Chapter 1 Introduction to Boundary element Method - 1D Example

For reference: Hong-Ki Hong and Jeng-Tzong Chen, *Boundary Element Method,* Chapter 1 Introduction to Boundary element Method - 1D Example, Lecture Notes, Department of Civil Engineering, Taiwan University, Taipei, Taiwan, September, 1993.

Summary.

1.1 Introduction

When I first heard of the boundary element method(BEM), I got an impression that there come a novel method for solving boundary value problems(BVP), by which one needed to work only on the boundary of the domain of a boundary value problem, and then he knew everything not only on the boundary but also in the domain. "What a wonderful method!" I thought, excited. "Is it possible?" "How can it be?" I was puzzled. Yes, it is, and this lecture series are going to present to you the method.

A boudary value problem is a problem, given some data on the boundary B, to solve for the rest of information on the boundary B and in the domain D. Due to various complexities, boundary value problems do not always have closed-form solutions, nor analytic solutions. It is often inevitable to resort to some kinds of approximations and numerical methods, among which the finite difference method(FDM), the finite element method(FEM), and the boundary element method(BEM) are three popular, systematic ones. These three are in fact certain types of discretizations of three different kinds of formulations of an identical bondary value problem, as shown in Table 1. Refering to Fig.1, a boundary value problem usually appears in a form like

$$\text{algebraic/differential equations} \quad \forall x \in D \tag{1.1}$$

$$\text{algebraic and/or differential equations} \quad \forall x \in B \tag{1.2}$$

Equations (1.1) and (1.2) are respectively called the governing equations and boundary conditions of the boundary value problem.

In fact, a boundary value problem is not necessarily described by differential equations. The reason why it may be in terms of differentail equations is that a

Table 1: BEM as a solution method for a BVP

BVP	differential formulation	FDM
	integral formulation	FEM
	boundary integral formulation	BEM

local view is adopted. In establishing Eq.(1.1), one looks at a typical point x, or more precisely, the neighborhood of a typical point x in the domain, and considers the changes of primary variables in such an infinitesimally small, differential element around x according to some established laws for the primary and secondary variables, thus resulting in differential equations for describing the behaviors of the primary variables at the point x. Take notices on the following correspondences of mathematical symbols and terminologies and physical entitities:

1. $x \leftrightarrow$ a typical point $\leftrightarrow$ an arbitrary point,
2. $dx, \mathcal{N}(x) \leftrightarrow$ neighborhood of $x \leftrightarrow$ a differential element.

If, instead, a global point of view is adopted, there result integral equations for the boundary value problem, that is

$$\text{integral equations.} \tag{1.3}$$

The formulation of the integral equations may directly based upon some established laws interpreted globally, or may be transformed from Eqs. (1.1) and (1.2). The differential formulation given in Eqs. (1.1) and (1.2) and the integral formulation given in Eq.(1.3) are often called the strong form and the weak form, respectively.

In additional to the above two formulations, some boundary value problems may be represented in forms like

$$\text{boundary integral equations.} \tag{1.4}$$

The formulation of the boundary integral equations are almost all transformed from the differential formulation of Eqs.(1.1) and (1.2) with the aid of fundamental solutions of Eq.(1.1) and some integral identities for two fields derived from governed Stokes' formula. Not every differential formulation has available a corresponding boundary integral formulation.

To be more precise, in the rest of this and the next three chapters, and even chapter 5, we shall single out Laplace's equation to be the chosen specific form of Eq.(1.1) and thus facilitate our study in a concrete way. First in the next section we shall study a 1D example of the problems of Laplace's equation to gain a feeling of transforming a differential formulation to its corresponding boundary integral formulation, and to have a rough knowledge of how the boundary element method works. Seven stages are to be identified in working out the example. Then in the rest of the chapter deeper investigations are exercised on some key stages of the derivation of a BIE, and their physical meanings are explored. In more details we shall study in Section 1.3, Laplace's equations in a one-dimensional domain, its possibly related boundary conditions, its various applications, its primary and secondary variables, established laws in application fields on which the derivations are based; in Section 1.4, the fundamental solution of Laplace's equations and its physical meaning; in Section 1.5, the derivation and physical meaning of Green's second identity; and then in Section 1.6, transformation from the differential formulation to a boundary integral formulation by using the results of Sections 1.3-1.5. In all the four Sections (1.3-1.6) every effect is made to provide physical meanings in various applications fields. In Sections 1.3-1.5, we also explore 2D and 3D cases. One of the purposes for doing this is to enhance the understanding of common features and subtle differences of 1D, 2D and 3D cases. However, in Section 1.6 only the 1D case is covered, and 2D and 3D cases which need ample study is left to the next chapter.

