# An Iterative BEM for the Inverse Problem of Detecting Corrosion in a Pipe*

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**Abstract**

In this paper, we consider an inverse problem of determining the corrosion occurring in an inaccessible interior part of a pipe from the measurements on the outer boundary. The problem is modelled by the Laplace equation with an unknown term $\gamma$ in the boundary condition on the inner boundary. Based on the Maz'ya iterative algorithm, a regularized BEM method is proposed for obtaining approximate solutions for this inverse problem. The numerical results show that our method can be easily realized and is quite effective.

## 1 Introduction

Detecting the corrosion inside a pipe is one of the most important topics in engineering, especially in the safety administration of the nuclear power station. There are several ways to do this. In this paper, we will discuss the mathematical theory and numerical algorithm for a method of detecting the corrosion by electrical fields. More exactly, we consider an

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inverse problem of determining the corrosion occurring in an inaccessible interior part of a pipe from the measurements on the outer boundary. Our goal is to determine information about the corrosion that possibly occurs on an interior surface of the pipe, which is an 'inaccessible' part, and we collect electrostatic data on the part of the exterior surface of the pipe, which is an 'accessible' part.

In the case that the thickness of the pipe is sufficiently small when compared with the radius of the pipe and the Cauchy data are given on the whole outer boundary, this inverse problem can be treated by the Thin Plate Approximation method (TPA). The algorithm and numerical analysis can be found in [7]. But this algorithm works only under the assumption that the thickness is small enough when compared with the radius of the pipe. The case, in which the Cauchy data are given on part of the outer boundary and the smallness assumption is abandoned, has not been studied and it is obvious that it is of great importance for practice problems.

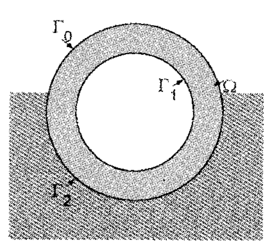
The main difficulty for this inverse problem is the ill-posedness of the inverse problem. The measured data are given only on part of the outer boundary and we want to determine an unknown function in the inner boundary. Because of the ill-posedness, the errors in measured data will be enlarged in the numerical treatment if we do not treat it suitably. In this paper, based on the Maz'ya iterative method, we propose a new BEM algorithm for this inverse problem. It can be easily realized. The numerical results show the efficiency of this method.

This paper is organized as follows:

1. Formulation of the inverse problem,
2. The iterative boundary element method,
3. Numerical examples,
4. Conclusions.

# 2 Formulation of the inverse problem

Suppose a domain $\Omega = \{x \,|\, r_1 < |x| < r_2\} \subset \mathbb{R}^2$ (see Figure 2.1) and the boundaries $\Gamma_1 = \{x|\,|x| = r_1\}$and $\Gamma_2 = \{x|\,|x| = r_2\}$.



Assume that $\Omega$ is a metallic body with constant conductivity. In the domain $\Omega$, we consider an electrostatic field. The electric potential $u$ satisfies the Laplace's equation in $\Omega$, i.e.,

$$\Delta u = 0, \qquad \text{in} \quad \Omega. \tag{2.1}$$

Let $\Gamma_0$ be an open set of the outer boundary $\Gamma_2$ of $\Omega$ which is an 'accessible' part. On $\Gamma_0$, the Dirichlet data and the Neumann data of the electric potential $u$ are given, i.e.,

$$u(x) = \phi(x), \qquad x \in \Gamma_0, \tag{2.2}$$

$$u_\nu(x) = \psi(x), \qquad x \in \Gamma_0, \tag{2.3}$$

where $u_\nu$ is the outer normal derivative of $u$ on the boundary.

We denote the rest part of the exterior boundary of $\Omega$ by $\widetilde{\Gamma}_2$,

$$\widetilde{\Gamma}_2 = \Gamma_2 \setminus \Gamma_0.$$

We assume that the corrosion only happened on the interior boundary of the domain $\Omega$ and the corrosion can be described by a non-negative function $\gamma$ in the boundary condition on the interior boundary. That is,

$$u_\nu + \gamma u = 0, \qquad \text{on} \quad \Gamma_1, \tag{2.4}$$

where $\gamma \geq 0$ represents the corrosion damage.

The inverse problem we discuss in this paper is to find the unknown coefficient $\gamma$ from the Cauchy data $\phi$ and $\psi$ on $\Gamma_0$.

We will treat this inverse problem by the following steps:

**Step 1:** Get the Cauchy data on the interior circle by solving the Cauchy problem for Laplace's equations.

We use the iterative boundary element method to solve the Cauchy problem:

$$\begin{cases} \Delta u(x) = 0, & x \in \Omega, \\ u(x) = \phi(x), & x \in \Gamma_0, \\ u_n(x) = \psi(x), & x \in \Gamma_0. \end{cases} \tag{2.5}$$

Our goal is to get the Cauchy data on $\Gamma_1$:

$$u(x) = \phi_1(x), \qquad x \in \Gamma_1; \qquad u_n(x) = \psi_1(x), \qquad x \in \Gamma_1.$$

**Step 2:** Get the impedance $\gamma$ from the Cauchy data on the interior circle.

For the boundary condition

$$u_n + \gamma u = 0, \qquad x \text{ on } \Gamma_1,$$

$\gamma$ can be obtained by

$$\gamma = -\frac{u_n}{u}\bigg|_{\Gamma_1} = -\frac{\psi_1}{\phi_1}, \qquad \text{if } \phi_1 \neq 0.$$

**Remark 2.1.** It can be proved that the measure of the zero set $\{\phi_1 = 0\}$ can not be non-zero. Therefore, our method is valid in the case of $\phi_1 \neq 0$.

