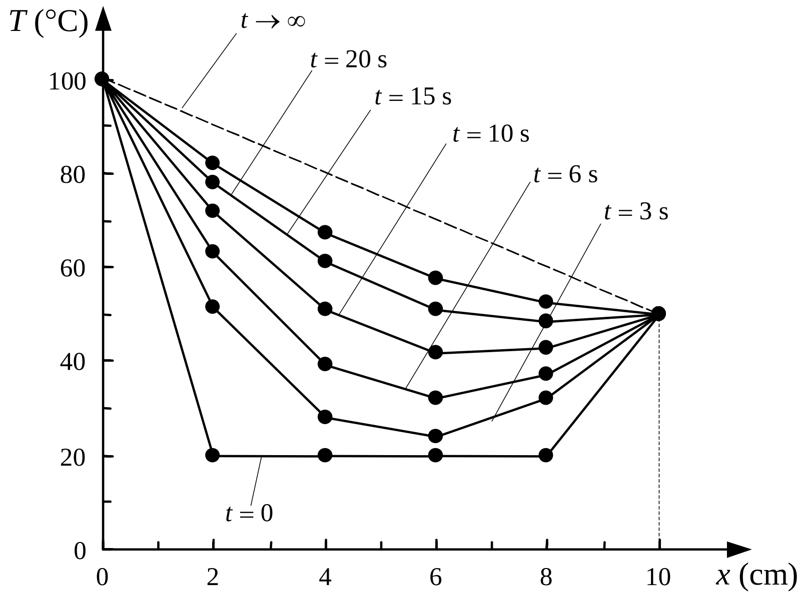


FIGURE 7.7 Result for Ex. 7.2

- For the explicit method, a more accurate result can be obtained if Δx and Δt approach zeros. However, *stability* of the results is only obtained if:

$$\lambda \le \tfrac{1}{2}$$

$$\text{or} \quad \Delta t \le \tfrac{1}{2}\frac{(\Delta x)^2}{k}$$

If the stability condition is not fulfilled, the results are contaminated by *oscillation*.

- To prevent oscillation, the **implicit method** can be used, where the second order spatial derivative is approximated at time l+1:

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}$$

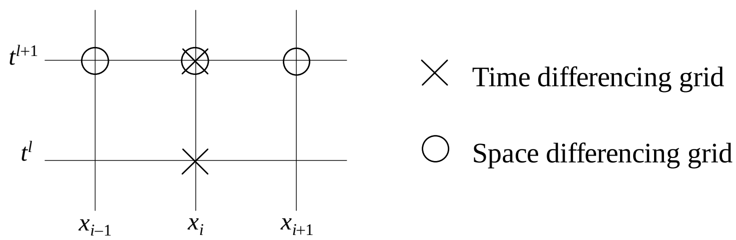


FIGURE 7.8 Computational grid for the implicit method

Hence, Eq. (7.10) becomes

$$k\frac{T_{i+1}^{l+1}-2T_i^{l+1}+T_{i-1}^{l+1}}{(\Delta x)^2}=\frac{T_i^{l+1}-T_i^l}{\Delta t} \quad (7.13)$$

$$-\lambda T_{i-1}^{l+1}+(1+2\lambda)T_i^{l+1}-\lambda T_{i+1}^{l+1}=T_i^l \quad (7.14)$$

Eq. (7.14) leads to n simultaneous linear equations having n unknowns.

Example 7.3

Repeat Ex. 7.2 using the implicit method.

Solution

From Ex. 7.2, $\lambda = 0.104375$. From Eq. (7.14):

$$-0.1044(100)+[1+2(0.1044)]T_1^1-0.1044T_2^1=20$$

$$[1+2(0.1044)]T_1^1-0.1044T_2^1=30.44$$

$$\vdots$$

Hence, the following system of linear equations can be formed:

$$\begin{bmatrix} 1.2088 & -0.1044 & 0 & 0 \\ -0.1044 & 1.2088 & -0.1044 & 0 \\ 0 & -0.1044 & 1.2088 & -0.1044 \\ 0 & 0 & -0.1044 & 1.2088 \end{bmatrix}\begin{Bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{Bmatrix}=\begin{Bmatrix} 30.44 \\ 20 \\ 20 \\ 25.22 \end{Bmatrix}$$

$$\mathbf{T}=[26.9634,\ 20.6256,\ 20.2800,\ 22.6152]^{\mathrm{T}}$$

- The combination of the explicit and implicit approaches produces the *semi-implicit* method, and one of its kind is the **Crank-Nicolson method**:

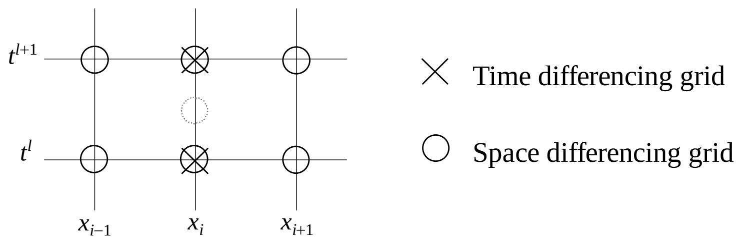


FIGURE 7.9 Computational grid for the Crank-Nicolson method

In this method, the finite difference term for the spatial derivative is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{2}\left[\frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{(\Delta x)^2} + \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{(\Delta x)^2}\right] \tag{7.15}$$

Hence, Eq. (7.10) becomes

$$-\lambda T_{i-1}^{l+1} + 2(1+\lambda)T_i^{l+1} - \lambda T_{i+1}^{l+1} = \lambda T_{i-1}^{l} + 2(1-\lambda)T_i^{l} + \lambda T_{i+1}^{l} \tag{7.16}$$

Example 7.4

Repeat Ex. 7.2 using the Crank-Nicolson method.

Solution

From Ex. 7.2, $\lambda = 0.104375$. From Eq. (7.16):

$$-0.10437(100) + 2(1+0.10437)T_1^1 - 0.10437T_2^1$$
$$= 0.10437(100) + 2(1-0.10437)20 + 0.10437(20)$$
$$2.20874T_1^1 - 0.10437T_2^1 = 58.7866$$

By considering other nodes, the following system of linear equations can be formed:

$$\begin{bmatrix} 2.2087 & -0.1044 & 0 & 0 \\ -0.1044 & 2.2087 & -0.1044 & 0 \\ 0 & -0.1044 & 2.2087 & -0.1044 \\ 0 & 0 & -0.1044 & 2.2087 \end{bmatrix} \begin{Bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{Bmatrix} = \begin{Bmatrix} 58.7866 \\ 40 \\ 40 \\ 48.3496 \end{Bmatrix}$$

$$\mathbf{T} = [27.5778,\ 20.3652,\ 20.1517,\ 22.8424]^{\mathrm{T}}$$

7.4 Hyperbolic Equations

- **Hyperbolic PDEs** are generally related to *transient problems with convection*, e.g. the 1-D wave equation.

- Consider a 1-D wave equation, which is a hyperbolic PDE:

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \qquad \textbf{(7.17)}$$

- One of the methods is the **MacCormack's technique**, which is an *explicit* finite-difference technique and is second-order-accurate in both space and time. By using the Taylor series:

$$u_i^{t+\Delta t} = u_i^t + \left(\frac{\partial u}{\partial t}\right)_{\text{avg}} \Delta t \qquad \textbf{(7.18)}$$

- This method consists of two steps: **predictor** and **corrector**. In **predictor step**, use forward difference in the right-hand side:

$$\left(\frac{\partial u}{\partial t}\right)_i^t = -a\left(\frac{u_{i+1}^t - u_i^t}{\Delta x}\right) \qquad \textbf{(7.19)}$$

Thus, from the Taylor series, the predicted value of *u* is:

$$\overline{u}_i^{t+\Delta t} = u_i^t + \left(\frac{\partial u}{\partial t}\right)_i^t \Delta t \qquad \textbf{(7.20)}$$

- In **corrector step**, by replacing the spatial derivatives with rearward differences:

$$\left(\frac{\overline{\partial u}}{\partial t}\right)_i^{t+\Delta t} = -a\left(\frac{\overline{u}_i^{t+\Delta t} - \overline{u}_{i-1}^{t+\Delta t}}{\Delta x}\right) \qquad \textbf{(7.21)}$$

The average of time derivative can be obtained using

$$\left(\frac{\partial u}{\partial t}\right)_{\text{avg}} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)_i^t + \left(\frac{\overline{\partial u}}{\partial t}\right)_i^{t+\Delta t}\right] \qquad \textbf{(7.22)}$$

- Hence the final, "corrected" value at time $t+\Delta t$ is:

$$u_i^{t+\Delta t} = u_i^t + \left(\frac{\partial u}{\partial t}\right)_{\text{avg}} \Delta t$$

- The accuracy of the solution for a hyperbolic PDE is dependent on truncation and round off errors, and the term representing it is called *artificial viscosity* $\frac{1}{2} a\, \Delta x (1 - \nu)$.