1.2 BEM - 1D example of problems of Laplace's equation

Now we study a 1D problem of Laplace's equation. First we give its differential formulation. The governing equation is

$$\frac{d^2u(x)}{dx^2} = 0, \quad \forall x \in (0,1). \tag{1.5}$$

Consider three cases of boundary conditions.

Case 1: Dirichlet boundary condition

$$u(0) = 100, \quad u(1) = 0.$$

Case 2: Neumann boundary condition

$$v(0) = 100, \quad v(1) = 100.$$

Case 3: Mixed boundary condition

$$u(0) = 100, \quad v(1) = 0.$$

In the above u is the primary variabe, x is a field point, dx is the neighborhood of x or the differential element at x, $u(x)$ is the primary field, and

$$v := k\frac{du}{dx}$$

is the secondary variable, k is a constant coefficient, which $= 1$ in the present cases, and $v(x)$ is the secondary field.

The above three boundary value problems are very simple and have closed-form solutions.

Case 1: $u(x) = 100 - 100x$.
Case 2: $u(x) = 100x + c$, where c is an arbitrary constant.
Case 3: $u(x) = 100$.

Nevertheless, for our present purposes, we now solve it by using the BEM/BIE method step by step.

Stage 1: *Find the fundamental solution of the governing equation.*

The fundamental solution $U(x, \bar{x})$ to Eq.(1.5) is a function of x with a parameter $\bar{x}$ defined as

$$U(x, \bar{x}) = -\frac{1}{2}|x - \bar{x}|, \tag{1.6}$$

which satisfies

$$-\frac{\partial^2 U(x, \bar{x})}{\partial x^2} = \delta(x - \bar{x}), \quad \forall x, \bar{x} \in \mathbf{R}, \tag{1.7}$$

where $\delta(x - \bar{x})$ is Dirac's delta function. Define

$$V(x, \bar{x}) := \frac{\partial U(x, \bar{x})}{\partial x} = \frac{1}{2} - H(x - \bar{x}).$$

where $H(x - \bar{x})$ is heaviside's unit step function.

Stage 2: *Derive a formula of two field variables which converts domain integrals into boundary integrals.*

Integrating by parts twice, we have

$$\int_0^1 u \frac{d^2 U}{dx^2} dx = \left(u \frac{dU}{dx} - U \frac{du}{dx} \right) |_0^1 + \int_0^1 U \frac{d^2 u}{dx^2} dx, \tag{1.8}$$

which indeed is the 1D version of Green's second identity.

Stage 3: *Obtain the BIE for a domain point.*

Substitution of Eqs.(1.5), and (1.7) into Eq.(1.8) yields

$$-u(\bar{x}) = \left(u(x) \frac{\partial U(x, \bar{x})}{\partial x} - U(x, \bar{x}) \frac{du(x)}{dx} \right) |_{x=0}^{x=1}, \tag{1.9}$$

or upon interchanging the barred and unbarred symbols,

$$-u(x) = \left(u(\bar{x}) \frac{\partial U(\bar{x}, x)}{\partial \bar{x}} - U(\bar{x}, x) \frac{du(\bar{x})}{d\bar{x}} \right) |_{\bar{x}=0}^{\bar{x}=1}, \tag{1.10}$$

or introducing the symbols v and V,

$$-u(x) = u(1)V(1, x) - U(1, x)v(1) - u(0)V(0, x) + U(0, x)v(0), \tag{1.11}$$

where

$$U(\bar{x}, x) = -\frac{1}{2} \mid \bar{x} - x \mid, \tag{1.12}$$

$$V(\bar{x}, x) = \frac{\partial U(\bar{x}, x)}{\partial \bar{x}} = \frac{1}{2} - H(\bar{x} - x). \tag{1.13}$$

Equation (1.10) is the BIE for a domain point x. Since the boundary of a 1D domain is simply two end points, we see in Eq.(1.10) boundary terms with no boundary integral signs. And since there are no nonhomogeneous terms in the governing equation (1.5), we see in Eq.(1.10) no volume integrals.