# 3 The iterative boundary element method for this Cauchy problem

In this section we will give the iterative boundary element method (see [9], [10],[11]) for the Cauchy problem in Step 1. We will prove the convergence rate only under the regularity assumption. Some numerical simulation results for the Cauchy problem are also presented.

## 3.1 Description of the algorithm

In [11], V.A. Kozlov, V.G. Maz'ya and A.V.Fomin proposed the algorithm as follows:
**1.** Specify an initial boundary guess $u_0$ on $\Gamma_1$ and $\widetilde{\Gamma}_2$.
**2.** Solve the well-posed mixed boundary value problem:

$$\begin{cases} \Delta U^{(0)}(x) = 0, & x \in \Omega, \\ U_n^{(0)} = \psi, & x \in \Gamma_0, \\ U^{(0)} = u_0, & x \in \Gamma_1 \cup \widetilde{\Gamma}_2. \end{cases} \tag{3.1}$$

to determine $U^{(0)}(x)$ for $x \in \Omega$ and $q_0 = U_n^{(0)}(x)$ for $x \in \Gamma_1 \cup \widetilde{\Gamma}_2$.
**3** (i). Suppose that the approximation $q_k$ is obtained. We can solve the mixed boundary value problem:

$$\begin{cases} \Delta U^{(2k+1)} = 0, & x \in \Omega, \\ U^{(2k+1)} = \phi, & x \in \Gamma_0, \\ U_n^{(2k+1)} = q_k, & x \in \Gamma_1 \cup \widetilde{\Gamma}_2. \end{cases} \tag{3.2}$$

Then we can determine $U^{(2k+1)}(x)$ for $x \in \Omega$ and $u_{k+1} = U^{(2k+1)}(x)$ for $x \in \Gamma_1 \cup \widetilde{\Gamma}_2$.

(ii) By $u_{k+1}$, we can obtain $U^{(2k+2)}(x)$ for $x \in \Omega$ and $q_{k+1} = U_n^{(2k+2)}(x)$ for $x \in \Gamma_1 \cup \widetilde{\Gamma}_2$ by solving the mixed boundary value problem:

$$\begin{cases} \Delta U^{(2k+2)} = 0, & x \in \Omega, \\ U_n^{(2k+2)} = \psi, & x \in \Gamma_0, \\ U^{(2k+2)} = u_{k+1}, & x \in \Gamma_1 \cup \widetilde{\Gamma}_2. \end{cases} \tag{3.3}$$

**4.** Repeat step 3 for $k \geq 0$ until a prescribed stopping criterion is satisfied.
The stopping criterion we will use in this paper is $\|u_{k+1} - u_k\|_{L^2(\Gamma_1 \cup \Gamma_2)} \leq \varepsilon$, where $\varepsilon$ is a small positive number.

**Remark 3.1.** The mixed boundary value problems (3.2) and (3.3) are well-posed problems.

We solve the mixed boundary value problems (3.2) and (3.3) by the boundary element method, which can be found in a lot of guide books on the boundary element method, for example, [1]. In the following, we give only the outline of the iterative BEM form.

Consider the following mixed boundary value problem in two-dimensional case:

$$\begin{cases} \Delta u = 0, & \text{in } \Omega, \\ u = f, & \text{on } \Gamma_D, \\ u_n = g, & \text{on } \Gamma_N. \end{cases} \tag{3.4}$$

As we have known, the foundational integral formula of the harmonic function

$$u(M_i) = \int_\Gamma \left( u^* \frac{\partial u}{\partial \nu} - u \frac{\partial u^*}{\partial \nu} \right) d\Gamma, \qquad M_i \in \Omega, \tag{3.5}$$

where $u^* = \frac{1}{2\pi} \ln \frac{1}{r_{MM_i}}$ represents the foundational solution of the Laplace's equation. And the boundary integral formula is :

$$c_i u(M_i) = \int_\Gamma \left( u^* \frac{\partial u}{\partial \nu} - u \frac{\partial u^*}{\partial \nu} \right) d\Gamma, \qquad M_i \in \partial\Omega. \tag{3.6}$$

Equation (3.6) can be discretized as follows:

$$c_i u_i + \sum_{j=1}^{N} \int_{\Gamma_j} u q^* d\Gamma - \sum_{j=1}^{N} \int_{\Gamma_j} u^* q d\Gamma = 0. \tag{3.7}$$