- The effect of artificial viscosity leads to *numerical dissipation*, which is originated by the even-order derivatives in the truncated term, but it improves stability.

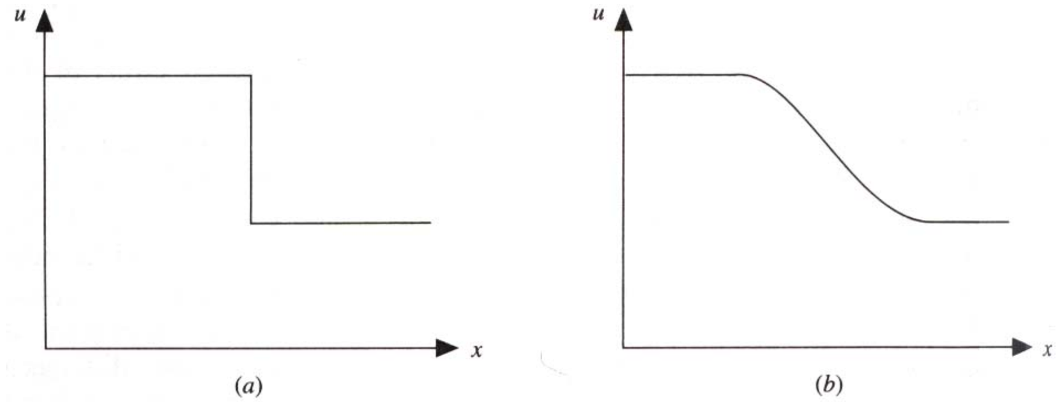


FIGURE 7.10 Numerical dissipation: (a) $t = 0$, (b) $t > 0$

- Another opposite effect is known as *numerical dispersion*, which is originated by the odd-order derivatives in the truncated term and causes 'wiggles'.

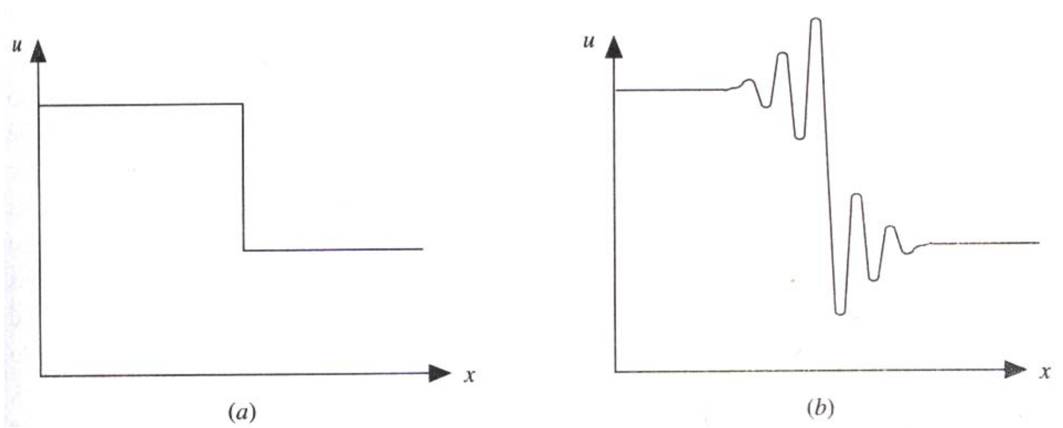


FIGURE 7.11 Numerical dispersion: (a) $t = 0$, (b) $t > 0$

7.5 Finite Element Method — An Introduction

- The **Finite Element Method** (FEM) is a computer aided mathematical technique used to obtain an approximate numerical solution of a response of a physical system which is subjected to an external loading.
- By using this technique, the computational domain which is theoretically a continuum, is being discretised in form of simple geometries.
- The *mesh* is the computational domain which is an assembly of discrete elemental blocks known as *finite elements*, and the vertices defining the elements are called *nodes*.
- Governing equation is employed at each element to form a set of algebraic equations — *local* system.
- Local equations assembled to form a *global* system which is the solved to yield a vector of variables.