Stage 4: *Let x approach to the boundary and obtain the BIE for a boundary point.*

Accordingly, as x approaches the left boundary point,

$$\begin{aligned}
\lim_{x\to 0} U(0, x) &= 0, \\
\lim_{x\to 0} U(1, x) &= -\frac{1}{2}, \\
\lim_{x\to 0} V(0, x) &= \frac{1}{2}, \\
\lim_{x\to 0} V(1, x) &= -\frac{1}{2},
\end{aligned}$$

and as x approaches the right boundary point,

$$\begin{aligned}
\lim_{x\to 1} U(0, x) &= -\frac{1}{2}, \\
\lim_{x\to 1} U(1, x) &= 0, \\
\lim_{x\to 1} V(0, x) &= \frac{1}{2}, \\
\lim_{x\to 1} V(1, x) &= -\frac{1}{2}.
\end{aligned}$$

Hence we have for the boundary point $x = 0$ the BIE,

$$-u(0) = u(1)(-\frac{1}{2}) + \frac{1}{2}v(1) - u(0)\frac{1}{2} + 0, \tag{1.14}$$

and for the boundary point $x = 1$ the BIE,

$$-u(1) = u(1)(-\frac{1}{2}) - 0 - u(0)\frac{1}{2} - \frac{1}{2}v(0). \tag{1.15}$$

Stage 5: *Choose a finite number of boundary points, discritize the BIE's for the boundary points, and put the discritized BIE's in matrix form.*

For 2D and 3D problems the boundary integrals have to be discretized first, but for 1D problems no discretization is needed. Arrange Eqs.(1.14) and (1.15) in a matrix form,

$$\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix} = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix} \begin{Bmatrix} v(0) \\ v(1) \end{Bmatrix}. \tag{1.16}$$

Stage 6: *With the aid of the prescribed boundary conditions solve the matrix equation.*

Case 1: Plugging in the prescribed Dirichlet boundary conditions,

$$\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} 100 \\ 0 \end{Bmatrix} = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix} \begin{Bmatrix} v(0) \\ v(1) \end{Bmatrix},$$

or

$$\begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix} \begin{Bmatrix} v(0) \\ v(1) \end{Bmatrix} = \begin{Bmatrix} 50 \\ -50 \end{Bmatrix},$$

the solution of which is readily found,

$$\begin{Bmatrix} v(0) \\ v(1) \end{Bmatrix} = \begin{Bmatrix} -100 \\ -100 \end{Bmatrix}.$$

This matches the exact solution, $u(x) = 100 - 100x$ and $v(x) = -100$.

Case 2: Plugging in the prescribed Neumann boundary conditions,

$$\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix} = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix} \begin{Bmatrix} 100 \\ 100 \end{Bmatrix},$$

or

$$\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix} = \begin{Bmatrix} -50 \\ 50 \end{Bmatrix},$$

the solution of which is readily found,

$$\begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix} = \begin{Bmatrix} c \\ 100 + c \end{Bmatrix}.$$

This matches the exact solution, $u(x) = 100x + c$.

Case 3: Plugging in the prescribed mixed boundary conditions,

$$\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} 100 \\ u(1) \end{Bmatrix} = \begin{bmatrix} 0 & -0.5 \\ 0.5 & 0 \end{bmatrix} \begin{Bmatrix} v(0) \\ 0 \end{Bmatrix},$$

or

$$\begin{bmatrix} 0 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} v(0) \\ u(1) \end{Bmatrix} = \begin{Bmatrix} -50 \\ 50 \end{Bmatrix},$$

the solution of which is readily found,

$$\begin{Bmatrix} v(0) \\ u(1) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 100 \end{Bmatrix}.$$

This matches the exact solution, $u(x) = 100$ and $v(x) = 0$.

Stage 7: *Solve for the domain points by using the BIE for a domain point.*

From Eqs.(1.10) and (1.12), for a domain point $x \in (0, 1)$, we have

$$u(x) = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} u(0) \\ u(1) \end{Bmatrix} + \begin{bmatrix} \frac{x}{2} & \frac{x-1}{2} \end{bmatrix} \begin{Bmatrix} v(0) \\ v(1) \end{Bmatrix}.$$

Now consider the three boundary value problems.

Case 1:

$$u(x) = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} 100 \\ 0 \end{Bmatrix} + \begin{bmatrix} \frac{x}{2} & \frac{x-1}{2} \end{bmatrix} \begin{Bmatrix} -100 \\ -100 \end{Bmatrix} = 100 - 100x.$$

Case 2:

$$u(x) = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} c \\ 100 + c \end{Bmatrix} + \begin{bmatrix} \frac{x}{2} & \frac{x-1}{2} \end{bmatrix} \begin{Bmatrix} 100 \\ 100 \end{Bmatrix} = 100x + c.$$

Case 3:

$$u(x) = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} 100 \\ 100 \end{Bmatrix} + \begin{bmatrix} \frac{x}{2} & \frac{x-1}{2} \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = 100.$$

For all three cases the BIE solutions appear to be identical with the closed-form solutions.