The values of $u$ and $q$ in the integrands of (3.7) are constant within each element, and $u$ and $q$ consequently can be taken out of the integrals. This gives

$$c_i u_i + \sum_{j=1}^{N} \left( \int_{\Gamma_j} q^* d\Gamma \right) u_j - \sum_{j=1}^{N} \left( \int_{\Gamma_j} u^* d\Gamma \right) q_j = 0. \tag{3.8}$$

With the given boundary condition, we can rearrange equation (3.8) with all the unknowns on the left-hand side and a vector on the right-hand side obtained by multiplying matrix elements with the known values. This gives

$$\begin{aligned} c_i u_i &+ \sum_{j=1}^{m} \left( \int_{\Gamma_j} q^* d\Gamma \right) u_j - \sum_{j=m+1}^{N} \left( \int_{\Gamma_j} u^* d\Gamma \right) q_j \\ &= \sum_{j=m+1}^{N} \left( \int_{\Gamma_j} q^* d\Gamma \right) u_j - \sum_{j=1}^{m} \left( \int_{\Gamma_j} u^* d\Gamma \right) q_j. \end{aligned} \tag{3.9}$$

The whole set of equations can be expressed in a matrix form as

$$\mathbf{A}\begin{pmatrix}\mathbf{q}_D\\ \mathbf{u}_N\end{pmatrix}=\mathbf{B}\begin{pmatrix}\mathbf{u}_D\\ \mathbf{q}_N\end{pmatrix}$$

where $\mathbf{u}_D$, $\mathbf{q}_D$ represent the Dirichlet and Neumann data on $\Gamma_D$ and $\mathbf{u}_N$,$\mathbf{q}_N$ represent the Dirichlet and Neumann data on $\Gamma_N$.

The step 3 of our iterative method can be presented as:

(i) solving the following linear equations :

$$\mathbf{A}\begin{pmatrix}\psi_{k+1}\\ u_{k+1}\end{pmatrix}=\mathbf{B}\begin{pmatrix}\phi\\ q_k\end{pmatrix}$$

and get $u_{k+1}$ that will be needed in the next equations.

(ii) With $u_{k+1}$, we can get $q_{k+1}$ by solving

$$\mathbf{B}\begin{pmatrix}\phi_{k+1}\\ q_{k+1}\end{pmatrix}=\mathbf{A}\begin{pmatrix}\psi\\ u_{k+1}\end{pmatrix}.$$

Our boundary element method gives a problem about computing linear equations twice in every iterative. It is easy to realize it by the technique of Matrix computing.

## 3.2 Convergence analysis

In this section we give the convergence analysis under the regularity assumption on the unknown potential $u$.

First of all, we simplify the **subproblem 1** as the following Cauchy problem for Laplace equation:

Let $\Omega\subset R^2$ be an open bounded set and $\Gamma_1$, $\Gamma_2$ be two parts of the boundary $\partial\Omega$, satisfying $\overline{\Gamma_1\cup\Gamma_2}=\partial\Omega$ and $\Gamma_1\cap\Gamma_2=\emptyset$.

$$\begin{cases}\Delta u=0, & x \text{ in } \Omega,\\ u=f, & x \text{ on } \Gamma_1,\\ u_\nu=g, & x \text{ on } \Gamma_1,\end{cases}\tag{3.10}$$

where $\nu$ is the unit outer derivative vector.

Given the Cauchy data $(f,g)\in H^{1/2}(\Gamma_1)\times H_{00}^{1/2}(\Gamma_1)'$, we assume that there exists an $H^1$-solution of problem (3.10). We are mainly interested in the determination of the Neumaan trace.

The following work is to introduce an operator $T:H_{00}^{1/2}(\Gamma_2)'\to H_{00}^{1/2}(\Gamma_2)'$ and represent the above iterative. Refer to [8].

We can simplify our iterative method as

$$\begin{cases}\Delta\omega=0 \text{ in } \Omega; & \omega|_{\Gamma_1}=f; & \omega_{\nu_A}|_{\Gamma_2}=\phi,\\ \Delta\nu=0 \text{ in } \Omega; & \nu_{\nu_A}|_{\Gamma_1}=g; & \nu|_{\Gamma_2}=\psi.\end{cases}$$

We define the operators $L_n : H_{00}^{1/2}(\Gamma_2)' \to H^1(\Omega)$ and $L_d : H^{1/2}(\Gamma_2) \to H^1(\Omega)$ by

$$L_n(\phi) := \omega \in H^1(\Omega),$$
$$L_d(\psi) := \nu \in H^1(\Omega).$$

Define the Neumann trace operator $\gamma_n : H^1(\Omega) \to H_{00}^{1/2}(\Gamma_2)'$, $\gamma_n(u) := u_{\nu_A}|_{\Gamma_2}$ and the Dirichlet trace operator $\gamma_d : H^1(\Omega) \to H^{1/2}(\Gamma_2)$, $\gamma_d(u) := u|_{\Gamma_2}$.