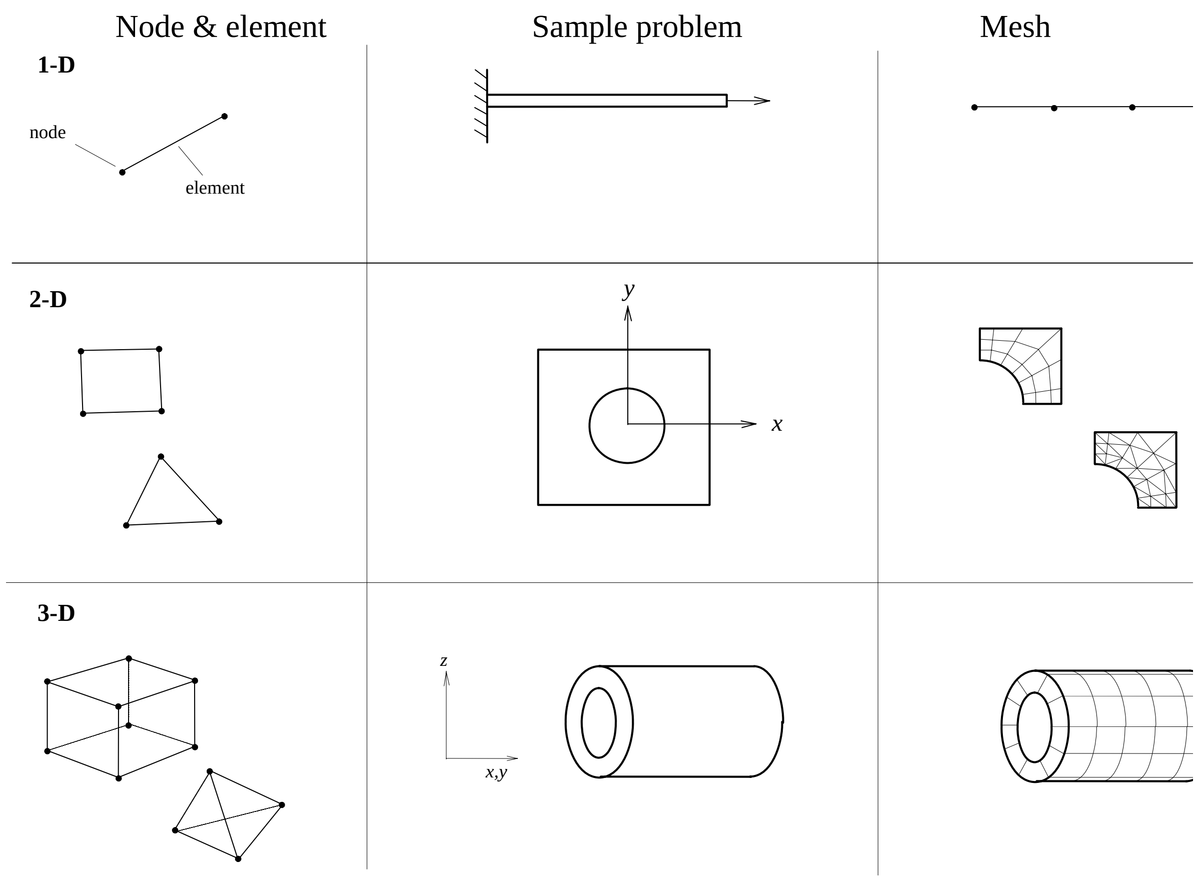


FIGURE 7.10 Examples of elements and their applications

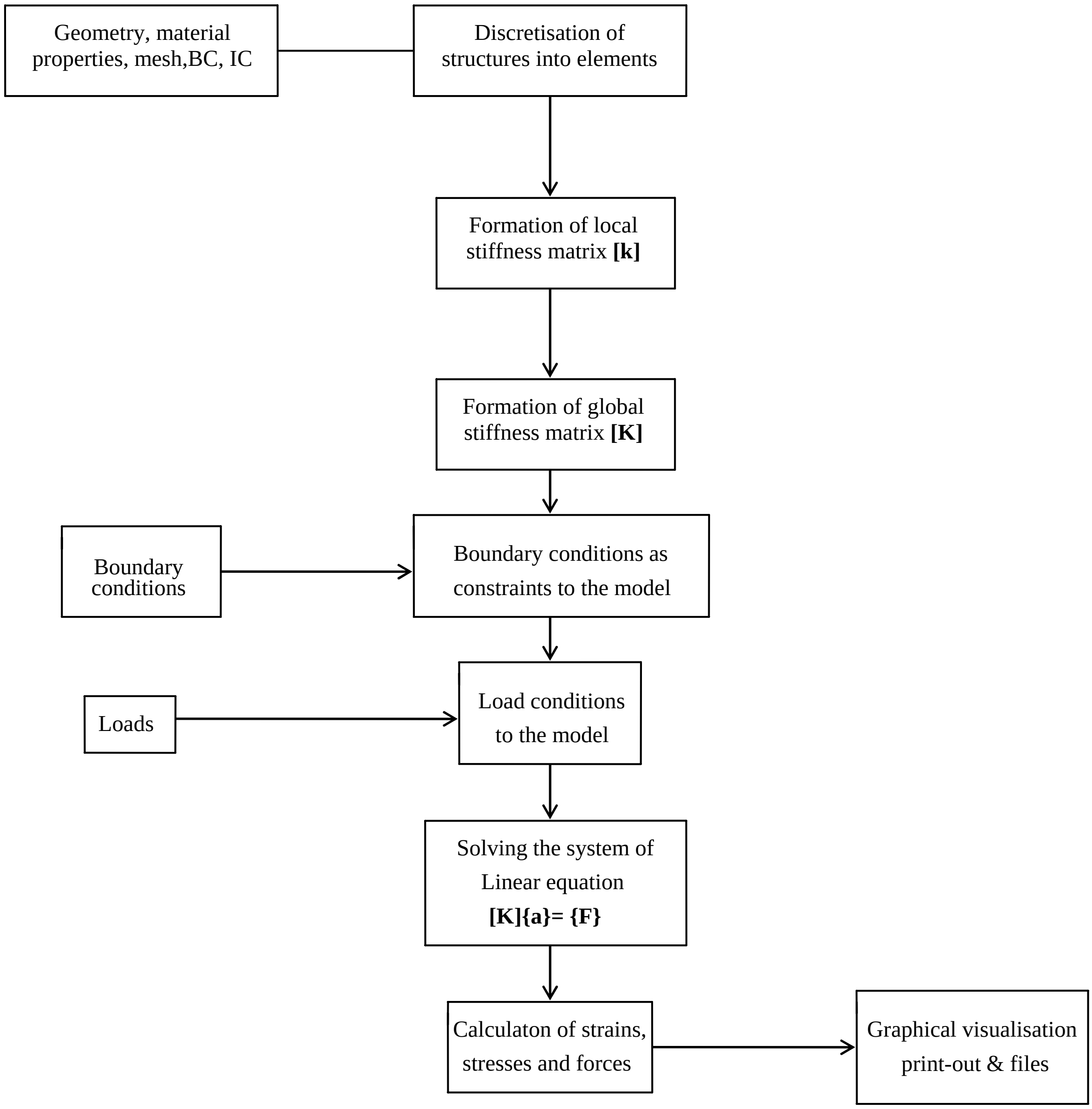


FIGURE 7.11 Algorithm for a force analysis using FEM

- Consider a 1-D steady state heat conduction:

$$k\frac{\partial^2 T}{\partial x^2} + Q(x) = 0 \qquad \textbf{(7.17)}$$

Eq. (7.17) needs appropriate boundary conditions such that:

$$\begin{aligned} T_{x=0} &= T_0 \\ q_{x=L} &= h(T_L - T_\infty) \end{aligned} \qquad \textbf{(7.18)}$$

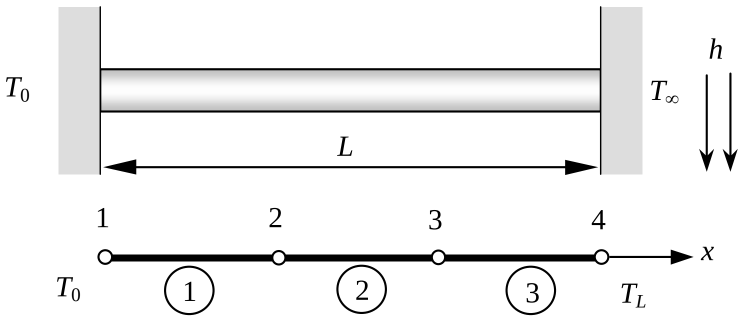


FIGURE 7.12 Boundary condition for a 1D steady state heat conduction

For the first element:

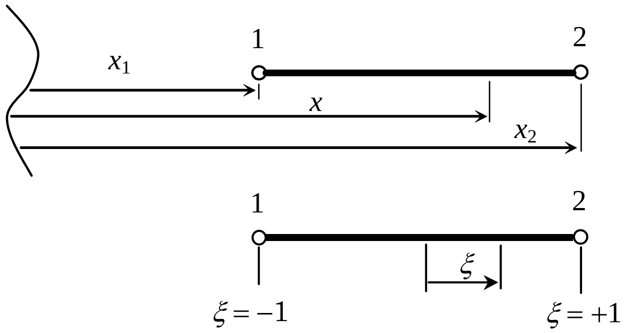


FIGURE 7.13 The first element in the finite element method

The transformation of the coordinate system to a local system is given by an *isoparametric coordinate* ξ, i.e.

$$\xi = \frac{2}{x_2 - x_1}\left(x - x_1\right) - 1 \qquad \textbf{(7.19)}$$

Thus, in order to calculate the temperature at the middle section, a linear interpolation function or a linear *shape function* can be used:

$$N_1(\xi) = \frac{1-\xi}{2}$$
$$N_2(\xi) = \frac{1+\xi}{2} \qquad \textbf{(7.20)}$$

Hence, the temperature can be interpolated using the shape function as followed:

$$T(\xi) = N_1 T_1 + N_2 T_2 = \mathbf{N}\mathbf{T}^e \qquad \textbf{(7.21)}$$

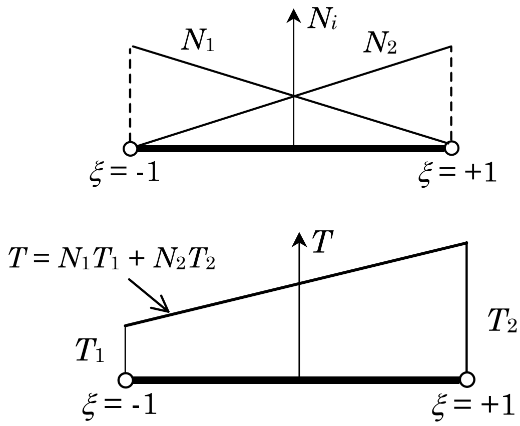


FIGURE 7.14 Linear shape function

Differentiation of Eq. (7.21) gives

$$d\xi = \frac{2}{x_2 - x_1}dx$$

Using a chain rule:

$$\begin{aligned}\frac{dT}{dx} &= \frac{dT}{d\xi}\frac{d\xi}{dx} \\ &= \frac{1}{x_2 - x_1}\left[-1, \quad 1\right]\mathbf{T}^e \\ &= \mathbf{BT}^e\end{aligned}$$

where

$$B = \frac{1}{x_2 - x_1}\left[-1, \quad 1\right]$$

- In energy form, the heat conduction problem can be represented by

$$\int_{\Omega_i} T\left\{k\frac{\partial^2 T}{\partial x^2} + Q(x)\right\}d\Omega = 0$$

$$\Pi_T = \int_0^L \tfrac{1}{2}k\left(\frac{dT}{dx}\right)^2 dx - \int_0^L TQ(x)dx + \tfrac{1}{2}h\left(T_L - T_\infty\right)^2 \qquad \textbf{(7.22)}$$

Discretisation of Eq. (7.22) gives

$$\Pi = \sum_e \tfrac{1}{2}\mathbf{T}^{e\mathrm{T}}\left[\frac{k_e l_e}{2}\int_{-1}^{1}\mathbf{B}^{\mathrm{T}}\mathbf{B}d\xi\right]\mathbf{T}^e - \sum_e\left[\frac{Q_e l_e}{2}\int_{-1}^{1}\mathbf{N}d\xi\right]\mathbf{T}^e + \tfrac{1}{2}h\left(T_L - T_\infty\right)^2 \qquad \textbf{(7.23)}$$

Hence, the stiffness matrix for the element is

$$\mathbf{k} = \frac{k_e l_e}{2}\int_{-1}^{1}\mathbf{B}^{\mathrm{T}}\mathbf{B}d\xi = \frac{k_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{7.24}$$

and, the heat rate vector is

$$\mathbf{r} = \frac{Q_e l_e}{2}\int_{-1}^{1}\mathbf{N}\,d\xi = \frac{Q_e l_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{7.25}$$