1.3 Laplace's equation and related boundary conditions

In the preceding section, we have seen mathematical and numerical manipulations of the BEM of the problem of 1D Laplace's equation. In this and the following three sections, we shall attach to them vivid physical meanings. Our central tool for this purpose is the rod problem. In this section, we shall see how its differential formulation, including the governing differential equation and related boundary conditions, is derived from established laws. In the following two sections, we shall see the corresponding physical meanings of the fundamental solution, Eq.(1.6), and the 1D version of Green's second identity, Eq.(1.8), recalling we have seen in the preceding section that the two play the central roles of converting the differential formulation to the BIE, Eq.(1.10). These parellel presentations of mathematic and physical aspects we believe will greatly enhance understanding and promote applications, and hopefully induce more creativity. Therefore, to amplify the effects of parellel study, we shall cite in the three sections more application examples in additional to rod.

Consider the axial deformation of a straight rod of constant cross section with cross section area A, made of homogeneous, linearly elastic material with Young's modulus E, subjected to axial loading $b(x)$, in which x is the coordinate. Based on the strain-displacement relation, Hooke's law and stress resultant, the internal force v and the displacement u have the following relation:

$$v = EA\frac{du}{dx}. \tag{1.17}$$

Consider a differential element dx located between cross sections at x and $x + dx$. Based on axial equilibrium of internal forces $v(x)$ at the cross section x and $v(x+dx)$ at the cross section $x+dx$, and the loading $b(x)dx$ apllied on the differential element dx, and noting

$$v(x + dx) = v(x) + \frac{dv(x)}{dx}(dx) + O(dx)^2,$$

we have

$$-\frac{dv}{dx} = b.$$

So the governing equation for the rod problem is

$$- EA\frac{d^2u}{dx^2} = b, \tag{1.18}$$

which has been derived for and is hence valid for any domain point x. It is a 1D Poission's equation, which reduces to Laplace's equation in case of $b = 0$, i.e. if there are no body forces applied in the domain.

We now formulate boundary conditions for the rod problem. In case the boundary point (or cross section) $x = a$ is constrained, either enforced to displace a certain distance d or fixed ($d = 0$), we say we have a Dirichlet boundary condition

$$u(a) = d \tag{1.19}$$

at the boundary point $x = a$. In case the boundary point $x = a$ is loaded by a (concentrated) load f, we say we have a Neumann boundary condition

$$v(a) = EA\frac{du}{dx}|_{x=a} = f \tag{1.20}$$

at the boundary point $x = a$. In case a spring support with spring constant k_b is installed at the boundary point $x = a$, we say we have a Robin boundary condition

$$v(a) = EA\frac{du}{dx}|_{x=a} = k_b u(a) \tag{1.21}$$

at the boundary point $x = a$.

Exercise. Transform the above three cases of BVP's to rod problems with constant axial rigidity, subjected to enforced support movements and/or to applied loads, all with definite quantities and dimensions. Give schematic drawings of the three cases of BVP's, from which plot the diagrams of applied loading, internal force, and displacement. See Fig.2.

Applied loading can be classified as either distributed or concentrated; hence, in the loading diagram, there should be two scales, one for distributed loading and the other for concentrated loading. For example, the loading $b(x)$ of a rod of $4m$ long under uniformly distributed loading of $1N/m$ and a concentrated load of $3N$ at the midpoint can be expressed as $b(x) = 1 + 3\,\delta(x-2)N/m$. So in the loading diagram, in additional to the conventional arrangement of the axial coordinate in the unit of m as the abscissa and distributed loading in the unit of N/m as the ordinate, we need one more ordinate to present concentrated loads in the unit of N, otherwise concentrated loads would always appear to go to infinity in the unit of N/m at their application points, failing to show their strengths in the unit of N. At the application point of a concentrated load the internal force diagram has discontinuity with the quantity of the concentrated load as its jump value. Corespondingly the displacement diagram has a kink at the application point of the concentrated load.

As a byproduct of the above expression we thus have had a strong feeling of Dirac's delta function, since, for example, $b(x) = \delta(x - \bar{x})$ just means a unit concentrated load applied at the point $x = \bar{x}$.