So we can rewrite the iterative as

$$\begin{cases} \omega = L_n(\phi_k); & \psi = \gamma_d(\omega), \\ \nu = L_d(\psi_k); & \phi_{k+1} = \gamma_n(\nu). \end{cases}$$

If we define $T := \gamma_n \circ L_d \circ \gamma_d \circ L_n$, we conclude that T is an affine operator on $H_{00}^{1/2}(\Gamma_2)$, which satisfies

$$\phi_{k+1} = T(\phi_k) = T^{k+1}(\phi_0).$$

That means we are able to describe the iterative with the powers of the operator T. As $L_n$ and $L_d$ are both affine, we can write

$$L_n(\cdot) = L_n^l(\cdot) + \omega_f, \qquad L_d(\cdot) = L_d^l(\cdot) + \nu_g,$$

where the $H^1(\Omega, P)$-functions $\omega_f$ and $\nu_g$ depend only on $f$ and $g$, respectively.

With these definitions we have

$$\begin{aligned} \phi_{k+1} = T(\phi_k) &= \underbrace{\gamma_n \circ L_d^l \circ \gamma_d \circ L_n^l(\phi_k)}_{T_l(\phi_k)} + \underbrace{\gamma_n \circ L_d^l \circ \gamma(\omega_f) + \gamma_n(\nu_g)}_{z_{f,g}} \\ &= T_l^{k+1}(\phi_0) + \Sigma_{j=0}^k T_l^j(z_{f,g}). \end{aligned}$$

From [8], we know the operator $T_l$ is positive, self adjoint, injective, regularly asymptotic in $H_{00}^{1/2}$ and non expansive. In [8] the convergence of this iterative method is presented. Under the source condition which is not so obvious for the engineers. Here we only use regularity assumptions in the convergence analysis. Since our problem is in an annular domain, the following theorems are discussed in the annular domain. But the results can be extended into a general domain.

Firstly, we define the Sobolev spaces of periodic functions

$$H_{per}^s(-\pi, \pi) := \{\phi(y) = \sum_{j \in Z} \phi_j e^{ijk} \Big| \sum_{j \in Z} (1 + j^2)^s \phi_j^2 < \infty\}, s \in R. \quad (3.11)$$

Before we give the theorems, we introduce the following logarithmic-type source conditions:

$$f(\lambda) = \begin{cases} (ln(exp(1)\lambda^{-1}))^{-p}, & \lambda > 0, \\ 0, & \lambda = 0. \end{cases} \quad (3.12)$$

**Theorem 3.2.** *Set $\Omega$ be an annular domain, $\Omega \subset R^2$. Let $(f,g)$ be consistent Cauchy data and assume that the solution $\overline{\phi}$ of the Cauchy problem* (3.10) *satisfies*

$$\overline{\phi} - \phi_0 \in H^1_{per},$$

*where $\phi_0 \in H$ is some initial guess. Let $\mu > 2$, $(f_\epsilon, g_\epsilon)$ be some given noisy data with $\|z_\epsilon - z_{f,g}\| \leq \epsilon$, $\epsilon > 0$ and $k(\epsilon, z_\epsilon)$ be the stopping rule determined by the discrepancy principle*

$$k(\epsilon, z_\epsilon) = \min\{k \in N | \|z_\epsilon - (I - T_l)\phi_k^\epsilon\| \leq \mu\epsilon\}. \tag{3.13}$$

*Then there exists a constant $C$, depending on $\phi_0$ only such that*

$$i) \quad \|\overline{\phi} - \phi_k^\epsilon\| \leq C(lnk)^{-1},$$
$$ii) \quad \|z_\epsilon - (I - T_l)\phi_k^\epsilon\| \leq Ck^{-1}(lnk)^{-1},$$

*for all iteration index $k$ satisfying $1 \leq k \leq k(\epsilon, z_\epsilon)$.*

**Theorem 3.3.** *Set $k_\epsilon = k(\epsilon, z_\epsilon)$. Under the assumption of Theorem 3.2 we have*

$$i) \quad k_\epsilon(ln(k_\epsilon)) = O(\epsilon^{-1}),$$
$$ii) \quad \|\overline{\phi} - \phi_{k_\epsilon}^\epsilon\| \leq O((-ln\sqrt{\epsilon})^{-1}).$$

The next lemma is most important for the proof of the theorems.

**Lemma 3.4.** *Set $\Omega$ be an annular domain, $\Omega \subset R^2$. Then the solution $\overline{\phi}$ of the Cauchy problem* (3.10) *in this domain satisfies*

$$\overline{\phi} - \phi_0 \in H^1_{per}, \tag{3.14}$$

*where $\phi_0 \in H$ is some initial guess and $H^1_{per}$ is the Sobolev spaces of periodic functions defined as in* (3.11). *This regularity assumption is equivalent to choosing some $\psi \in H^0_{per}$ satisfying*

$$\overline{\phi} - \phi_0 = f(I - T_l)\psi,$$

*where $f$ is the logarithmic-type source conditions* (3.12).

*Proof.* For simplicity, we consider Cauchy problem (3.10) in the annular domain

$$\Gamma_1 = \{(R, \theta); \theta \in (-\pi, \pi)\}, R > 1,$$

$$\Gamma_2 = \{(1, \theta); \theta \in (-\pi, \pi)\},$$

where $f(\theta) = \sum_{j=1}^N a_j \sin(j\theta)$, $g(\theta) = \sum_{j=1}^N b_j \sin(j\theta)$.