The global stiffness matrix can be produced via an assembly of all stiffness matrices for all elements, i.e.

$$\mathbf{K} \leftarrow \sum_e \mathbf{k}_e \tag{7.26}$$

Likewise, the global heat rate vector is an assembly by local heat rate vectors:

$$\mathbf{R} \leftarrow \sum_e \mathbf{r}_e \tag{7.27}$$

To combine with the boundary condition $T_1 = T_0$, a *penalty* method can be used:

$$\begin{bmatrix} (K_{11} + C) & K_{12} & \cdots & K_{1L} \\ K_{12} & K_{22} & \cdots & K_{2L} \\ \vdots & \vdots & & \vdots \\ K_{L1} & K_{L2} & \cdots & K_{LL} + h \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_L \end{Bmatrix} = \begin{Bmatrix} R_1 + CT_0 \\ R_2 \\ \vdots \\ R_L + hT\infty \end{Bmatrix} \tag{7.28}$$

or, in a matrix form

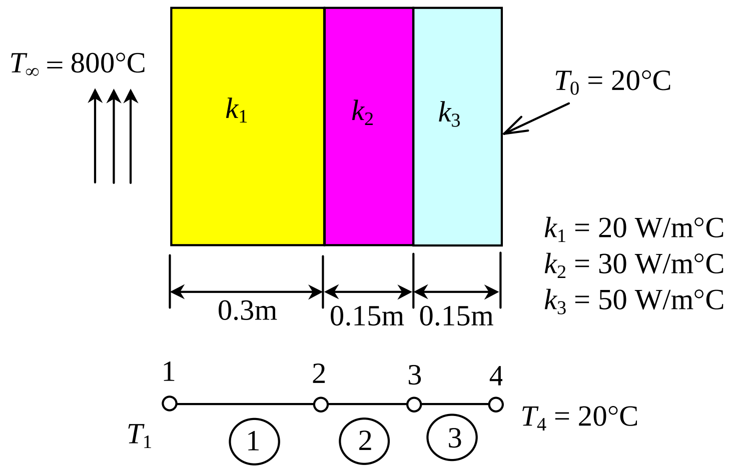
$$\mathbf{K}\cdot\mathbf{T} = \mathbf{R} \tag{7.29}$$

where the penalty parameter C can be estimated as:

$$C = \max\left|\mathbf{K}_{ij}\right| \times 10^4 \tag{7.30}$$

Example 7.5

Fig. 7.15 shows plate of three composites, where the temperature at the right-hand side is $T_0 = 20°\text{C}$ and the left-hand side is subjected to convection with $T_\infty = 800°\text{C}$ and $h = 25\ \text{W/m}^2°\text{C}$. Obtain a temperature distribution across the plate using the finite element method.



RAJAH 7.15 Plate of three composites used in Ex. 7.5

Solution

Divide the domain into three elements and construct local stiffness matrices for all elements:

$$\mathbf{k}^1 = \frac{20}{0.3}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{k}^2 = \frac{30}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\mathbf{k}^3 = \frac{50}{0.15}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Combine all local stiffness matrices to form a global stiffness matrix:

$$\mathbf{K} = 66.7\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 5 \end{bmatrix}$$

whereas the heat rate vector is

$$\mathbf{R} = \left[25 \times 800, \quad 0, \quad 0, \quad 0\right]^{\mathrm{T}}$$

The penalty parameter C can be estimated by

$$C = \max\left|\mathbf{K}_{ij}\right| \times 10^4 = 66.7 \times 8 \times 10^4$$

Hence, the finite element system can be solved as followed:

$$66.7\begin{bmatrix} 1.375 & -1 & 0 & 0 \\ -1 & 4 & -3 & 0 \\ 0 & -3 & 8 & -5 \\ 0 & 0 & -5 & 80{,}005 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 25 \times 800 \\ 0 \\ 0 \\ 10{,}672 \times 10^4 \end{Bmatrix}$$

$$\mathbf{T} = \left[304.6, \quad 119.0, \quad 57.1, \quad 20.0\right]^{\mathrm{T}}$$

7.6 Boundary Element Method — An Introduction

- The **Boundary Element Method** (BEM) is a relatively new for PDE, where it consists only *boundary elements*, either line (2-D) or surface (3-D) elements.