Exercise. We have already derived the governing equation for the rod problem and its related various boundary conditions. Do the same things for the taut string, shear beam, soil layer problems. The last problem refers to static earth pressure induced in soil layer. Study Tables 2 and 3, and further augment the tables by filling in more application problems.

Table 2: Physical problems for the 1D Laplace's equation

	problem	primary			secondary			coefficient		
		u	common symbol	common unit	v	common symbol	common unit	k	common symbol	common unit
(1)	rod	axial displacement	u	mm	axial internal force	v	N	axial rigidity	EA	N
(2)	taut string	lateral displacement	w	mm	vertical component of string tension	V	N	horizontal component of string tension	H	N
(3)	shear beam	lateral displacement	w	mm	shear force	V	N	shear rigidity	GA	N
(4)	soil layer	horizontal displacement	w	mm	shear stress	τ	MPa	shear modulus	G	MPa
(5)										

Referring to Tables 4 and 6, problems (6)-(10), etc. also have 1D versions
$v = k\frac{du}{dx}$
Throughout k is assumed constant

Table 3: Natural laws for the 1D Laplace's equation and meanings of some related boundary conditions

	G.E.		B.C.		
	$v = k\frac{du}{dx}$	$\frac{dv}{dx} = 0$	$u = 0$	$v = 0$	$v = k_b u$
(1)	Hooke's law σ-ϵ relation and stress resultant	axial force equilibrium	fixed	free	spring
(2)	sag configuration and tension components	lateral force equilibrium	fixed	slide	spring
(3)	Hooke's law w-γ relation and stress resultant	shear force equilibrium	fixed	free	spring
(4)	Hooke's law w-γ relation	shear stress equilibrium	fixed fixed	free free	compliant basement
(5)					

Table 4: Physical problems for the 2D Laplace's equation

	problem	primary			secondary			coefficient		
		u	common symbol	common unit	$\mathbf{v}$	common symbol	common unit	k	common symbol	common unit
(2)	taut membrane	lateral displacement	w	mm	vertical componenet of membrane tension	V	N/m	horizontal componeent of membrane tension	H	N/m
(5)	torsion									
(6)										

Referring to Table 6, problems (6)-(10), etc. also have 2D versions

$\mathbf{V} = k\nabla u$

Throughout k is assumed constant

Exercise. Derive the governing equation for the taut membrance problem and its related various boundary conditions. Do the same things for the problem of torsion of a solid cylinder. Study Tables 4 and 5, and further augment the tables by filling in more application problems.

Table 5: Natural laws for the 2D Laplace's equation and meanings of some related boundary conditions

	G.E.		B.C.		
	$v = k\nabla u$	$\nabla \cdot v = 0$	$u = 0$	$v = 0$	$v = k_b u$
(2)	sag configuration and tension components		fixed	slide	spring
(5)					
(6)					
$v = \mathbf{n} \cdot \mathbf{v} = k\frac{\partial u}{\partial n}$					

Exercise. Derive the governing equation for the potential flow problem and its related various boundary conditions. Do the same things for the seepage, Darcy flow, steady state heat conduction, steady state diffusion, electrostatics, magnetostatics problems, respectively. Study Tables 6 and 7, and further augment the tables by filling in more application problems.

Table 6: Physical problems for the 3D Laplace's equation

	problem	primary			secondary			coefficient		
		u	common symbol	common unit	$\mathbf{v}$	common symbol	common unit	k	common symbol	common unit
(6)	potential flow	potential function	ϕ	m^2s^{-1}	flux	$\mathbf{q}$	ms^{-1}	1		dimensionless
(7)	seepage									
(8)	Darcy flow									
(9)	steady state heat conduction									
(10)	steady state diffusion									
(11)	electrostatics									
(12)	magnetostatics									
(13)										
Referring to Tables 2 and 4, problems (6)-(10), etc. also have 1D and 2D versions $\mathbf{v} = k\nabla u$ Throughout k is assumed constant										

Table 7: Natural laws for the 3D Laplace's equation and meanings of some related boundary conditions

	G.E.		B.C.		
	$\mathbf{v} = k\nabla u$	$\nabla \cdot \mathbf{v} = 0$	$u = 0$	$v = 0$	$v = k_b u$
(6)					
(7)					
(8)					
(9)					
(10)					
(11)					
(12)					
(13)					
$v = \mathbf{n} \cdot \mathbf{v} = k\frac{\partial u}{\partial n}$					