Given the Neumann data

$$\phi_0(\theta) = \sum_{j=1}^{N} \phi_{0,j} \sin(j\theta),$$

we can get

$$(T_l\phi_0)(\theta) = \sum_{j=1}^{\infty} \lambda_j \phi_{0,j} \sin(j\theta),$$

where

$$\lambda_j = \frac{(R^j - R^{-j})^2}{(R^j + R^{-j})^2}.$$

For $\overline{\phi} - \phi_0 \in H^1_{per}$, there exists $a_j, (j = 1 \cdots N)$ satisfying $\sum_{j=1}^{N} a_j^2 < \infty$,

$$\overline{\phi} - \phi_0 = \sum_{j=1}^{N} a_j j^{-1} \sin(jy).$$

So we get

$$\sum_{j=1}^{N} (1 + j^2) a_j^2 j^{-2} < \infty.$$

To the logarithmic-type source conditions (3.12), the source condition is to find some $\psi \in H^0_{per}$, satisfying

$$\overline{\phi} - \phi_0 = f(I - T_l)\psi.$$

So our problem comes into finding this $\psi$.
Set $\psi = \sum_{j=1}^{N} b_j \sin(jy)$, then

$$b_j = \frac{a_j}{jf(1 - \lambda_j)}.$$

From the estimate

$$\begin{aligned} ln\left(\frac{exp(1)}{1-\lambda_j}\right) &\geq 1 - ln\left(exp(1)\left[1 - \frac{R^j - R^{-j}}{R^j + R^{-j}}\right]\right) \\ &= -ln\left(\frac{2R^{-j}}{R^j + R^{-j}}\right) \\ &\geq 2jlnR - 1, \end{aligned}$$

$$\begin{aligned} ln\left(\frac{exp(1)}{1-\lambda_j}\right) &\leq 1 + ln\left(\frac{1}{1 - \frac{R^j - R^{-j}}{R^j + R^{-j}}}\right) \\ &= 1 + ln\left(\frac{R^j + R^{-j}}{2R^{-j}}\right) \\ &\leq 2jlnR + 1 - ln2, \end{aligned}$$

we have

$$2jlnR - 1 \leq \frac{1}{f(1-\lambda_j)} \leq 2jlnR + 1 - ln2$$

And with$\sum_{j=1}^{N} a_j^2 < \infty$, we can obtain $\sum_{j=1}^{N} b_j^2 < \infty$, i.e., $\psi \in H^0_{per}$. □

**Lemma 3.5.** *Let $(f, g)$ be consistent Cauchy data and assume that the solution $\overline{\phi}$ of the fixed point equation satisfies the source condition*

$$\overline{\phi} - \phi_0 = f(I - T_l)\psi, \quad for \quad some \quad \psi \in H,$$

*where $\phi_0 \in H$ is some initial guess and $f$ is the function defined in (3.12) with $p \geq 1$. Let $\mu > 2$, $(f_\epsilon, g_\epsilon)$ be some given noisy data with $\|z_\epsilon - z_{f,g}\| \leq \epsilon$, $\epsilon > 0$ and $k(\epsilon, z_\epsilon)$ the stopping rule determined by the discrepancy principle. Then there exists a constant $C$,depending on $p$ and $\|\psi\|$only such that*

$$\begin{aligned} &i) \quad \|\overline{\phi} - \phi_k^\epsilon\| \leq C(lnk)^{-p}, \\ &ii) \quad \|z_\epsilon - (I - T_l)\phi_k^\epsilon\| \leq Ck^{-1}(lnk)^{-p}, \end{aligned}$$

*for all iteration index $k$ satisfying $1 \leq k \leq k(\epsilon, z_\epsilon)$.*

**Lemma 3.6.** *Set $k_\epsilon = k(\epsilon, z_\epsilon)$. Under the assumption of Lemma 3.5 we have*

$$\begin{aligned} &i) \quad k_\epsilon(ln(k_\epsilon))^p = O(\epsilon^{-1}), \\ &ii) \quad \|\overline{\phi} - \phi_{k_\epsilon}^\epsilon\| \leq O((-ln\sqrt{\epsilon})^{-p}). \end{aligned}$$

The proof of lemma 3.5, lemma 3.6 can be found in [4].

With all the lemmas above, it is easy to give the proof. Theorem 3.2 can be deduced by lemma 3.4 and lemma 3.5. Theorem 3.3 can be deduced by lemma 3.4 and lemma 3.6.