- Consider a Laplace equation for a steady-state potential flow problems (ψ is the stream function):

$$\nabla^2 \psi = 0$$

- For a multi-dimensional case, this equation leads to an analytical solution:

$$\psi^* = \frac{1}{2\pi} \ln \frac{1}{r} \qquad \textbf{(7.31)}$$

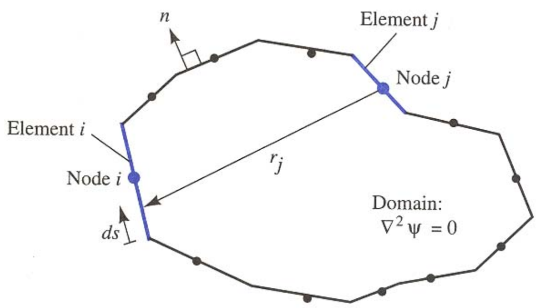


FIGURE 7.15 Boundary elements

- Eq. (7.31) can be discretised to yield:

$$\tfrac{1}{2}\psi_i + \sum_{j=1}^{N} \psi_j \left(\int_j \frac{\partial \psi^*}{\partial n} ds \right) = \sum_{j=1}^{N} \left(\frac{\partial \psi}{\partial n} \right)_j \left(\int_j \psi^* ds \right) \qquad \textbf{(7.32)}$$

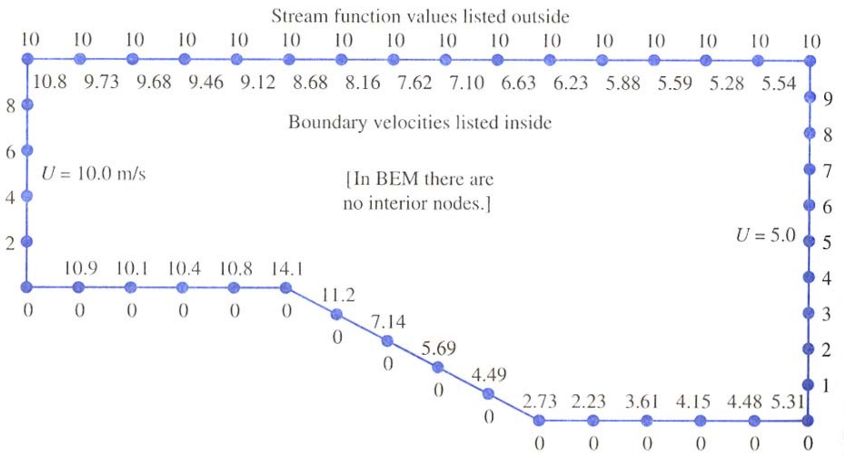
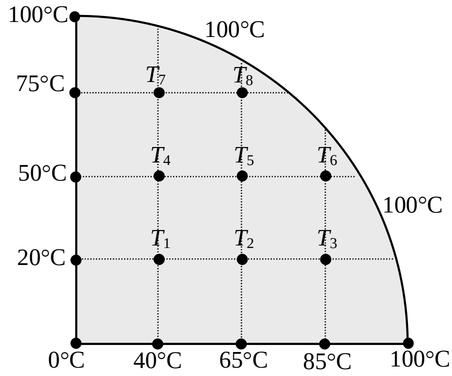


FIGURE 7.16 Example of an analysis using BEM

Exercises

1. A quarter of disc having a radius of 8 cm as shown below has a variation of temperature at the boundary aligned with the principal axes, while the temperature is fixed at 100°C at its outer radius of 8 cm. If the temperature distribution follows the Laplace equation:

$$\nabla^2 T = 0$$



By using the grid as shown:

a. Obtain a system of algebraic equations using the finite difference method,

b. Solve the system to obtain T_i .

2. By using the implicit technique, solve the following heat conduction problem:

$$\frac{\partial}{\partial t} T(x,t) = \alpha \frac{\partial^2}{\partial x^2} T(x,t), \quad 0 < x < 10, \quad t > 0$$

using the following boundary conditions:

$$T(0,t) = 0, \quad T(10,t) = 100, \quad T(x,0) = 0.$$

Use the constant $\alpha = 10$, the time step $\Delta t = 0.1$, and a model of 10 grid including the grid at the boundary. Compar this solution with the solution obtained by using $\Delta t = 0.3$.