1.4 Fundamental solution

For the rod problem, the abovementioned fundamental solution can be transformed to

$$U(x, \bar{x}) = -\frac{1}{2EA}|x - \bar{x}| + c_1 x + c_2, \tag{1.22}$$

which satisfies

$$-EA\frac{\partial^2 U(x,\bar{x})}{\partial x^2} = \delta(x-\bar{x}), \quad \forall x, \bar{x} \in \mathbf{R}. \tag{1.23}$$

It is now obvious that the fundamental solution of the rod problem is simply the displacement diagram of an infinitely long rod subjected to a unit concentrated load at a point $\bar{x}$. In other words, it is the influence function representing flexibility of an otherwise identical rod of infinite length. So the fundamental solution $U(x,\bar{x})$ of the 1D Laplace's equation is the influence function with domain extended unbounded and with a unit coefficient — the effect on the primary field at the field point x due to the cause of a unit concentrated source at the source point $\bar{x}$. The graphs of Dirac's delta function $\delta(x-\bar{x})$, the derivative of the fundamental solution $V(x,\bar{x})$, and the fundamental solution $U(x,\bar{x})$ with $\bar{x}$ fixed are respectively the diagrams of the concentrated source, and the induced secondary and primary fields.

Exercise. Plot the loading, internal force, and displacement diagrams of an infinitely long rod with unit axial rigidity subjected to a unit concentrated load applied at the point $\bar{x}$. They display respectively $b(x) = \delta(x-\bar{x})$, $v(x) = V(x,\bar{x})$ and $u(x) = U(x,\bar{x})$. Note the unit concentrated source with scale different from distributed sources, and also note discontinuity in the derivative of the fundamental solution and kink in the fundamental solution.

The c_1 in Eq.(1.22) is a real constant, and the c_2 can be any arbitrary constant or the plus or minus infinity. It is natural there comes out two integration constants in the solution to a differential equation of order two. They could be determined by the conditions at infinity. If we set

$$c_1 = 0, \quad c_2 = \frac{\infty}{2EA},$$

the displacements at the plus and minus infinity vanish. This set of constant values gives the most natural physical explanation of the fundamental solution: the primary variable (e.g. displacement) has singularity at the source point (e.g. the application point of the concentrated load), tends to zero at infinity, and is symmetric with respect to the source point. However, in practice, we usually set

$$c_1 = 0, \quad c_2 = 0,$$

to facilitate calculation and also to retain the merit of the symmetry about the source point. As will be shown later in the next section the values of the two constants do not alter the final form of the BIE derived.

Exercise. Give other explanations of the fundamental solution of the 1D Laplace's equation. Give appropriate explanations of the fundamental solutions of the 2D and 3D Laplace's equations.

1.5 Green's second identity

The state of a rod of length ℓ is described by the set of the displacement u, the internal force v, and the loading b, which satisfy the governing equation (1.18) and some appropriate boundary conditions, Eqs.(1.19), (1.20) and/or (1.21). For the rod problem, Betti's theorem of reciprocality of energy and work correlates two states (u, v, b) and (u^*, v^*, b^*) of a rod by

$$\int_0^\ell (ub^* - u^*b)dx + (uv^* - u^*v)|_0^\ell = 0. \tag{1.24}$$

Upon noting that the state (u, v, b) satisfies Eqs.(1.17) and (1.18) and the state (u^*, v^*, b^*) satisfies similar equations, Eq.(1.24) becomes

$$\int_0^\ell (u\frac{du^*}{dx} - u^*\frac{du}{dx})dx = (u\frac{du^*}{dx} - u^*\frac{du}{dx})|_0^\ell, \tag{1.25}$$

which is exactly the 1D version of Green's second identity, as we have derived previously in Eq.(1.8). Hence Green's second identity used in the derivation of the BIE for the problem of Laplace's equation may be understood vividly as the reciprocity theorem for the rod problem.

Exercise. Examine the meanings of Green's second identity in the various application fields as summarized in the foregoing tables and try to state and prove appropriate theorems for the problems.

Exercise. Prove that the constants c_1 and c_2 in the fundamental solution of Eq.(1.22) does not alter the derived BIE.

1.6 Physical meaning of the BIE for rod

Through the study in the last three sections, we may clearly see the physical meaning of the BIE for rod, describing the rod state (u, v, b) at point x by the reciprocality theorem with the aid of the state (U, V, δ) of an otherwise grantidentitical rod of infinite length subjected to a unit concentrated load applied at the point x.

Exercise. Examine the meanings of the BIE for the 1D Laplace's equation problem for various application fields as summarized in the Tables 2 and 3.