## 3.3 Numerical experiment for the Maz'ya iteration

In this section, we will test the previous algorithm to calculate a few examples with Matlab. For simplicity, we set the domain $\Omega$ with interior radius 1 and outer radius $1+b$ in the following experiments. The number of the boundary element is $n$. Since we use the quadratic elements, we take $n$ nodes on the outer circle and also $n$ nodes on the interior circle. And set the number of nodes whose data are given to be $m$. we consider a harmonic function:

$$u(x, y) = \log\left[(x - 0.5)^2 + (y - 0.5)^2\right].$$

We use the prescribed algorithm to get the unknown data on the boundary, and then use the harmonic basic integral formulation to calculate the data on the circle with the radius $1 + a(a \leq b)$. In the following numerical experiment the noise level is $\delta$ noisy. The figures on the left show the exact solution compared with the approximate solution, and the dot line represents the approximate solution The real line represents the exact solution. The figures on the right side are the curves of the absolute errors. We use the stopping rule as $\|u_{k+1} - u_k\|_{L^2(\Gamma_1 \cup \Gamma_2)} \leq 10^{-3}$.

**Example 1.** In this experiment we take $n = 100,\ 200,\ m = 50, 100, b = 1, a = 0.5$ and $\delta = 0.01$, respectively.

$n$=100, $m$=50:

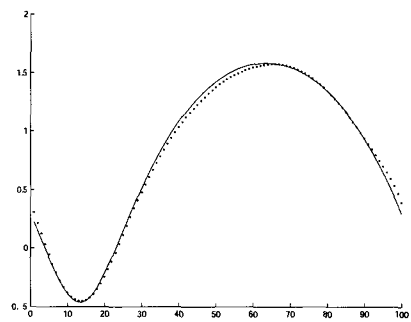


Figure 3.1

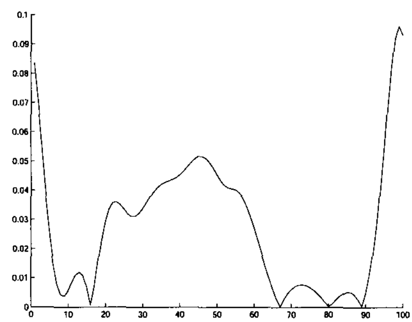


Figure 3.2

$n$=200, $m$=100:

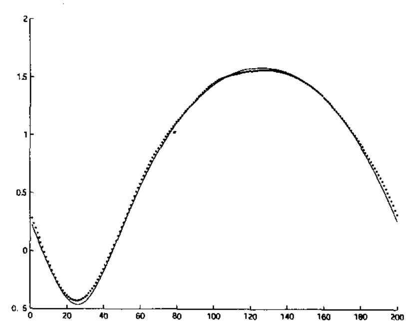


Figure 3.3

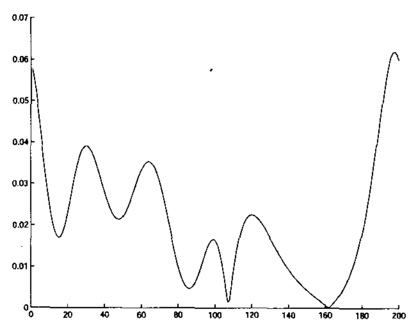


Figure 3.4

So if you want higher precision you should use more element during the process of this iterative.

**Example 2.** In this experiment we set $n = 100$, $m = 30$, $b = 1$, $a = 0.5$ and $\delta = 0.01,\ 0.001$, respectively.

$n$=100, $m$=30, $\delta = 0.01$:

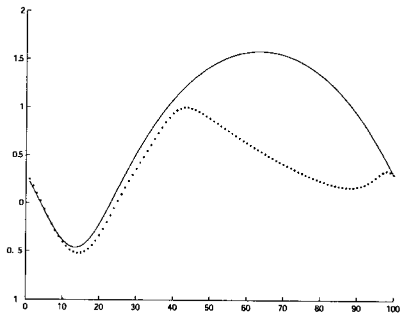



Figure 3.5

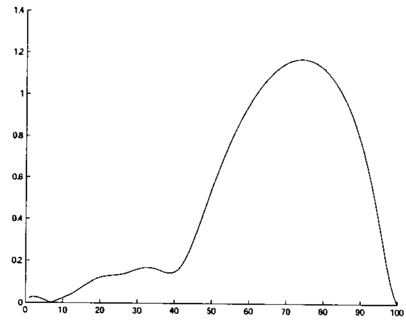



Figure 3.6

$n$=100, $m$=30, $\delta = 0.001$:

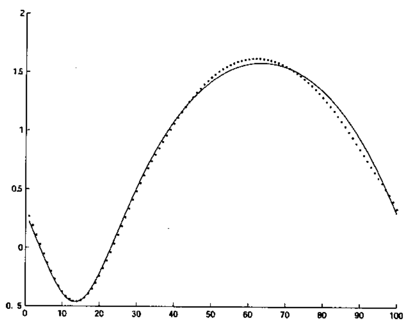



Figure 3.7

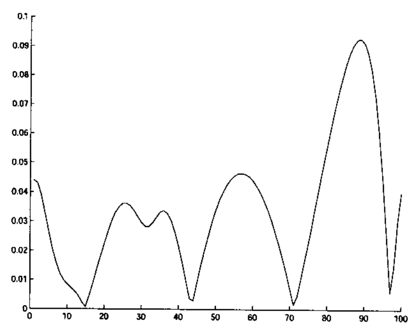



Figure 3.8

The numerical results show that the subproblem 1 is ill-posed in the Hadamard sense, i.e., the solution does not depend continuously on the data, which means the small errors in the measurement of the voltages on the boundary can produce unbounded errors in the solution.

**Example 3.** In this experiment we set $n = 100$, $m = 50$, $a = 0.1,\ 0.25,\ 0.5$, $b$=1 and $\delta = 0.01$ ( $a = 0.5$ is shown in the pervious example) *a*=0.25:

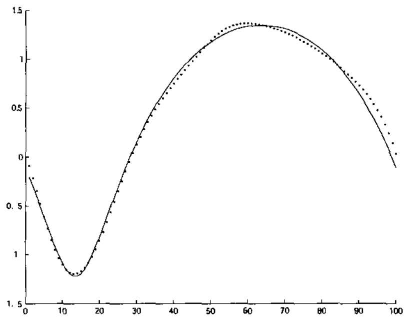


Figure 3.9

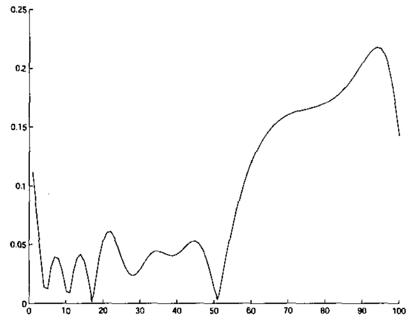


Figure 3.10

*a*=0.1:

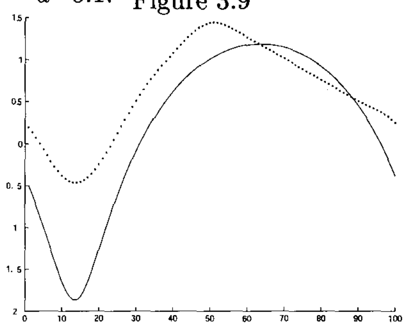


Figure 3.11

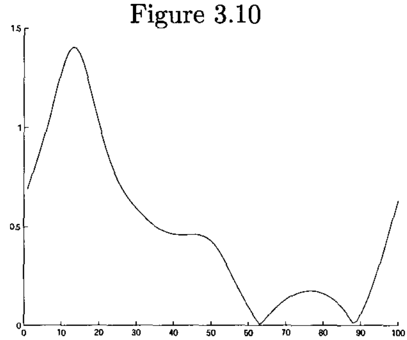


Figure 3.12

From the numerical simulation, it can be seen that the precision decreases as $a$ decreases.

# 4 Numerical results for the inverse problem

In this section we will use the iterative algorithm to treat our inverse problem, and give some numerical examples.

In the following test, we choose the ring domain as

$$\Omega = \{(x,y)|1 \leq \sqrt{x^2+y^2} \leq 2\}.$$

**Example 1.** In this test we recover the continuous piecewise linear function:

$$\gamma(\theta) = \begin{cases} 1 & \text{when} \quad \theta < 1, \\ 4\theta - 3 & \text{when} \quad 1 \leq \theta < 1.5, \\ -2\theta + 6 & \text{when} \quad 1.5 \leq \theta < 2.5, \\ 1 & \text{when} \quad 2.5 \leq \theta < 4.5, \\ 3\theta - \frac{25}{2} & \text{when} \quad 4.5 \leq \theta < 5.5, \\ -6\theta + 37 & \text{when} \quad 5.5 \leq \theta < 6, \\ 1 & \text{otherwise.} \end{cases}$$

We get the data $\phi$, $\psi$ from the solution of the direct problem by the boundary element:

$$\begin{cases} \triangle U = 0, & x \in \Omega, \\ U_n = -1, & x \in \Gamma_0 \cup \Gamma_1, \\ U_n + \gamma U = 0, & x \in \Gamma_2. \end{cases}$$

The follwing are the result figures.

(i)Set $m$=100, $n$=100

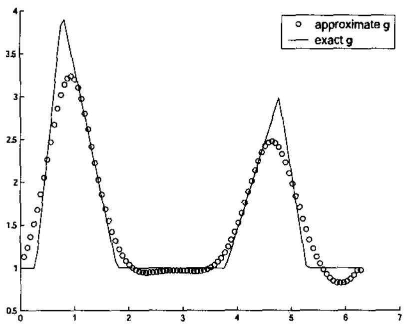



Figure 4.1

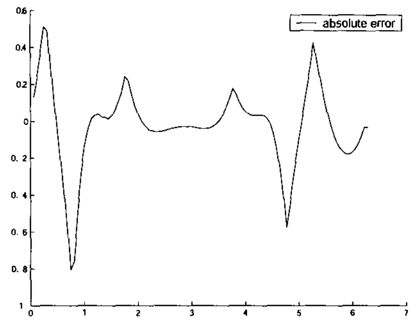



Figure 4.2

(ii)Set $m$=50, $n$=100

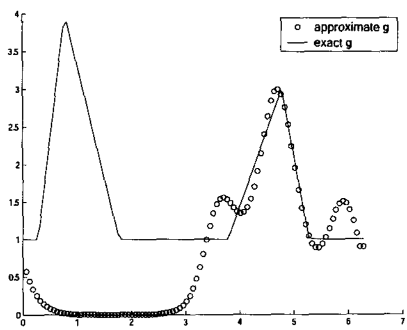



Figure 4.3

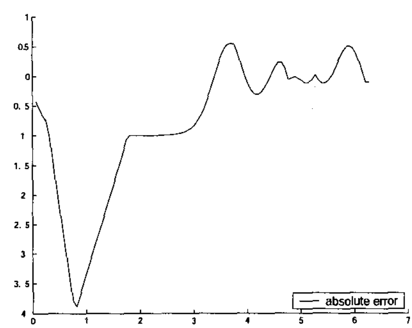



Figure 4.4

**Example 2.** We consider the harmonic function:

$$u(x, y) = y^3 - x^2y + x^2 - y^2 + 6,$$

whose polar coordinates form is

$$u(r, \theta) = r^3(\sin^3\theta - \cos^2\theta \sin\theta) + r^2 \cos 2\theta + 6.$$

It is easy to know that the coefficient $\gamma$ on the inner circle is

$$\gamma(\theta) = -\frac{3(\sin^3\theta - \cos^2\theta\sin\theta) + 2cos2\theta}{\sin^3\theta - \cos^2\theta\sin\theta + \cos 2\theta + 6}.$$

The figure on the left side is a comparison between the approximate $\gamma$ and the exact $\gamma$. The right one is the absolute error curve.

(i)Set $m$=100, $n$=100

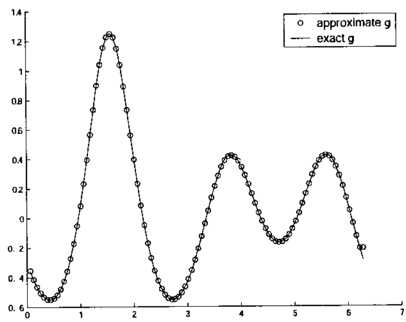



Figure 4.5

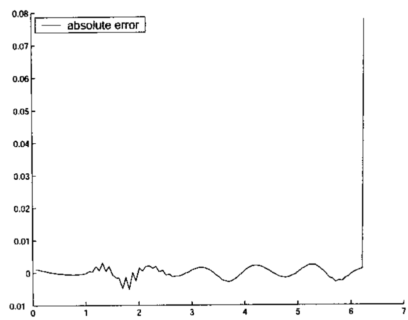



Figure 4.6

(ii)Set $m$=50, $n$=100

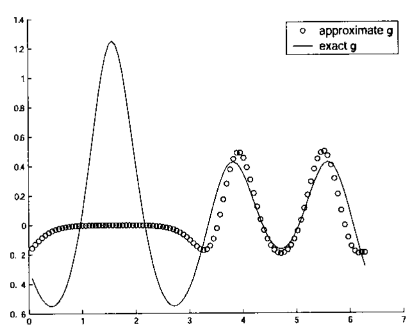



Figure 4.7

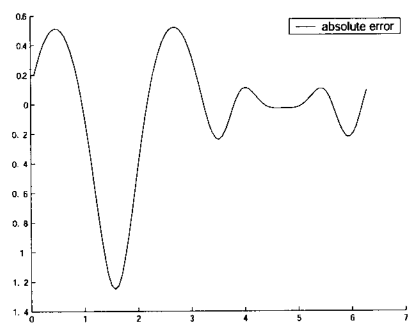



Figure 4.8

(iii)Set m=100, n=100 and add 5% random noisy on the Cauchy data.

It can be seen from the numerical results that there is a lot of noise in the numerical solution which means our method to this inverse problem is so sensitive. Any small error of the data may lead the iterative method not to converge. Another thing is that, if the Cauchy data is only given on part of the boundary, we can only obtain the local solution.

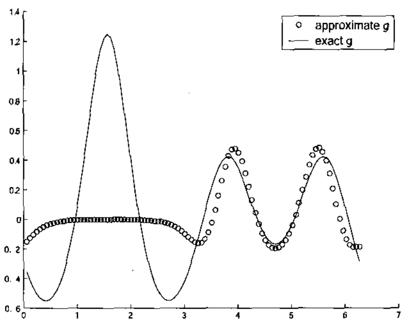



Figure 4.9

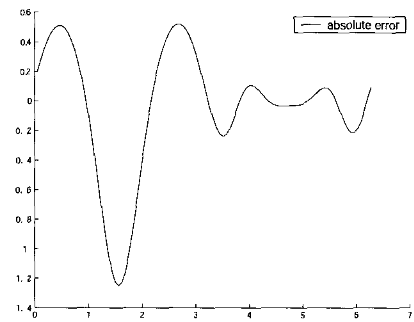



Figure 4.10

# 5 Conclusions

In this paper, we have investigated an inverse problem in detecting corrosion in a pipe. The problem has be modelled by Laplace equation with the unknown coefficient in the boundary condition.We deduce a numerical method to solve it and test the result with numerical experiments.

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